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REGULARITY AND CONTROLLABILITY FOR SEMILINEAR CONTROL SYSTEM

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We deal with the regularity and the approximate controllability for the semilinear system with time delay in Hilbert space. With the Lipschitz continuity of nonlinear operator f from $\mathcal{R} \times V$ to H, we will establish the problem for regularity result for the retarded functional differential equation. It is shown the equivalence between the reachable set of the semilinear system and that of its corresponding linear system.

Key Words : Regularity; Semilinear Control System; Reachable Set; Approximate Controllability

1. INTRODUCTION

In this paper, we deal with control problem for semilinear parabolic type equation in Hilbert space H as follows.

$$\frac{d}{dt} x(t) = A_0 x(t) + A_1 x(t-h) + \int_{-h}^{0} a(s)\, A_2 x(t+s)\, ds + f(t, x(t)) + B_0 u(t), \quad t \in (0, T]. \quad \text{... (1.1)}$$

Let A_0 be the operator associated with a sesquilinear form defined on $V \times V$ satisfying Gårding's inequality :

$$(A_0 u, v) = -a(u, v), \quad u, v \in V,$$

where V is a Hilbert space such that $V \subset H \subset V^*$. Then A_0 generates an analytic semigroup in both

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H and V^* and so the equation (1.1) may be considered as an equation in both H and V^*. Let the operators A_1 and A_2 be a bounded linear operators from V to V^* and $a(\cdot)$ be Hölder continuous. The nonlinear operator f from $\mathcal{R} \times V$ to H is Lipschitz continuous.

Under ensuring the wellposedness and regularity of solution of the eq. (1.1) we prove a result for the approximate controllability of (1.1) for sufficiently small time under suitable assumptions. There are many literatures which deal with structural properties for the linear system (the case where $f = 0$)[1-4] and Nakagiri[3] has dealt with structural properties and solution semigroups associated with (1.1). The control problem of general initial value problem without delay term was discussed frequently [3, 5-7]. With the aid of the solution semigroup and fundamental solution of (1.1) that was constructed[4], eq. (1.1) can be transformed into an abstract equation

$$\frac{d}{dt} z(t) = Az(t) + F(z(t)) + Bu(t) \qquad \dots (1.2)$$

in the product space $Z = H \times L^2(-h, 0; V)$. Here, $z(t) = (x(t), x_t(\cdot))$ where $x(t)$ is a solution of (1.1),

$$Ag = \left(A_0 g^0 + A_1 g^1(-h) + \int_{-h}^{0} a(s) A_2 g^1(s)\, ds,\ \frac{dg^1(\cdot)}{ds} \right),$$

$$F(z(t)) = (f(t, x(t)), 0) \text{ and } Bu = (B_0 u, 0).$$

Therefore, we can also apply the result[7] to this system, but we want to obtain more general conditions for retarded system (1.1) without time restriction[7]. For the control problem (1.2) where the operator A generates an analytic semigroup was dealt with in M. Yamamoto and Park[8]. But our case of the eq. (1.2) is that the operator A is the infinitesimal generator of C_0-semigroup. In [3, 5, 6 & 8], the authors showed the approximate controllable under assumption that the nonlinear term $f(t, x(t))$ is uniformly bounded. Now we note that it is known that the C_0-semigroup generated by A associated with the equation (1.1) is not compact operator in general (see theorem 5.3 in [1]). The control problem of (1.2) that the semigroup generated by A is compact operator was obtained by Naito[5, 6 & 9] using topological degree theory.

The first part of this paper is to give the wellposedness and regularity in sections 2 and 3. This approach is closed to that [1, 2] mentioned above. For the semilinear system (1.1), we will give the result by using the intermediate property and the contraction mapping principle. Next, under more generalized range condition of the controller than of in [5-7 & 10], we establish that the approximate controllability for semilinear system is equivalent to that of its corresponding linear system in section 4.

2. Wellposedness and Regularity

We consider the control problem for the following retarded functional differential equation of parabolic type with nonlinear term

$$\frac{d}{dt} x(t) = A_0 x(t) + A_2 x(t-h) + \int_{-h}^{0} a(s) A_2 x(t+s)\, ds + f(t, x(t)) + B_0 u(t), \qquad \dots (2.1)$$

$$x(0) = g^0, \quad x(s) = g^1(s), \quad s \in [-h, 0). \qquad \dots (2.2)$$

in Hilbert space in H. Let V be another Hilbert space such that $V \subset H \subset V^*$. Therefore, for the simplicity, we may regard that $\| u \|_* \leq | u | \leq \| u \|$ for all $v \in V$ where the notations $| \cdot |$, $\| \cdot \|$ and $\| \cdot \|_*$ denote the norms of H, V and V^* respectively as usual. Let $a(u, v)$ be a bounded sesquilinear form defined in $V \times V$ satisfying Gårding's inequality

$$Re\, a(u, u) \geq c_0 \| u \|^2 - c_1 | u |^2, \quad c_0 > 0, \quad c_1 \geq 0. \qquad \text{... (2.3)}$$

Let A_0 be the operator associated with a sesquilinear form

$$(A_0 u, v) = - a(u, v), \quad u, v \in V. \qquad \text{... (2.4)}$$

Then the operator A_0 is a bounded linear from V to V^*. The operators A_1 and A_2 are bounded linear operators from V to V^* such that they map $D(A_0)$ into H. We may assume that $(D(A_0), H)_{1/2, 2} = V$ satisfying

$$\| u \| \leq C_1 \| u \|_{D(A_0)}^{1/2} | u |^{1/2} \qquad \text{... (2.5)}$$

for some a constant $C_1 > 0$ where $(D(A_0), H)_{\theta, p}$ denotes the real interpolation space between $D(A_0)$ and H. The function $a(\cdot)$ is assumed to be a real valued Hölder continuous in $[-h, 0]$ and the controller operator B_0 is a bounded linear operator from some Banach space U to H. Let f be a nonlinear mapping from $\mathcal{R} \times V$ into H. We assume that for any $x_1, x_2 \in V$ there exists a constant $L > 0$ such that

$$| f(t, x_1) - f(t, x_2) | \leq L \| x_1 - x_2 \| \qquad \text{... (2.6)}$$

$$f(t, 0) = 0. \qquad \text{... (2.7)}$$

We may assume that (2.3) holds for $c_1 = 0$ as noting that $A_0 + c_1$ is an isomorphism from V to V^* if $c_1 \neq 0$.

Theorem 2.1 — *Under the above assumptions for the nonlinear mapping f, there exists a unique solution x of (2.1) and (2.2) such that*

$$x \in L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \subset C([0, T]; H)$$

for any $g = (g^0, g^1) \in Z = H \times L^2(-h, 0; V)$. Moreover, there exists a constant C such that

$$\| x \|_{L^2(0, T; V) \cap W^{1,2}(0, T; V^*)} \leq C(| g^0 | + \| g^1 \|_{L^2(-h, 0; V)} + \| u \|_{L^2(0, T; U)}),$$

where

$$\| \cdot \|_{L^2(0, T; V) \cap W^{1,2}(0, T; V^*)} = \max \{ \| \cdot \|_{L^2(0, T; V)}, \| \cdot \|_{W^{1,2}(0, T; V^*)} \}.$$

The proof will be shown a little later on. From now on, we consider the estimate of a solution of the problem (2.1) and (2.2) in accordance with the result of Theorem 3.3 of [1] if it exists.

Lemma 2.1 — Let $T > 0$. Then

$$H = \left\{ x \in V^* : \int_0^T \| A_0 e^{tA_0} x \|_*^2 \, dt < \infty \right\},$$

where $\| \cdot \|_*$ is the norm of the element of V^*.

PROOF : Put $u(t) = e^{tA_0} x$ for $x \in H$. From

$$\frac{1}{2} \frac{d}{dt} | u(t) |^2 = Re \, (\dot{u}(t), u(t)) = Re \, (A_0 u(t), u(t))$$

$$= - \; Re \;\; a(u(t), u(t)) \le - c_0 \| u(t) \|^2,$$

it follows

$$\frac{1}{2} \frac{d}{dt} | u(t) |^2 + c_0 \| u(t) \|^2 \le 0.$$

By integrating over t yields

$$\frac{1}{2} | u(t) |^2 + c_0 \int_0^t \| u(s) \|^2 \, ds \le \frac{1}{2} | x |^2.$$

Hence, we obtain that

$$\int_0^T \| A_0 e^{tA_0} x \|_*^2 \, dt \le \int_0^T \| u(s) \|^2 \, ds < \infty.$$

Conversely, suppose that $x \in V^*$ and $\int_0^T \| A_0 e^{tA_0} x \|_*^2 \, dt < \infty$. Put $u(t) = e^{tA_0} x$. Then since A_0 is an isomorphism operator from V to V^* there exists a constant $c > 0$ such that

$$\int_0^T \| u(t) \|^2 \, dt \le c \int_0^T \| A_0 u(t) \|_*^2 \, dt = c \int_0^T \| A_0 e^{tA_0} x \|_*^2 \, dt.$$

From the assumptions and $\dot{u}(t) = A_0 e^{tA_0} x$ it follows

$$u \in L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \subset C([0, T]; H).$$

Therefore, $x = u(0) \in H$.

Identifying the antidual of H with H we may consider $V \subset H \subset V^*$. The realization of A_0 in H which is the restriction of A_0 to

$$D(A_0) = \{u \in V : A_0 u \in H\}$$

is also denoted by A_0. It is known that A_0 generates an analytic semigroup in both H and V^*. Replacing intermediate space F in the paper [1] with the space H, we can derive the results of G. Kunisch and Sinestrari[1] regarding term by term to deduce the following result.

Proposition 2.1 — Let $g = (g^0, g^1) \in Z = H \times L^2(-h, 0; V)$ and $f \in L^2(0, T; V^*)$. Then for each $T > 0$, a solution x of the equation (2.1) and (2.2) belongs to

$$L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \subset C([0, T]; H).$$

Moreover, for some constant C_T we have

$$\|x\|_{L^2(0,T;V) \cap W^{1,2}(0,T;V^*)} \le C_T(|g^0| + \|g^1\|_{L^2(-h,0;V)} + \|f\|_{L^2(0,T;V^*)} + \|u\|_{L^2(0,T;U)}).$$

PROOF OF THEOREM 2.1 : Let us fix $T \in (0, h)$ such that

$$C_1 C_T L (T/\sqrt{2})^{1/2} < 1. \qquad \text{... (2.8)}$$

For $i = 1, 2$, we consider the following equation.

$$\frac{d}{dt} y_i(t) = A_0 y_i(t) + A_1 y_i(t-h) + \int_{-h}^{0} a(s) A_2 y_i(t+s)\, ds + f(t, x_i(t)) + B_0 u(t), \quad t \in (0, T]$$

$$y_i(0) = g^0, \quad y_i(s) = g^1(s), \quad s \in [-h, 0).$$

Then

$$\frac{d}{dt}(y_1(t) - y_2(t)) = A_0(y_1(t) - y_2(t)) + A_1(y_1(t-h) - y_2(t-h)) + \int_{-h}^{0} a(s) A_2(y_1(t+s) - y_2(t+s))\, ds + f(t, x_1(t)) - f(t, x_2(t)), \quad t \in (0, T]$$

$$y_1(0) - y_2(0) = 0, \quad y_1(s) - y_2(s) = 0, \quad s \in [-h, 0).$$

From Theorem 3.3 of [1] and (2.6) it follows that

$$\|y_1 - y_2\|_{L^2(0,T;D(A_0)) \cap W^{1,2}(0,T;H)} \le C_T \|f(\cdot, x_1) - f(\cdot, x_2)\|_{L^2(0,T;H)},$$

$$\|f(\cdot, x_1) - f(\cdot, x_2)\|_{L^2(0,T;H)} \le L \|x_1 - x_2\|_{L^2(0,T;V)}.$$

Using the Hölder inequality we also obtain that

$$\| y_1 - y_2 \|_{L^2(0,T;H)} = \left\{ \int_0^T | y_1(t) - y_2(t) |^2 dt \right\}^{1/2}$$

$$= \left\{ \int_0^T \left| \int_0^t (\dot{y}_1(\tau) - \dot{y}_2(\tau)) d\tau \right|^2 dt \right\}^{1/2}$$

$$\le \left\{ \int_0^T t \int_0^t | \dot{y}_1(\tau) - \dot{y}_2(\tau) |^2 d\tau dt \right\}^{1/2}$$

$$\le \frac{T}{\sqrt{2}} \| y_1 - y_2 \|_{W^{1,2}(0,T;H)}. \qquad \text{... (2.9)}$$

Therefore, in terms of (2.5) and (2.9) we have

$$\| y_1 - y_2 \|_{L^2(0,T;V)} \le C_1 \| y_1 - y_2 \|^{1/2}_{L^2(0,T;D(A_0))} \| y_1 - y_2 \|^{1/2}_{L^2(0,T;H)}$$

$$\le C_1 \| y_1 - y_2 \|^{1/2}_{L^2(0,T;D(A_0))} \left(\frac{T}{\sqrt{2}} \right)^{1/2} \| y_1 - y_2 \|^{1/2}_{W^{1,2}(0,T;H)}$$

$$\le C_1 \left(\frac{T}{\sqrt{2}} \right)^{1/2} \| y_1 - y_2 \|_{L^2(0,T;D(A_0)) \cap W^{1,2}(0,T:H)}$$

$$\le C_1 C_T \left(\frac{T}{\sqrt{2}} \right)^{1/2} \| f(\cdot, x_1) - f(\cdot, x_2) \|_{L^2(0,T;H)}$$

$$\le C_1 C_T L \left(\frac{T}{\sqrt{2}} \right)^{1/2} \| x_1 - x_2 \|_{L^2(0,T;V)}.$$

So by virtue of the condition (2.8) the contraction principle gives that the equation of (2.1) and (2.2) has a unique solution in $[-h, T]$.

Let $x(\cdot)$ be a solution of (2.1) and (2.2) and $y(\cdot)$ be a solution of following equation.

$$\frac{d}{dt} y(t) = A_0 y(t) + A_1 y(t-h) \int_{-h}^{0} a(s) A_2 y(t+s)\, ds + B_0 u(t), \qquad t \in (0, T]$$

$$y(0) = g^0, \quad y(s) = g^1(s), \quad s \in [-h, 0).$$

Consider the following problem :-

$$\frac{d}{dt}(x(t)-y(t)) = A_0(x(t)-y(t)) + A_1(x(t-h)-y(t-h))$$

$$+ \int_{-h}^{0} a(s)\, A_2(x(t+s)-y(t+s))\, ds + f(t, x(t)),$$

$$x(0) - y(0) = 0,\ x(s) - y(s) = 0 \qquad s \in [-h, 0).$$

In virtue of Theorem 3.3 of [1] we have

$$\| x-y \|_{L^2(0,T;D(A_0)) \cap W^{1,2}(0,T;H)} \le C_T \| f(\cdot, x) \|_{L^2(0,T;H)}$$

$$\le C_T L \| x \|_{L^2(0,T:V)}$$

$$\le C_T L (\| x-y \|_{L^2(0,T;V)} + \| y \|_{L^2(0,T;V)}).$$

Combining (2.5), (2.9) and the above inequality we have

$$\| x-y \|_{L^2(0,T;V)} \le C_1 \| x-y \|^{1/2}_{L^2(0,T;D(A_0))} \| x-y \|^{1/2}_{L^2(0,T;H)}$$

$$\le C_1 \| x-y \|^{1/2}_{L^2(0,T;D(A_0))} \left\{ \frac{T}{\sqrt{2}} \| x-y \|_{W^{1,2}(0,T;H)} \right\}^{1/2}$$

$$\le C_1 \left(\frac{T}{\sqrt{2}} \right)^{1/2} \| x-y \|_{L^2(0,T;D(A_0)) \cap W^{1,2}(0,T;H)}$$

$$\le C_1 \left(\frac{T}{\sqrt{2}} \right)^{1/2} C_T L (\| x-y \|_{L^2(0,T;V)} + \| y \|_{L^2(0,T;V)}).$$

Therefore, we have

$$\| x-y \|_{L^2(0,T;V)} \le \frac{C_1 C_T L \left(\frac{T}{\sqrt{2}} \right)^{1/2}}{1 - C_1 C_T L \left(\frac{T}{\sqrt{2}} \right)^{1/2}} \| y \|_{L^2(0,T;V)},$$

$$\| x \|_{L^2(0,T;V)} \le \frac{1}{1 - C_1 C_T L \left(\frac{T}{\sqrt{2}} \right)^{1/2}} \| y \|_{L^2(0,T;V)}. \qquad \ldots (2.10)$$

Combining Proposition 2.1 and (2.10) we obtain

$$\| x \|_{L^2(0,T;V) \cap W^{1,2}(0,T;V^*)} \le C_T (| g_0 | + \| g^1 \|_{L^2(0,T;V)}$$

$$+ \| f(\cdot, x) \|_{L^2(0,T;V^*)} + \| u \|_{L^2(0,T:U)})$$

$$\le C_T (| g^0 | + \| g^1 \|_{L^2(0,T;V)} + L \| x \|_{L^2(0,T;V)} + \| u \|_{L^2(0,T;U)})$$

$$\le C_T(|g_0| + \|g^1\|_{L^2(0,T;V)} + \|u\|_{L^2(0,T:U)})$$

$$+ \frac{L}{1 - C_1 C_T L\left(\frac{T}{\sqrt{2}}\right)^{1/2}} \|y\|_{L^2(0,T;V)}$$

$$\le C_T(|g_0| + \|g^1\|_{L^2(0,T;V)} + \|u\|_{L^2(0,T:U)})$$

$$+ \frac{LC_T}{1 - C_1 C_T L\left(\frac{T}{\sqrt{2}}\right)^{1/2}} (|g^0| + \|g^1\|_{L^2(0,T;V)} + \|u\|_{L^2(0,T;U)})$$

$$\le C(|g_0| + \|g^1\|_{L^2(0,T;V)} + \|u\|_{L^2(0,T:U)}).$$

Since the condition (2.8) is independent of initial value, the solution of (2.1) and (2.2) can be extended to the interval $[-h, nT]$ for every natural number n, and so the proof is complete.

3. Solutions of Semilinear Retarded System

In this section we first consider the fundamental solution of retarded system. The fundamental solution $W(t)$ of the eqs. (2.1) and (2.2) is defined as follows :

$$\frac{d}{dt} W(t) = A_0 W(t) + A_1 W(t-h) + \int_{-h}^{0} a(s) A_2 W(t+s) ds, \quad t > 0,$$

$$W(0) = I, \quad W(s) = 0, \quad s \in [-h, 0).$$

Since we are assuming that $a(\cdot)$ is Hölder continuous, as is seen, [3] the fundamental solution exists. Let A_0 generate an analytic semigroup $G(t)$ on H. Then as is seen[4], the fundamental solution $W(t)$ is a unique solution of

$$W(t) = G(t) + \int_0^t G(t-s)\,(A_1 W(s-h) + \int_{-h}^{0} a(\tau) A_2 W(\tau + s)\, d\tau)\, ds, \; t \ge 0, \qquad \text{... (3.1)}$$

$$W(s) = 0, \quad s \in [-h, 0). \qquad \text{... (3.2)}$$

It is also known that $W(t)$ is strongly continuous and $AW(t)$ and $dW(t)/dt$ are strongly continuous except at $t = nh$, $n = 0, 1, 2, ...$ Therefore we may assume that

$$|W(t)| \le M, \quad t \ge 0,$$

where M is a constant. The solution of (2.1) and (2.2) is expressed by

$$x(t) = W(t)\, g^0 + \int_{-h}^{0} U_t(s)\, g^1(s)\, ds + \int_0^t W(t-\tau) f(\tau, x(\tau))\, d\tau,$$

$$U_t(s) = W(t-s-h)A_1 + \int_{-h}^{s} W(t-s+\sigma)\, a(\sigma)\, A_2 d\sigma$$

in the sense of Nakagiri[3].

Lemma 3.1 — Let $f \in L^2(0, T; H)$ and $x(t) = \int_0^t W(t-s) f(s)\, ds$. Then there exists a constant C such that

$$\| x \|_{L^2(0, T; V)} \le C\sqrt{T}\, \| f \|_{L^2(0, T; H)}.$$

PROOF : By the similar way of Theorem 2.3 of[1] it holds that

$$\| x \|_{L^2(0, T; D(A_0))} \le C_T \| f \|_{L^2(0, T; H)}. \qquad \ldots (3.3)$$

By using Hölder inequality,

$$\| x \|^2_{L^2(0, T; H)} = \int_0^T \left| \int_0^t W(t-s) f(s)\, ds \right|^2 dt$$

$$\le M^2 \int_0^T \left(\int_0^t | f(s) |\, ds \right)^2 dt$$

$$\le M^2 \int_0^T t \int_0^t | f(s) |^2\, ds\, dt$$

$$\le M^2 \frac{T^2}{2} \int_0^t | f(s) |^2\, ds.$$

Therefore,

$$\| x \|_{L^2(0, T; H)} \le MT \| f \|_{L^2(0, T; H)}. \qquad \ldots (3.4)$$

Combining (3.3) and (3.4) we have that

$$\| x \|^2_{L^2(0, T; V)} \le C_T MT \| f \|^2_{L^2(0, T; H)}.$$

Let $Z = H \times L^2(-h, 0; V)$ be the state space and be a product Hilbert space with the norm

$$\| g \|_Z = \left(| g^0 |^2 + \int_{-h}^{0} \| g^1(s) \|^2\, ds \right)^{1/2}, \quad g = (g^0, g^1) \in Z.$$

Let $g \in Z$ and $x(t; g, f, u)$ be a solution of the equation (2.1) and (2.2) associated with nonlinear term f and control $B_0 u$ at time t.

Lemma 3.2 — Let $x_u(t) = x(t; g, f, u)$. Then for $T > 0$ there exists a constant C such that

(1) $\| f(\cdot, x_u) \|_{L^2(0,T;H)} \le C(\| g \|_Z + \| u \|_{L^2(0,T;U)})$,

(2) $\| f(\cdot, x_{u_1}) - f(\cdot, x_{u_2}) \|_{L^2(0,T;H)}$

$$\le LC\sqrt{T}/(1 - LC\sqrt{T}) \| B_0(u_1 - u_2) \|_{L^2(0,T;H)}.$$

PROOF : (1) From Theorem 2.1 it follows that

$$\| f(\cdot, x(\cdot)) \|_{L^2(0,T;H)} \le L \| x \|_{L^2(0,T;V)}$$

$$\le LC(\| g \|_Z + \| u \|_{L^2(0,T;U)}).$$

(2) From Lemma 3.1 it follows that

$$\| f(\cdot, x_{u_1}) - f(\cdot, x_{u_2}) \|_{L^2(0,T;H)} \le L \| x_{u_1} - x_{u_2} \|_{L^2(0,T;V)}$$

$$\le L \left\| \int_0^t W(t-s) \{ f(s, x_{u_1}(s)) - f(s, x_{u_2}(s)) \} \, ds \right\|_{L_t^2(0,T;V)}$$

$$+ L \left\| \int_0^t W(t-s) B \{ u_1(s) - u_2(s) \} \, ds \right\|_{L_t^2(0,T;V)}$$

$$\le LC\sqrt{T} \| f(\cdot, x_{u_1}) - f(\cdot, x_{u_2}) \|_{L^2(0,T;H)}$$

$$+ LC\sqrt{T} \| B_0(u_1 - u_2) \|_{L^2(0,T;U)}$$

where we set $\| f(t) \|_{L_t^2(0,T;V)} = \| f \|_{L^2(0,T;V)}$.

Remark 3.1 : In view of the result of Proposition 2.1 considered as an equation in V^*, we can define the solution semigroup for the problem (1.1) and (1.2) as follows :

$$S(t)g = (x(t; g, 0, 0), x_t(\cdot; g, 0, 0)) \qquad \text{... (3.5)}$$

for every $g = (g^0, g^1) \in Z$ where $x(t; g, 0, 0)$ is the solution of (2.1) and (2.2) with $f(t, x) = 0$ and $B_0 = 0$ and $x_t(s; g, 0, 0) = x(t + s; g, 0, 0)$ defined in $[-h, 0]$. Then the operator $S(t)$ is a C_0-semigroup on Z and the infinitesimal generator A of $S(t)$ is characterized by

$$D(A) = \{ g = (g^0, g^1) : g^0 \in H, \ g^1 \in L^2(-h, 0; V),$$

$$g^1(0) = g^0, A_0 g^0 + A_1 g^1(-h) + \int_{-h}^0 a(s) A_2 g^1(s) \, ds \in H \},$$

$$Ag = \left(A_0 g^0 + A_1 g^1(-h) + \int_{-h}^0 a(s) A_2 g^1(s) \, ds, \frac{dg^1(\cdot)}{ds} \right).$$

As is seen in[11], eq. (2.1) and (2.2) can be transformed into an abstract equation

$$z'(t) = Az(t) + F(z(t)) + Bu(t), \qquad \text{... (3.6)}$$

$$z(0) = g, \qquad \text{... (3.7)}$$

where the operator A is the infinitesimal generator of C_0-semigroup $S(t)$, $F(z(t)) = (f(t, x(t)), 0)$ and $Bu = (B_0 u, 0)$. Then the mild solution of initial problem (3.6) and (3.7) is represented by

$$z(t; g, f, Bu) = S(t)g + \int_0^t S(t-s)\, F(z(s))\, ds + \int_0^t S(t-s)\, Bu(s)\, ds.$$

4. Reachable Sets

We define reachable sets for the system (2.1) and (2.2) as follows :

$$L_T(g) = \{x(T; g, 0, u) : u \in L^2(0, T; U)\},$$

$$R_T(g) = \{x(T; g, f, u) : u \in L^2(0, T; U)\}.$$

We define a bounded linear operator from $L^2(0, T; H)$ to H by

$$\hat{W} p = \int_0^T W(T-s)\, p(s)\, ds$$

for $p \in L^2(0, T; H)$.

The system (2.1) and (2.2) is approximately controllable on $[0, T]$ if $\overline{R_T(g)} = H$, that is, for any $\varepsilon > 0$ and $x \in H$ there exists a control $u \in L^2(0, T; U)$ such that

$$\left| x - W(T)g^0 - \int_{-h}^0 U_T(s)\, g^1(s)\, ds - \hat{W} f(\cdot, x_u) - \hat{W} B_0 u \right| < \varepsilon.$$

We need the following hypotheses :

(B) For any $\varepsilon > 0$ and $p \in L^2(0, T; H)$ there exists a $u \in L^2(0, T; U)$ such that

$$| \hat{W} p - \hat{W} B_0 u | < \varepsilon,$$

$$\| B_0 u \|_{L^2(0, T; H)} \leq q \| p \|_{L^2(0, T; H)}$$

where q is a constant independent of p.

(E) $1 - 2LC\sqrt{T} > 0$, where the constants L and C are in (2.6) and in Theorem 2.1, respectively.

Remark 4.1 : The assumption (E) is a similar conception with those in Naito [p. 133 in 9] and George[12]. The constant C depends on T, L, M, $\| A_1 \|$ and $\| A_2 \|$ as is seen in Theorem 3.3 of [1]. The assumption (E) may be satisfied when L, T take sufficiently small values. But, in view of section 3.2 in [4], we may assume $\| W(t) \| < M' e^{\omega t}$, $M' > 0$. So, it may be possible to control the state at a large time T when the constant ω is negative, i.e., it occurs when the system satisfies the stability condition as in [13].

Remark 4.2 : Let us consider the following equation :

$$\frac{d}{dt}x(t) = A_0x(t) + A_0x(t-h) + f(t, x(t)) + B_0u(t),$$

$$x(0) = g^0, \quad x(s) = g^1(s), \quad s \in [-h, 0).$$

Then the fundamental solution $W(t)$ is represented by

$$W(t) = \sum_{j=0}^{n} \frac{(t-j)^j}{j!} A_0^j\, G(t-j), \quad t \in [n, n+1],\ n = 1, 2, \ldots$$

Thus we can substitute the assumption (B) with the following hypothesis :

(B′) For any $\varepsilon > 0$ and $p \in L^2(0, T; H)$ there exists a $u \in L^2(0, T; U)$ such that

$$\left| \int_0^t G(t-s)\, p(s)\, ds - \int_0^t G(t-s)\, B_0u(s)\, ds \right| < \varepsilon, \quad 0 \le t \le T,$$

$$\| B_0u \|_{L^2(0, t; H)} \le q \| p \|_{L^2(0, t; H)}.$$

Remark 4.3 : The mild solution $x(t)$ is also given by

$$x(t) = G(t)\, g^0 + \int_0^t G(t-s) \left\{ A_1x(s-h) + \int_{-h}^{0} a(\tau)\, A_2x(s+\tau)\, d\tau + f(s, x(s)) + B_0u(s) \right\} ds$$

for $t \ge 0$. As is seen in Fattorini [14; Remark 3.4 and Theorem 3.10] it is known that $L_T(g)$ with $A_1 = A_2 = 0$ is independent of both initial data g and time T.

Here, we note that the quantity condition of the constant $\hat{q}$ in Zhou[16] is not necessary. It is easily seen that if the range of the operator B_0 is dense in H then the condition is satisfied. Our concern is based on more general assumption than that in [10, 13, 16 & 17] which associated with the solution semigroup (3.5) without delay term in section 3. In [10; Example 2] and [15; Example 2] it is introduced simple examples of the control operator B that satisfies assumption (B).

Theorem 4.1' — *Let us assume hypotheses (B) and (E). Then we have that* $\overline{R_T(g)} = \overline{L_T(g)}$.

PROOF : Since the solution $x(T; g, f, u)$ belongs to $L^2(0, T; V)$, from the assumptions (2.6) and (2.7) we have $f(\cdot, x) \in L^2(0, T; V)$. Thus it implies that $\overline{R_T(g)} \subset \overline{L_T(g)}$ by virtue of the assumption (B). Now we will show that $\overline{L_T(g)} \subset \overline{R_T(g)}$. Let $z_T \in \overline{L_T(g)}$. Then for any given $\varepsilon > 0$ there exists $u \in L^2(0, T; U)$ such that

$$\left| z_T - W(T)\, g^0 - \int_{-h}^{0} U_T(s)\, g^1(s)\, ds - \hat{W} B_0u \right| \le \frac{\varepsilon}{2^3}. \qquad \ldots (4.1)$$

Let $v_1 \in L^2(0, T; U)$ is arbitrarily fixed. By assumption (B) there exists $v_2 \in L^2(0, T; U)$ such that

$$| \hat{W}(B_0 u - f(\cdot, x_{v_1})) - \hat{W} B_0 v_2 | \le \frac{\varepsilon}{2^3}. \qquad \text{... (4.2)}$$

From (4.1) and (4.2) it follows that

$$\left| z_T - W(T) g^0 - \int_{-h}^{0} U_T(s) g^1(s)\, ds - \hat{W} f(\cdot, x_{v_1}) - \hat{S} B_0 v_2 \right| \le \frac{\varepsilon}{2^2}. \qquad \text{... (4.3)}$$

We can choose $w_2 \in L^2(0, T; U)$ such that

$$| \hat{W}(f(\cdot, x_{v_2}) - f(\cdot, x_{v_1})) - \hat{W} B_0 w_2 | \le \frac{\varepsilon}{2^3}. \qquad \text{... (4.4)}$$

Therefore, from Lemma 3.2 it obtains that

$$\| B_0 w_2 \|_{L^2(0, T; H)} \le q \| f(\cdot, x_{v_2}) - f(\cdot, x_{v_1}) \|_{L^2(0, T; H)}$$

$$\le q \frac{LC\sqrt{T}}{1 - LC\sqrt{T}} \| B_0 v_2 - B_0 v_1 \|_{L^2(0, T; H)}.$$

Let us define $v_3 = v_2 - w_2$ in $L^2(0, T; U)$. Then from (4.3) and (4.4)

$$\left| z_T - W(T) g^0 - \int_{-h}^{0} U_T(s) g^1(s)\, ds - \hat{W} f(\cdot, x_{v_2}) - \hat{W} B_0 v_3 \right| \le \left(\frac{1}{2^2} + \frac{1}{2^3} \right) \varepsilon.$$

Define $v_n = v_{n-1} - w_{n-1}$ by induction. Then we have

$$\left| z_T - W(T) g^0 - \int_{-h}^{0} U_T(s) g^1(s)\, ds - \hat{W} f(\cdot, x_{v_n}) - B_0 v_{n+1} \right| \le \left(\frac{1}{2^2} + \dots + \frac{1}{2^{n+1}} \right) \varepsilon \le \frac{1}{2} \varepsilon$$

and

$$\| B_0 v_{n+1} - B_0 v_n \|_{L^2(0, T; H)}$$

$$\le q \frac{LC\sqrt{T}}{1 - LC\sqrt{T}} \| B_0 v_n - B_0 v_{n-1} \|_{L^2(0, T; H)}.$$

From the assumption (E), the sequence $\{B_0 v_n\}$ is Cauchy sequence and hence converges in $L^2(0, T; H)$. Thus there exists some integer N such that for all $n \ge N$ we have that

$$| \hat{W} B_0 v_{n+1} - \hat{W} B_0 v_n | \le \frac{1}{2} \varepsilon.$$

Therefore, it follows that

$$\left| z_t - W(T)\, g^0 - \int_{-h}^{0} U_T(s)\, g^1(s)\, ds - \hat{W} f(\cdot, x_{v_n}) - \hat{W} B_0 v_n \right|$$

$$\leq \left| z_t - W(T)\, g^0 - \int_{-h}^{0} U_T(s)\, g^1(s)\, ds - \hat{W} f(\cdot, x_{v_n}) - \hat{W} B_0 v_{n+1} \right| + | \hat{W} B_0 v_{n+1} - \hat{W} B_0 v_n |$$

$$\leq \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon \leq \varepsilon$$

for all $n \geq N$. Hence, we have proof that $\overline{L_T(g)} \subset \overline{R_T(g)}$.

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