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# ON THE PERFECT SYSTEMS OF THE SPECHT MODULES OF THE WEYL GROUPS OF TYPE $C_n$

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In this paper, we construct the perfect systems for the root systems of type $C_n$, wherefore we present an algorithm which results in a basis for the Specht modules of the Weyl groups of type $C_n$ in particular, application of the algorithm conforms with known results in the representation theory of the hyperoctahedral groups, so the construction of the irreducible representations of $W(C_n)$ in terms of root systems is completed.

**Key Wrods : Specht Module, Weyl Groups, Root Systems, Perfect Systems**

## 1. INTRODUCTION

Although a great deal of progress has been made in generalizing the representation theory of symmetric groups to Weyl groups, this is not as combinatorial in nature as what is so useful in the case of symmetric groups. The first attempt at providing such a generalization has been given by Morris[1], where the basic combinatorial concepts such as tableau, tabloid, etc., which were successful for symmetric groups as exemplified in the work of James[2], were interpreted in the context of root systems of Weyl groups. In recent years, a further development of these ideas has appeared in Halicioglu and Morris[3] and Halicioglu[4, 5].

For Weyl groups, the symmetric groups and the hyperoctahedral groups were taken as role models. The familiar concepts of Young tableaux, tabloids, etc., which are so crucial in the development of the representation theory of the symmetric groups and hyperoctahedral groups, are seen to have equally familiar counterparts in the context of root systems. In this alternative approach, the Weyl groups of type $A_n$ and $C_n$ are used to motivate a possible generalization to Weyl groups in general.

For the construction of a basis for the Specht modules of Weyl groups, Halicioglu[4] has considered the root systems of simply laced type only (i.e., $A_n$ $(n \geq 1)$, $D_n$ $(n \geq 4)$, $E_6, E_7, E_8$) and also parabolic subsystems only.

In this paper, we construct a basis for the Specht modules of the Weyl groups of type $C_n$, wherefore we present an algorithm which is a modification of Algorithm 4.1 of Halicioglu[4], so the construction of the irreducible representations of $W(C_n)$ in terms of root systems is completed.

## 2. Construction of Specht Modules

The construction of the representations of the Weyl groups in terms of root systems follows the processes of useful systems, good systems, very good systems and perfect systems, respectively. We now give a brief resume of the main results of Halicioglu and Morris[3] and Halicioglu[4] in a form suitable for our later purposes, with emphasis on the key concepts above. Full details may be found in the cited papers.

2.1. Let $\Phi$ be a root system relating to the Weyl group $W = W(\Phi)$ with simple system $\pi$ and Dynkin diagram $\Gamma$. Let $\Psi$ be a subsystem of $\Phi$ with simple system $J \subset \Phi^+$ and Dynkin[5] diagram $\Delta$. If $\Psi = \sum_{i=1}^{k} \Psi_i$, where $\Psi_i$ are the indecomposable components of $\Psi$, then let $J_i$ be a simple system in $\Psi_i$ $(i = 1, \ldots, k)$ and $J = \sum_{i=1}^{k} J_i$. Let $\Psi^{\perp}$ be the largest subsystem in $\Phi$ orthogonal to $\Psi$ and let $J^{\perp} \subset \Phi^+$ be the simple system of $\Psi^{\perp}$.

Let $\Psi'$ be a subsystem of $\Phi$ which is contained in $\Phi \backslash \Psi$, with simple system $J' \subset \Phi^+$ and Dynkin diagram $\Delta'$. If $\Psi' = \sum_{i=1}^{l} \Psi_i'$, where $\Psi_i'$ are the indecomposable components of $\Psi'$, then let $J_i'$ be a simple system in $\Psi_i'$ $(i = 1, \ldots, l)$ and $J' = \sum_{i=1}^{l} J_i'$. Let $\Psi'^{\perp}$ be the largest subsystem in $\Phi$ orthogonal to $\Psi'$ and let $J'^{\perp} \subset \Phi^+$ be the simple system of $\Psi'^{\perp}$.

Let $\bar{J}$ stand for the ordered set $\{J_1, \ldots, J_k ; J_1', \ldots, J_l'\}$, where in addition the elements in each $J_i$ and $J_i'$ are ordered, and put $\mathcal{T}_\Delta = \{w\bar{J} \mid w \in W\}$.

The pair $\bar{J} = \{J, J'\}$ is called a useful system in $\Phi$ if

$$W(J) \cap W(J') = \langle e \rangle \text{ and } W(J^{\perp}) \cap W(J'^{\perp}) = \langle e \rangle .$$

The elements of $\mathcal{T}_\Delta$ are called $\Delta$-tableaux, the $J_i$ and $J_i'$ are called the rows and columns of the useful system respectively.

This construction is a natural extension of the concept of a Young tableau in the representation theory of symmetric groups (for a fuller explanation, see Halicioglu and Morris[3]). We may also interpret this for the special case $W(C_n)$ with the help of the work of Morris[1] as follows.

2.2. Let $\Phi = C_n$ with simple system $\pi = \{\alpha_i = e_i - e_{i+1}$ $(i = 1, \ldots, n-1)$, $\alpha_n = 2e_n\}$. By Dynkin[6], let $\Psi = \sum_{i=1}^{r} A_{\lambda_i} + \sum_{j=1}^{s} C_{\mu_j}$ $\left( \sum_{i=1}^{r} (\lambda_i + 1) + \sum_{j=1}^{s} \mu_j = n \right)$, then let $J_{\lambda_i}^{(1)}$ and $J_{\mu_j}^{(2)}$ be simple systems in $A_{\lambda_i}$ $(i = 1, \ldots, r)$ and $C_{\mu_j}$ $(j = 1, \ldots, s)$ respectively and $J = J^{(1)} + J^{(2)}$, where

$J^{(1)} = \sum_{i=1}^{r} J_{\lambda_i}^{(1)}$ and $J^{(2)} = \sum_{j=1}^{s} J_{\mu_j}^{(2)}$, and let $\Psi' = \sum_{i=1}^{r'} C_{\lambda'_i} + \sum_{j=1}^{s'} A_{\mu'_j}$ $\left( \sum_{i=1}^{r'} \lambda'_i + \sum_{j=1}^{s'} (\mu'_j + 1) = n \right)$, then let $J'^{(1)}_{\lambda'_i}$ and $J'^{(2)}_{\mu'_j}$ be simple systems in $C_{\lambda'_i}$ $(i = 1, ..., r')$ and $A_{\mu'_j}$ $(j = 1, ..., s')$ respectively and $J' = J'^{(1)} + J'^{(2)}$, where $J'^{(1)} = \sum_{i=1}^{r'} J'^{(1)}_{\lambda'_i}$ and $J'^{(2)} = \sum_{j=1}^{s'} J'^{(2)}_{\mu'_j}$.

Inspired by the concept of a double Young tableau in Morris[1], we identify $\bar{J}$ with the ordered double set $\{(J^{(1)}; J'^{(1)}), (J^{(2)}; J'^{(2)})\}$ given by

$$\left\{ \left( J_{\lambda_1}^{(1)}, ..., J_{\lambda_r}^{(1)} ; J'^{(1)}_{\lambda'_1}, ..., J'^{(1)}_{\lambda'_{r'}} \right), \left( J_{\mu_1}^{(2)}, ..., J_{\mu_s}^{(2)} ; J'^{(2)}_{\mu'_1}, ..., J'^{(2)}_{\mu'_{s'}} \right) \right\},$$

where in addition the elements in each $J_{\lambda_i}^{(1)}, J_{\mu_j}^{(2)}, J'^{(1)}_{\lambda'_i}$ and $J'^{(2)}_{\mu'_j}$ are ordered.

Namely, for $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_r \geq 0$, $\mu_1 \geq \mu_2 \geq ... \geq \mu_s \geq 0$ and $\sum_{i=1}^{r} (\lambda_i + 1) + \sum_{j=1}^{s} \mu_j = n$, let $\Psi = \sum_{i=1}^{r} A_{\lambda_i} + \sum_{j=1}^{s} C_{\mu_j}$ be a subsystem of $\Phi$ then $(\lambda, \mu) = (\lambda_1 + 1, ..., \lambda_r + 1, \mu_1, ..., \mu_s)$ is a pair of partitions of $n$, and so the corresponding Weyl subgroup is $W(A_{\lambda_1}) \times ... \times W(A_{\lambda_r}) \times W(C_{\mu_1}) \times ... \times W(C_{\mu_s})$ which is isomorphic to the subgroup $S_{\lambda_1 + 1} \times ... \times S_{\lambda_r + 1} \times O_{\mu_1} \times ... \times O_{\mu_s}$ of the hyperoctahedral group $O_n$.

Put $k_0 = 0$, $k_i = \lambda_1 + ... + \lambda_i + i$ $(i = 1, ..., r)$ and $l_0 = k_r = \sum_{i=1}^{r} (\lambda_i + 1)$, $l_j = l_0 + \mu_1 + ... \mu_j$ $(j = 1, ..., s)$, then

$$J_{k_i}^{(1)} = \{\alpha_{k_{i-1}+1}, \alpha_{k_{i-1}+2}, ..., \alpha_{k_i} - 1\}$$

$$= \{e_{k_{i-1}+1} - e_{k_{i-1}+2}, e_{k_{i-1}+2} - e_{k_{i-1}+3}, ..., e_{k_i} - 1 - e_{k_i}\}$$

is a simple system for $A_{\lambda_i}$ and therefore $J^{(1)} = \sum_{i=1}^{r} J_{k_i}^{(1)}$ is a simple system for $\sum_{i=1}^{r} A_{\lambda_i}$, and

$$J_{l_j}^{(2)} = \{\alpha_{l_{j-1}+1}, \alpha_{l_{j-1}+2}, ..., \alpha_{l_j} - 1, 2e_{l_j}\}$$

$$= \{e_{l_{j-1}+1} - e_{l_{j-1}+2}, e_{l_{j-1}+2} - e_{l_j} - 1 + 3, ..., e_{l_j} - 1 - e_{l_j}, 2e_{l_j}\}$$

is a simple system for $C_{\mu_j}$ and therefore $J^{(2)} = \sum_{j=1}^{s} J_{l_j}^{(2)}$ is a simple system for $\sum_{j=1}^{s} C_{\mu_j}$. Thus, $J = J^{(1)} + J^{(2)}$ is a simple system for $\Psi = \sum_{i=1}^{r} A_{\lambda_i} + \sum_{j=1}^{s} C_{\mu_j}$, and the subsystem $\Psi$ may be represented by the rows of the $(\lambda,\mu)$-tableau

$$t = \begin{pmatrix} 1 & 2 & \cdots & & k_1 & k_r+1 & k_r+2 & \cdots & & l_1 \\ k_1+1 & k_1+2 & \cdots & k_2 & & l_1+1 & l_1+2 & \cdots & l_2 & \\ k_2+1 & k_2+2 & \cdots & k_3 & , & l_2+1 & l_2+2 & \cdots & l_3 & \\ \cdot & \cdot & \cdots & & & \cdot & \cdot & \cdots & & \\ k_{r-1}+1 & k_{r-1}+2 & \cdots k_r & & & l_{s-1}+1 & l_{s-1}+2 & \cdots & n & \end{pmatrix}$$

as in Morris[1], the other $2^n n!$ $(\lambda,\mu)$-tableaux being obtained by allowing the elements of $O_n$ to act on this tableau. The orthogonal subsystem $\Psi^{\perp}$ is the root system determined by the elements in rows of length one in the first part of the $(\lambda,\mu)$-tableau $t$.

Let $\Psi' = \sum_{i=1}^{r'} C_{\lambda'_i} + \sum_{j=1}^{s'} A_{\mu'_j}$ be the subsystem of $\Phi$ with simple system $J' = J'^{(1)} + J'^{(2)}$, where $J' = J'^{(1)} + J'^{(2)}$ is represented by the columns of the $(\lambda,\mu)$-tableau $t$ (in the last section, we shall show how to determine the $J'$). Then the orthogonal subsystem $\Psi'^{\perp}$ is the root system determined by the elements in columns of length one in the second part of the $(\lambda,\mu)$-tableau $t$. Hence, $W(J) \cong R_t$ and $W(J') \cong C_t$, where $R_t$ (resp. $C_t$) is the row (resp. column) stabilizer of the $(\lambda,\mu)$-tableau $t$.

Since $W(J) \cap W(J') = \langle e \rangle$ and $W(J^{\perp}) \cap W(J'^{\perp}) = \langle e \rangle$ then $\bar{J} = \{(J^{(1)}; J'^{(1)}), (J^{(2)}; J'^{(2)})\}$ is a useful system in $\Phi$. The $J_{\lambda_i}^{(1)}$ and $J_{\lambda'_i}'^{(1)}$ ($J_{\mu_j}^{(2)}$, and $J_{\mu'_j}'^{(2)}$) are called the rows and columns of the first part (second part) of the useful system respectively. From now on, we will take into account this construction for the useful systems in $C_n$.

2.3. Two $\Delta$-tableaux $\bar{J}$ and $\bar{K}$ are row equivalent, written $\bar{J} \sim \bar{K}$, if there exists $w \in W(J)$ such that $\bar{K} = w\bar{J}$. The equivalence class which contains the $\Delta$-tableaux $\bar{J}$ is $\{\bar{J}\}$ and is called a $\Delta$-tabloid.

Let $\tau_\Delta$ be the set of all $\Delta$-tabloids, then we have $\tau_\Delta = \{\{\overline{dJ}\} \mid d \in D_\Psi\}$, where $D_\Psi = \{w \in W \mid w(\alpha) \in \Phi^+ \text{ for all } \alpha \in J\}$ is a distinguished set of coset representatives for $W(\Psi)$ in $W$.

The Weyl group $W$ acts on $\tau_\Delta$ according to

$$\sigma\{\overline{wJ}\} = \{\overline{\sigma wJ}\} \quad \text{for all } \sigma \in W.$$

Let $K$ be an arbitrary field and let $M^\Delta$ be the $K$-space whose basis elements are the $\Delta$-tabloids. Extending this action to be linear on $M^\Delta$ turns $M^\Delta$ into a $KW$-module.

Define $\kappa_{\bar{J}} \in KW$ and $e_{\bar{J}}$ by

$$\kappa_{\bar{J}} = \sum_{\sigma \in W(J')} (sgn\, \sigma)\sigma \text{ and } e_{\bar{J}} = \kappa_{\bar{J}}\{\bar{J}\},$$

where $sgn\, \sigma = (-1)^{l(\sigma)}$ with $l(\sigma)$ being the length of $\sigma$. Then $e_{\bar{J}}$ is called the $\Delta$-polytabloid associated with $\bar{J}$.

The Specht module $S^{\Delta, \Delta'}$ is the submodule of $M^{\Delta}$ generated by $e_{\overline{wJ}}$, where $w \in W$.

A useful system $\bar{J}$ in $\Phi$ is called a good system if $d\Psi \cap \Psi' = \phi$ for $d \in D_{\Psi}$ then $\{\overline{dJ}\}$ appears in $e_{\bar{J}}$. If $\bar{J}$ is a good system in $\Phi$ and the characteristic of $K$ is zero, then $S^{\Delta, \Delta'}$ is irreducible.

2.4. Let $\bar{J}$ be a good system in $\Phi$, and $w \in W$. A $\Delta$-tableau $\overline{wJ}$ is standard if $w \in D_{\Psi} \cap \dot{D}_{\Psi'}$. A $\Delta$-tabloid $\{\overline{wJ}\}$ is standard if there is a standard $\Delta$-tableau in the equivalence class $\{\overline{wJ}\}$. A $\Delta$-polytabloid $e_{\overline{wJ}}$ is standard if $\overline{wJ}$ is standard.

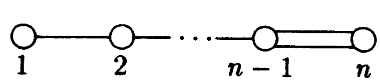
A good system $\bar{J}$ is called a very good system in $\Phi$ if, for all $d \in D_{\Psi} \cap D_{\Psi'}$ and $d' \in D_{\Psi}$, $d' = d\sigma\rho$ where $\sigma \in W(J'), \rho \in W(J)$ then $d \leq d'$. If $\bar{J}$ is a very good system in $\Phi$ then $\{e_{\overline{dJ}} \mid d \in D_{\Psi} \cap D_{\Psi'}\}$ is linearly independent over $K$. The question arises whether this set is a $K$-basis for $S^{\Delta, \Delta'}$. In that case, we say that $\bar{J}$ is a perfect system in $\Phi$ if the set $\{e_{\overline{dJ}} \mid d \in D_{\Psi} \cap D_{\Psi'}\}$ is a basis for $S^{\Delta, \Delta'}$.

Thus, given a subsystem $\Psi$ of $\Phi$ with simple system $J$, if we can determine a subsystem $\Psi'$ in $\Phi \setminus \Psi$ with simple system $J'$ such that $\bar{J}$ is a perfect system, not only is $S^{\Delta, \Delta'}$ an irreducible $KW$-module, but we also have a $K$-basis for $S^{\Delta, \Delta'}$ which consists of standard polytabloids. If $J'$ uniquely exists, then we call $J'$ the dual of $J$. For the construction of perfect systems, Halicioglu[4] has given an algorithm which is only valid for simply laced types (i.e. $A_n$ $(n \geq 1)$, $D_n$ $(n \geq 4)$, $E_6, E_7, E_8$) and their parabolic subsystems. In the following section, we modify it to cover type $C_n$.

## 3. Perfect Systems for the Root System of Type $C_n$

We now construct the perfect systems for the root system of type $C_n$, wherefore we present an algorithm which is a modification of Algorithm 4.1 of Halicioglu[4] and results in a suitable dual. In particular, application of the algorithm conforms with known results in the case of the hyperoctahedral groups in Morris[1].

Let $\Phi$ be a root system of type $C_n$ with simple system $\pi = \{\alpha_i = e_i - e_{i+1}$ $(i = 1, \ldots, n-1)$, $\alpha_n = 2e_n\}$. Then the Dynkin diagram for $\Phi$ is



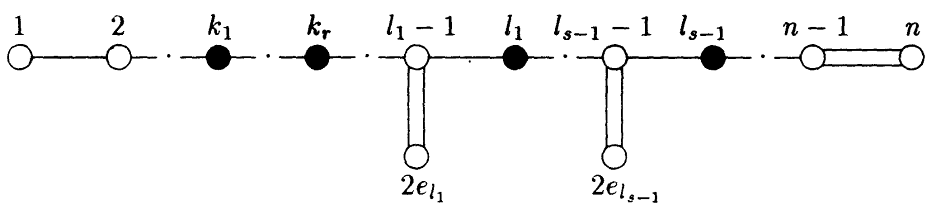
where the node corresponding to $\alpha_i$ $(i = 1, ..., n)$ is denoted by $i$.

*Definition* 3.1 — Let $\Phi = C_n$ with simple system $\pi$ as above and let

$$\pi_t = \{\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_t} \mid \{i_1, ..., i_t\} \subseteq \{1, 2, ..., n\}\}$$

be a subset of $\pi$. For $m \in \{1, ..., t\}$, the root $\alpha_{i_m} \in \pi_t$ is called the maximum root in $\pi_t$, written $\max\{\pi_t\} = \alpha_{i_m}$, if $\max\{i_1, ..., i_t\} = i_m$.

For $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_r \geq 0$, $\mu_1 \geq \mu_2 \geq ... \geq \mu_s \geq 0$ and $\sum_{i=1}^{r} (\lambda_i + 1) + \sum_{j=1}^{s} \mu_j = n$, let $\Psi = \sum_{i=1}^{r} A_{\lambda_i} + \sum_{j=1}^{s} C_{\mu_j}$ be a subsystem of $\Phi$ with simple system $J = J^{(1)} + J^{(2)}$ with the notation as in (2.2), then the Dynkin diagram $\Delta$ for $\Psi$ is




that is, the nodes $k_1, ..., k_r, l_1, ..., l_{s-1}$ have been deleted and the nodes $2e_{l_1}, ..., 2e_{l_{s-1}}$ have been added. If we do not consider the nodes $2e_{l_1}, ..., 2e_{l_{s-1}}$ which have been added to the Dynkin diagram of $\Phi$ then we may look upon the simple system $J$ of the subsystem $\Psi = \sum_{i=1}^{r} A_{\lambda_i} + \sum_{j=1}^{s} C_{\mu_j}$ as a subset of $\pi$. In particular, if $\Psi = \sum_{i=1}^{r} A_{\lambda_i}$ with $\sum_{i=1}^{r} (\lambda_i + 1) = n$ then we may look upon the subsystem $\Psi = \sum_{i=1}^{r} A_{\lambda_i}$ as a parabolic subsystem of $\Phi$ with simple system $J = J^{(1)} \subset \pi$.

Now, let $\Psi = \sum_{i=1}^{r} A_{\lambda_i} + \sum_{j=1}^{s} C_{\mu_j}$ be a subsystem of $\Phi$ with simple system $J$ such that

$$\pi_J = J \setminus \{\alpha \in J \mid \alpha = 2e_k,\ 1 \leq k \leq n-1\} \subseteq \pi$$

where $J = J^{(1)} + J^{(2)}$ with $J^{(1)} = \sum_{i=1}^{r} J_{\lambda_i}^{(1)}$ and $J^{(2)} = \sum_{j=1}^{s} J_{\mu_j}^{(2)}$ are simple systems for $\sum_{i=1}^{r} A_{\lambda_i}$ and $\sum_{j=1}^{s} C_{\mu_j}$ respectively.

For $J^{(2)} = \sum_{j=1}^{s} J_{\mu_j}^{(2)}$, let $J_{\mu_j}^{(2)*} = J_{\mu_j}^{(2)} \setminus \{\alpha \in J_{\mu_j}^{(2)} \mid \alpha = 2e_k,\ 1 \le k \le n\}$ where $j = 1, \ldots, s$, and let $J^{(2)*} = \sum_{j=1}^{s} J_{\mu_j}^{(2)*}$. Let $\pi \setminus \pi_J = \{\beta_1, \beta_2, \ldots, \beta_q\}$ be the set of all the deleted nodes from the Dynkin diagram of $\Phi$. For $1 \le p \le q$, let $\pi_{J^{(2)}} = \{\beta_p, \beta_{p+1}, \ldots, \beta_q\}$ be the subset of $\pi \setminus \pi_J$ such that each element in $\pi_{J^{(2)}}$ is connected to two components of $J^{(2)}$ (i.e. for each element $\gamma \in \pi_{J^{(2)}}$ there exist two components of $J^{(2)}$ which have a node connected to $\gamma$). Clearly if $\Psi$ is one of the subsystems of the form

$$\sum_{i=1}^{r} A_{\lambda_i},\ C_\mu\ (1 \le \mu < n),\ \sum_{i=1}^{r} A_{\lambda_i} + C_\mu\ (1 \le \mu < n)$$

then $\pi_{J^{(2)}} = \phi$ and if $\Psi = \sum_{j=1}^{s} C_{\mu_j}$ with $\sum_{j=1}^{s} \mu_j = n$ then $\pi_{J^{(2)}} = \pi \setminus \pi_J$.

Suppose that $\pi \setminus \pi_J = \{\beta_1, \ldots, \beta_{p-1}\} \cup \pi_{J^{(2)}}$. For $v = 1, 2, \ldots, p-1$ let

$$\Pi_v^{(1)} = \bigcup \{J_{\lambda_i}^{(1)} \mid (\alpha_{\lambda_i}^{(1)}, \beta_v) \ne 0 \text{ for some } \alpha_{\lambda_i}^{(1)} \in J_{\lambda_i}^{(1)}\},$$

that is, the components of $J^{(1)}$ which have a node connected to $\beta_v$.

For $u = p, p+1, \ldots, q$ let

$$\Pi_u^{(2)} = \bigcup \{J_{\mu_j}^{(2)*} \mid (\alpha_{\mu_j}^{(2)}, \beta_u) \ne 0 \text{ for some } \alpha_{\mu_j}^{(2)} \in J_{\mu_j}^{(2)*}\},$$

that is, the components of $J^{(2)*}$ which have a node connected to $\beta_u$.

*Algorithm* 3.1 — (a) For each $\beta_v$ ($v = 1, 2, \ldots, p-1$) let $\Phi_v^{(1)}$ be the root system with simple system $\Pi_v^{(1)} \cup \{\beta_v\}$ and let $n_v^{(1)}$ be the length of the longest positive root in $\Phi_v^{(1)}$.

(i) Define recursively $D_{v,t}^{(1)}$ ($t = 1, \ldots, n_v^{(1)}$) as follows : Put $D_{v,1}^{(1)} = \{\beta_v\}$ and $D_{v,t+1}^{(1)} = \{\tau_\alpha(\beta) \mid (\alpha, \beta) < 0,\ \alpha \in \Pi_v^{(1)},\ \beta \in D_{v,t}^{(1)}\}$.

(ii) For each $\beta_v$ ($v = 1, 2, \ldots, p-1$) such that $\beta_v \ne 2e_n$, let $\max\{\Pi_v^{(1)} \cup \{\beta_v\}\} = \alpha_i = e_i - e_{i+1}$ $(1 \le i \le n-1)$ be the maximum root in $\Pi_v^{(1)} \cup \{\beta_v\}$. For this maximum root, choose $2e_{i+1} \in \Phi$, and for $t = 1, \ldots, n_v^{(1)}$ put

$$D_{v,t} = \{\alpha \in D_{v,t}^{(1)} \mid (\alpha, 2e_{i+1}) < 0\}, \text{ and}$$

$$D'_{v,t} = \{\tau_\alpha(2e_{i+1}) \mid \alpha \in D_{v,t}\}.$$

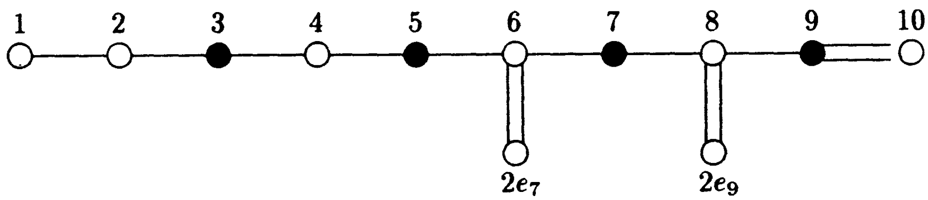
(b) For each $\beta_u \in \pi_{J^{(2)}}$ $(u = p, p+1, \ldots, q)$ let $\Phi_u^{(2)}$ be the root system with simple system $\Pi_u^{(2)} \cup \{\beta_u\}$ and let $n_u^{(2)}$ be the length of the longest positive root in $\Phi_u^{(2)}$

Define recursively $D_{u,t}^{(2)}$ $(t = 1, \ldots, n_u^{(2)})$ as follows : Put $D_{u,1}^{(2)} = \{\beta_u\}$ and $D_{u,t+1}^{(2)} = \{\tau_\alpha(\beta) \mid (\alpha, \beta) < 0,\ \alpha \in \Pi_u^{(2)},\ \beta \in D_{u,t}^{(2)}\}$.

*Example* 3.1 — Let $\Phi = C_{10}$ with simple system $\pi = \{\alpha_i = e_i - e_{i+1}$ $(i = 1, 2, \ldots, 9)$, $\alpha_{10} = 2e_{10}\}$. Let $\Psi = A_2 + A_1 + C_2 + C_2 + C_1$ be a subsystem of $C_{10}$ with simple system

$$J = J^{(1)} + J^{(2)} = \{e_1 - e_2, e_2 - e_3, e_4 - e_5\} \cup \{e_6 - e_7, 2e_7, e_8 - e_9, 2e_9, 2e_{10}\}.$$

Then the Dynkin diagram for $\Psi$ is




The set $\pi \setminus \pi_J = \{\alpha_3, \alpha_5\} \cup \pi_{J^{(2)}}$ where $\pi_{J^{(2)}} = \{\alpha_7, \alpha_9\}$ and $J^{(2)*} = \{e_6 - e_7, e_8 - e_9\}$.

(*i*) Consider $\alpha_3$ as a deleted node. Then by part (*a*) of Algorithm 3.1, $\Pi_3^{(1)} = \{e_1 - e_2, e_2 - e_3, e_4 - e_5\}$, and for $max\{\Pi_3^{(1)} \cup \{\alpha_3\}\} = e_4 - e_5$, choose $2e_5 \in \Phi$. Then

$$D_{3,1}^{(1)} = \{e_3 - e_4\} \qquad D'_{3,1} = \{\phi\}$$

$$D_{3,2}^{(1)} = \{e_2 - e_4, e_3 - e_5\} \qquad D'_{3,2} = \{2e_3\}$$

$$D_{3,3}^{(1)} = \{e_1 - e_4, e_2 - e_5\} \qquad D'_{3,3} = \{2e_2\}$$

$$D_{3,4}^{(1)} = \{e_1 - e_5\} \qquad D'_{3,4} = \{2e_1\}$$

(*ii*) Consider $\alpha_5$ as a deleted node. Then by part (*a*) of Algorithm 3.1, $\Pi_5^{(1)} = \{e_4 - e_5\}$, and for $max\{\Pi_5^{(1)} \cup \{\alpha_5\}\} = e_5 - e_6$, choose $2e_6 \in \Phi$. Then

$$D_{5,1}^{(1)} = \{e_5 - e_6\} \qquad D'_{5,1} = \{2e_5\}$$

$$D_{5,2}^{(1)} = \{e_4 - e_6\} \qquad D'_{5,2} = \{2e_4\}$$

(*iii*) Consider $\alpha_7 \in \pi_{J^{(2)}}$ as a deleted node. Then by part (*b*) of Algorithm 3.1 we have

$\Pi_7^{(2)} = \{e_6 - e_7, e_8 - e_9\}$

$D_{7,1}^{(2)} = \{e_7 - e_8\}$

$D_{7,2}^{(2)} = \{e_6 - e_8, e_7 - e_9\}$

$D_{7,3}^{(2)} = \{e_6 - e_9\}.$

(*iv*) Consider $\alpha_9 \in \pi_{J^{(2)}}$ as a deleted node. Then by part (*b*) of Algorithm 3.1 we have

$\Pi_9^{(2)} = \{e_8 - e_9\}$

$D_{9,1}^{(2)} = \{e_9 - e_{10}\}$

$D_{9,2}^{(2)} = \{e_8 - e_{10}\}.$

On the other hand, the subsystem $\Psi = A_2 + A_1 + C_2 + C_2 + C_1$ corresponds to the pair of partitions $(\lambda, \mu) = (32,221)$ of 10. Thus the subsystem $\Psi = A_2 + A_1 + C_2 + C_2 + C_1$ is represented by the rows of the tableau

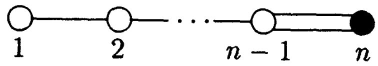
$$t = \begin{pmatrix} 1 & 2 & 3 & & 6 & 7 \\ 4 & 5 & & , & 8 & 9 \\ & & & & 10 & \end{pmatrix}.$$

Then its row stabilizer $R_t$ is $S_{\{1,2,3\}} \times S_{\{4,5\}} \times O_{\{\pm 6, \pm 7\}} \times O_{\{\pm 8, \pm 9\}} \times O_{\{\pm 10\}}$ and its column stabilizer $C_t$ is $O_{\{\pm 1, \pm 4\}} \times O_{\{\pm 2, \pm 5\}} \times O_{\{\pm 3\}} \times S_{\{6,8,10\}} \times S_{\{7, 9\}}$, as in Morris[1]. Now, put $J'^{(1)} = D_{3,3}^{(1)} + D'_{3,2} + D'_{5,1} + D'_{5,2}$ and $J'^{(2)} = D_{7,2}^{(2)} + D_{9,2}^{(2)}$ then $J'^{(1)} + J'^{(2)} = \{e_1 - e_4, 2e_4, e_2 - e_5, 2e_5, 2e_3\} \cup \{e_6 - e_8, e_8 - e_{10}, e_7 - e_9\}$ is linearly independent over $R$. If we put $J' = J'^{(1)} + J'^{(2)}$ then the corresponding subsystem $\Psi'$ is $C_2 + C_2 + C_1 + A_2 + A_1$ with simple system $J' = J'^{(1)} + J'^{(2)}$. The subsystem $\Psi' = C_2 + C_2 + C_1 + A_2 + A_1$ is represented by the columns of the tableau $t$, and so $R_t \cong W(J)$ and $C_t \cong W(J')$. Thus, known results Morris[1] in the representation theory of $O_n$ gives that $\bar{J} = \{(J^{(1)}; J'^{(1)}), (J^{(2)}; J'^{(2)})\}$ is a perfect system in $C_{10}$.

*Remark* 3.1 : Let $\Psi$ be a subsystem of $\Phi$ with simple system $J = J^{(1)} + J^{(2)}$ as given earlier. Then the subsystem $\Psi$ is represented by the rows of the $(\lambda, \mu)$-tableau $t$ as in (2.2). Since the Algorithm 3.1 enables us to construct the subsystem $\Psi'$ such that its simple system $J'$ is represented by the columns of the $(\lambda, \mu)$-tableau $t$, the remainder of the paper shows that this is true in general. Furthermore, when we consider the construction of the useful system given in (2.2), we see that this work can be translated to the language of $(\lambda, \mu)$-tableaux in the hyperoctahedral groups context, that is, the key concepts (i.e. the useful systems, good systems, very good systems and perfect systems) of this paper are reduced to the standard $(\lambda, \mu)$-tableaux.

We now apply Algorithm 3.1 step by step to determine a subsystem $\Psi'$ in $\Phi\backslash\Psi$ with simple system $J' = J'^{(1)} + J'^{(2)}$ such that $\bar{J} = \{(J^{(1)}; J'^{(1)}), (J^{(2)}; J'^{(2)})\}$ is a perfect system in $\Phi$. For this, we consider the following possible cases:

(1) Let $\Psi = A_{n-1}$ be a subsystem of $\Phi$ with simple system $J = J^{(1)} = \{\alpha_1, \alpha_2, ..., \alpha_{n-1}\}$. Then the Dynkin diagram for $A_{n-1}$ is




Consider $\alpha_n$ as a deleted node. Then by applying part (*a*) of Algorithm 3.1 we obtain

$$\Pi_n^{(1)} = \{\alpha_1, \alpha_2, ..., \alpha_{n-1}\}$$

$$D_{n,1}^{(1)} = \{\alpha_n\} = \{2e_n\}$$

$$D_{n,t}^{(1)} = \{2e_{n-t+1}\} \ (t = 2, 3, ..., n).$$

Since $\alpha_n = 2e_n$, we do not consider the part (*a*) (*ii*) of Algorithm 3.1.

On the other hand, the subsystem $\Psi = A_{n-1}$ corresponds to the pair of partitions $(\lambda, \mu) = (n, 0)$ of $n$. Thus the subsystem $\Psi = A_{n-1}$ is represented by the row of the tableau

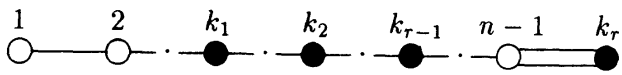
$$t = (1, 2, ..., n, \phi).$$

Then its row stabilizer $R_t$ is $S_{\{1, ..., n\}}$ and its column stabilizer $C_t$ is $O_{\{\pm 1\}} \times O_{\{\pm 2\}} \times ... \times O_{\{\pm n\}}$, as in Morris[1]. Now, put $J'^{(1)} = \sum_{t=1}^{n} D_{n,t}^{(1)}$ then $J'^{(1)}$ is linearly independent over $\boldsymbol{R}$. If we put $J' = J'^{(1)}$ then $J'$ is a simple system for $\Psi' = nC_1$. The subsystem $\Psi' = nC_1$ is represented by the columns of the tableau $t$, and so $R_t \cong W(J)$ and $C_t \cong W(J')$. It follows that $\bar{J} = \{(J^{(1)}; J'^{(1)}), (\phi, \phi)\}$ is a perfect system in $\Phi$. Then we have the following lemma.

*Lemma* 3.1 — Let $\Psi = A_{n-1}$ be a subsystem of $\Phi$ with simple system given by $J = J^{(1)} = \{\alpha_1, ..., \alpha_{n-1}\}$ and let $\Psi'$ be the subsystem of $\Phi$ with simple system $J' = J'^{(1)} = \sum_{t=1}^{n} D_{n,t}^{(1)}$. Then $\bar{J} = \{(J^{(1)}; J'^{(1)}), (\phi, \phi)\}$ is a perfect system in $\Phi$.

(2) For $r \geq 2, \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_r \geq 0$ and $\sum_{i=1}^{r} (\lambda_i + 1) = n$, let $\Psi = \sum_{i=1}^{r} A_{\lambda_i}$ be a subsystem of $\Phi$. Put $k_0 = 0, k_i = \lambda_1 + \lambda_2 + ... + \lambda_i + i \ (i = 1, ..., r)$ then $J_{k_i}^{(1)} = \{\alpha_{k_{i-1}+1}, \alpha_{k_{i-1}+2}, ..., \alpha_{k_i - 1}\}$ is a

simple system for $A_{\lambda_i}$ and therefore $J = J^{(1)} = \sum_{i=1}^{r} J^{(1)}_{k_i}$ is a simple system for $\Psi = \sum_{i=1}^{r} A_{\lambda_i}$. The Dynkin diagram for $\Psi$ is




By part (*a*) (*i*) of Algorithm 3.1 for $v = 1, 2, .., r$ we obtain $\Pi^{(1)}_{k_v}$ and $D^{(1)}_{k_v, t}\,(1 \le t \le \lambda_v + 1)$ as follows : $\Pi^{(1)}_{k_v} = J^{(1)}_{k_v} \cup J^{(1)}_{k_{v+1}}$ $(1 \le v \le r-1)$ and $\Pi^{(1)}_{k_r} = J^{(1)}_{k_r}$ ,

$$D^{(1)}_{k_{v,t}} = \left\{ \sum_{j=k_v+i-t}^{k_v+i-1} \alpha_j \mid k_v + i - 1 \le k_{v+1} - 1 \text{ for } i \in \{1, \ldots, t\} \right\} \begin{pmatrix} 1 \le v \le r-1 \\ 1 \le t \le \lambda_v + 1 \end{pmatrix},$$

$$D^{(1)}_{k_{r,t}} = \{2e_{k_r+1-t}\} \quad (1 \le t \le \lambda_r + 1).$$

By part (*a*) (*ii*) of Algorithm 3.1, for $1 \le v \le r-1$,

$$max\ \{\Pi^{(1)}_{k_v} \cup \{\alpha_{k_v}\}\} = \begin{cases} \alpha_{k_{v+1}} - 1 & \text{if } J^{(1)}_{k_v} \ne \phi \text{ and } J^{(1)}_{k_{v+1}} \ne \phi \\ \alpha_{k_v} & \text{if } J^{(1)}_{k_v} \ne \phi \text{ and } J^{(1)}_{k_{v+1}} = \phi \\ & \text{or if } \Pi^{(1)}_{k_v} = \phi . \end{cases}$$

For $1 \le v \le r-1$, if $J^{(1)}_{k_v} \ne \phi$ and $J^{(1)}_{k_{v+1}} \ne \phi$ then choose $2e_{k_{v+1}} \in \Phi$ and consider $D^{(1)}_{k_{v,t}}$ , where $1 \le t \le \lambda_v$. For $i \in \{1, 2, .., \lambda_{v+1} + 1\}$, we have $k_v + i \le k_{v+1}$ $(1 \le v \le r-1)$. If $\lambda_v = \lambda_{v+1}$ $(1 \le v \le r-1)$ then $k_v + i < k_{v+1}$ for all $i \in \{1, 2, ..., \lambda_v\}$, and so $(\alpha, 2e_{k_{v+1}}) = 0$ for all $\alpha \in D^{(1)}_{k_{v,t}}$ $(1 \le t \le \lambda_v)$. Thus $D_{k_{v,t}} = \phi\,(1 \le t \le \lambda_v)$ and so $D'_{k_{v,t}} = \phi\,(1 \le t \le \lambda_v)$.

If $\lambda_v > \lambda_{v+1}$ $(1 \le v \le r-1)$ then for $i = 1, 2, \ldots, \lambda_{v+1}$ we have $k_v + i < k_{v+1}$ and $(\alpha, 2e_{k_{v+1}}) = 0$ for all $\alpha \in D^{(1)}_{k_{v,t}}$ $(1 \le t \le \lambda_{v+1})$ and so $D_{k_{v,t}} = \phi\,(1 \le t \le \lambda_{v+1})$ and $D'_{k_{v,t}} = \phi$ $(1 \le t \le \lambda_{v+1})$.

For $i = \lambda_{v+1} + 1\,(1 \le v \le r-1)$ we have $k_v + i = k_{v+1}$ and $(\alpha, 2e_{k_{v+1}}) < 0$ for some $\alpha \in D^{(1)}_{k_{v,t}}$ $(\lambda_{v+1} + 1 \le t \le \lambda_v)$, and so

$$D_{k_{v,t}} = \left\{ e_{k_{v+1}-t} - e_{k_{v+1}} \right\} \quad (\lambda_{v+1} + 1 \le t \le \lambda_v)$$

$$D'_{k_{v,t}} = \left\{ 2e_{k_{v+1}-t} \right\} \quad (\lambda_{v+1} + 1 \le t \le \lambda_v).$$

For $1 \leq v \leq r-1$, if $J^{(1)}_{k_v} \neq \phi$ and $J^{(1)}_{k_{v+1}} = \phi$ or if $\Pi^{(1)}_{k_v} = \phi$ then choose $2e_{k_{v+1}} \in \Phi$ and so

$$D'_{k_{v,t}} = \left\{2e_{k_v+1-t}\right\} \ (1 \leq t \leq \lambda_v + 1).$$

On the other hand, the subsystem $\Psi = \sum_{i=1}^{r} A_{\lambda_i}$ corresponds to the pair of partitions $(\lambda, \mu) = (\lambda_1 + 1, ..., \lambda_r + 1, 0)$ of $n$. Thus the subsystem $\Psi = \sum_{i=1}^{r} A_{\lambda_i}$ is represented by the rows of the tableau

$$t = \left( \begin{matrix} 1 & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & k_1 \\ k_1+1 & k_1+2 & \cdot & \cdot & \cdot & k_2 & & \\ & & \cdot & \cdot & \cdot & & & \\ k_{r-1}+1 & k_{r-1}+2 & \cdot & \cdot & n & & & \end{matrix} , \phi \right).$$

Now, for $v = 1, 2, ..., r$, put

$$J'^{(1)} = \sum_{v=1}^{r-1} \left\{ D^{(1)}_{k_v, \lambda_v+1} + \sum_{t=1}^{\lambda_v} D'_{k_v, t} \right\} + \sum_{t=1}^{\lambda_r+1} D^{(1)}_{k_r, t}.$$

Then $J'^{(1)}$ is represented by the columns of the tableau $t$ and so $J'^{(1)}$ is linearly independent over $\mathbf{R}$. If $\Psi'$ is a subsystem of $\Phi$ with simple system $J' = J'^{(1)}$ then $R_t \cong W(J)$ and $C_t \cong W(J')$, where $R_t$ (resp. $C_t$) is the row (resp. column) stabilizer of the tableau $t^1 \to t$ It follows that $\bar{J} = \{(J^{(1)}; J'^{(1)}), (\phi, \phi)\}$ is a perfect system in $\Phi$. Then we have the following lemma.

*Lemma* 3.2 — For $r \geq 2$, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r \geq 0$ and $\sum_{i=1}^{r} (\lambda_i + 1) = n$, let $\Psi = \sum_{i=1}^{r} A_{\lambda_i}$ be a subsystem of $\Phi$. Let $k_0 = 0$ and $k_i = \lambda_1 + \ldots + \lambda_i + i$ $(i = 1, 2, ..., r)$. For $i = 1, ..., r$, let $J^{(1)}_{k_i} = \{\alpha_{k_{i-1}+1}, \alpha_{k_{i-1}+2}, ..., \alpha_{k_i} - 1\}$ be a simple system for $A_{\lambda_i}$ and let $J = J^{(1)} = \sum_{i=1}^{r} J^{(1)}_{k_i}$ be a simple system for $\Psi$.

Let $\Psi'$ be the subsystem of $\Phi$ with simple system $J' = J'^{(1)} = \sum_{v=1}^{r-1} \left\{ D^{(1)}_{k_v, \lambda_v+1} + \sum_{t=1}^{\lambda_v} D'_{k_v, t} \right\} + \sum_{t=1}^{\lambda_r+1} D^{(1)}_{k_r, t}$. Then $\bar{J} = \{(J^{(1)}; J'^{(1)}), (\phi, \phi)\}$ is a perfect system in $\Phi$.

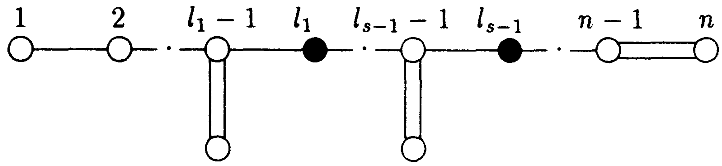
(3) For $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_s > 0$ and $\sum_{j=1}^{s} \mu_j = n$, let $\Psi = \sum_{j=1}^{s} C_{\mu_j}$ be a subsystem of $\Phi$.

If we put $l_0 = 0, l_j = \mu_1 + \mu_2 + \ldots + \mu_j$ $(j = 1, 2, ..., s)$ then

$$J^{(2)}_{l_j} = \{\alpha_{l_{j-1}+1}, \alpha_{l_{j-1}+2}, \dots, \alpha_{l_j - 1}, 2e_{l_j}\}$$

is a simple system for $C_{\mu_j}$ and therefore $J = J^{(2)} = \sum_{j=1}^{s} J^{(2)}_{l_j}$ is a simple system for $\Psi = \sum_{j=1}^{s} C_{\mu_j}$.

The Dynkin diagram for $\Psi$ is




that is, the nodes $l_1, l_2, \dots, l_{s-1}$ have been deleted.

By applying part (*b*) of Algorithm 3.1, for $u = 1, 2, \dots, s-1$ we have

$$D^{(2)}_{l_u, t} = \left\{ \sum_{j = l_u + i - t}^{l_u + i - 1} \alpha_j \mid l_u + i - 1 \le l_{\mu+1} - 1 \text{ for } i \in \{1, 2, \dots, t\} \right\}$$

where $1 \le t \le \mu_u$.

On the other hand, the subsystem $\Psi = \sum_{j=1}^{s} C_{\mu_j}$ corresponds to the pair of partitions $(\lambda, \mu)$ $= (0, \mu_1, \dots, \mu_s)$ of $n$. Thus the subsystem $\Psi = \sum_{j=1}^{s} C_{\mu_j}$ is represented by the rows of the tableau

$$t = \left( \phi, \quad \begin{matrix} 1 & 2 & \cdots & l_1 \\ l_1 + 1 & l_1 + 2 & \cdots & l_2 \\ \vdots & \vdots & & \\ l_{s-1} + 1 & l_{s-1} + 2 & \cdots & n \end{matrix} \right)$$

Now, for $u = 1, 2, \dots, s-1$, put

$$J'^{(2)} = \sum_{u=1}^{s-1} D^{(2)}_{l_u, \mu_u}.$$

Then $J'^{(2)}$ is represented by the columns of the tableau $t$ and so $J'^{(2)}$ is linearly independent over $\mathbf{R}$. If $\Psi'$ is a subsystem of $\Phi$ with simple system $J' = J'^{(2)}$ then $R_t \cong W(J)$ and $C_t \cong W(J')$, where $R_t$ (resp. $C_t$) is the row (resp. column) stabilizer of the tableau $t$.

It follows that $\bar{J} = \{(\phi, \phi), (J^{(2)}; J'^{(2)})\}$ is a perfect system in $\Phi$. Then we have the following lemma.

*Lemma* 3.3 — For $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_s > 0$ and $\sum_{j=1}^{s} \mu_j = n$, let $\Psi = \sum_{j=1}^{s} C_{\mu_j}$ be a subsystem of $\Phi$. Let $l_0 = 0$ and $l_j = \mu_1 + \ldots + \mu_j$ ($j = 1, 2, \ldots, s$). For $j = 1, \ldots, s$, let $J_{l_j}^{(2)} = \{\alpha_{l_{j-1}+1}, \alpha_{l_{j-1}+2}, \ldots, \alpha_{l_j} - 1, 2e_{l_j}\}$ be a simple system for $C_{\mu_j}$ and $J = J^{(2)} = \sum_{j=1}^{s} J_{l_j}^{(2)}$ be a simple system for $\Psi = \sum_{j=1}^{s} C_{\mu_j}$. Let $\Psi'$ be the subsystem of $\Phi$ with simple system $J' = J'^{(2)} = \sum_{u=1}^{s-1} D_{l_u, \mu_u}^{(2)}$. Then $\bar{J} = \{(\phi, \phi), (J^{(2)}; J'^{(2)})\}$ is a perfect system in $\Phi$.

(4) Finally, for $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r \geq 0$, $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_s > 0$ and $\sum_{i=1}^{r} (\lambda_i + 1) + \sum_{j=1}^{s} \mu_j = n$, let $\Psi = \sum_{i=1}^{r} A_{\lambda_i} + \sum_{j=1}^{s} C_{\mu_j}$ be a subsystem of $\Phi$ with simple system $J = J^{(1)} + J^{(2)}$ and Dynkin diagram $\Delta$ as given earlier.

By part (*a*) (*i*) of Algorithm 3.1, for $v = 1, 2, \ldots, r$ we obtain $\Pi_{k_v}^{(1)}$ and $D_{k_{v,t}}^{(1)}$ $(1 \leq t \leq \lambda_v + 1)$ as follows : $\Pi_{k_v}^{(1)} = J_{k_v}^{(1)} \cup J_{k_{v+1}}^{(1)}$ $(1 \leq v \leq r-1)$ and $\Pi_{k_v}^{(1)} = J_{k_r}^{(1)}$,

$$D_{k_v, t}^{(1)} = \left\{ \sum_{j=k_v+i-t}^{k_v+i-1} \alpha_j \mid k_v + i - 1 \leq k_{v+1} - 1 \text{ for } i \in \{1, \ldots, t\} \right\} \quad \begin{pmatrix} 1 \leq v \leq r-1 \\ 1 \leq t \leq \lambda_v + 1 \end{pmatrix},$$

$$D_{k_r, t}^{(1)} = \left\{ \sum_{j=k_r+1-t}^{k_r} \alpha_j \right\} \quad (1 \leq t \leq \lambda_r + 1).$$

By part (*a*) (*ii*) of Algorithm 3.1, for $1 \leq v \leq r-1$,

$$max\ \{\Pi_{k_v}^{(1)} \cup \{\alpha_{k_v}\}\} = \begin{cases} \alpha_{k_{v+1}-1} & \text{if } J_{k_v}^{(1)} \neq \phi \text{ and } J_{k_{v+1}}^{(1)} \neq \phi \\ \alpha_{k_v} & \text{if } J_{k_v}^{(1)} \neq \phi \text{ and } J_{k_{v+1}}^{(1)} = \phi \\ & \text{or if } \Pi_{k_v}^{(1)} = \phi. \end{cases}$$

For $1 \leq v \leq r-1$, if $J_{k_v}^{(1)} \neq \phi$ and $J_{k_{v+1}}^{(1)} \neq \phi$ then choose $2e_{k_{v+1}} \in \Phi$ and consider $D_{k_{v,t}}^{(1)}$, where

$1 \le t \le \lambda_v$. For $i \in \{1, 2, \dots, \lambda_{v+1}+1\}$, we have $k_v + i \le k_{v+1}$ $(1 \le v \le r-1)$. If $\lambda_v = \lambda_{v+1}$ $(1 \le v \le r-1)$ then $k_v + i < k_{v+1}$ for all $i \in \{1, 2, \dots, \lambda_v\}$, and so $(\alpha, 2e_{k_{v+1}}) = 0$ for all $\alpha \in D^{(1)}_{k_{v,t}}$ $(1 \le t \le \lambda_v)$. Thus $D_{k_{v,t}} = \phi$ $(1 \le t \le \lambda_v)$ and so $D'_{k_{v,t}} = \phi$ $(1 \le t \le \lambda_v)$.

If $\lambda_v > \lambda_{v+1}$ $(1 \le v \le r-1)$ then for $i = 1, 2, \dots, \lambda_{v+1}$ we have $k_v + i < k_{v+1}$ and $(\alpha, 2e_{k_{v+1}}) = 0$ for all $\alpha \in D^{(1)}_{k_{v,t}}$ $(1 \le t \le \lambda_{v+1})$ and so $D_{k_v,t} = \phi$ $(1 \le t \le \lambda_{v+1})$ and $D'_{k_v,t} = \phi$ $(1 \le t \le \lambda_{v+1})$.

For $i = \lambda_{v+1} + 1$ $(1 \le v \le r-1)$ we have $k_v + i = k_{v+1}$ and $(\alpha, 2e_{k_{v+1}}) < 0$ for some $\alpha \in D^{(1)}_{k_v,t}$ $(\lambda_{v+1} + 1 \le t \le \lambda_v)$, and so

$$D_{k_{v,t}} = \left\{ e_{k_{v+1}-t} - e_{k_{v+1}} \right\} \ (\lambda_{v+1} + 1 \le t \le \lambda_v)$$

$$D'_{k_{v,t}} = \left\{ 2e_{k_{v+1}-t} \right\} \ (\lambda_{v+1} + 1 \le t \le \lambda_v).$$

For $1 \le v \le r-1$, if $J^{(1)}_{k_v} \ne \phi$ and $J^{(1)}_{k_{v+1}} = \phi$ or if $\Pi^{(1)}_{k_v} = \phi$ then choose $2e_{k_v+1} \in \Phi$ and so

$$D'_{k_v,t} = \left\{ 2e_{k_v+1-t} \right\} \ (1 \le t \le \lambda_v + 1).$$

By part (*a*) (*ii*) of Algorithm 3.1, for $v = r$, $max\{\Pi^{(1)}_{k_r} \cup \{\alpha_{k_r}\}\} = \alpha_{k_r}$ and choose $2e_{k_r+1} \in \Phi$ and so $D'_{k_r,t} = \{2e_{k_r+1-t}\}$ $(1 \le t \le \lambda_r + 1)$.

By part (*b*) of Algorithm 3.1, for $u = 1, 2, \dots, s-1$ we have

$$D^{(2)}_{l_{u,t}} = \left\{ \sum_{j=l_u+i-t}^{l_u+i-1} \alpha_j \mid l_u + i - 1 \le l_{u+1} - 1 \text{ for } i \in \{1, 2, \dots, t\} \right\}$$

where $1 \le t \le \mu_u$.

On the other hand, the subsystem $\Psi = \sum_{i=1}^{r} A_{\lambda_i} + \sum_{j=1}^{s} C_{\mu_j}$ corresponds to the pair of partitions

$$(\lambda, \mu) = (\lambda_1 + 1, \dots, \lambda_r + 1, \mu_1, \dots, \mu_s) \text{ of } n.$$

Thus the subsystem $\Psi = \sum_{i=1}^{r} A_{\lambda_i} + \sum_{j=1}^{s} C_{\mu_j}$ is represented by the rows of the $(\lambda, \mu)$-tableau $t$ given in (2.2).

Now, for $v = 1, 2, ..., r$, put

$$J'^{(1)} = \sum_{v=1}^{r-1} \left\{ D^{(1)}_{k_v, \lambda_v + 1} + \sum_{t=1}^{\lambda_v} D'_{k_{v,t}} \right\} + \sum_{t=1}^{\lambda_r + 1} D'_{k_{r,t}},$$

and for $u = 1, 2, ..., s - 1$, put

$$J'^{(2)} = \sum_{u=1}^{s-1} D^{(2)}_{l_u, \mu_u}.$$

Then $J'^{(1)} + J'^{(2)}$ is represented by the columns of the $(\lambda, \mu)$-tableau $t$ given in (2.2) and so $J'^{(1)} + J'^{(2)}$ is linearly independent over $\boldsymbol{R}$. If $\Psi'$ is a subsystem of $\Phi$ with simple system $J' = J'^{(1)} + J'^{(2)}$ then $R_t \cong W(J)$ and $C_t \cong W(J')$, where $R_t$ (resp. $C_t$) is the row (resp. column) stabilizer of the $(\lambda, \mu)$-tableau $t$ given in (2.2).

It follows that $\bar{J} = \{(J^{(1)}; J'^{(1)}), (J^{(2)}; J'^{(2)})\}$ is a perfect system in $\Phi$. Then we have the following theorem.

***Theorem*** 3.1 — *For $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r \geq 0$, $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_s > 0$ and $\sum_{i=1}^{r} (\lambda_i + 1) + \sum_{j=1}^{s} \mu_j = n$, let $\Psi = \sum_{i=1}^{r} A_{\lambda_i} + \sum_{j=1}^{s} C_{\mu_j}$ be a subsystem of $\Phi$. Let $k_0 = 0$ and $k_i = \lambda_1 + \ldots + \lambda_i + i$ $(i = 1, 2, ..., r)$ and $l_0 = k_r$, $l_j = l_0 + \mu_1 + \ldots + \mu_j$ $(j = 1, 2, ..., s)$. For $i = 1, ..., r$, let $J^{(1)}_{k_i} = \{\alpha_{k_{i-1}+1}, \alpha_{k_{i-1}+2}, ..., \alpha_{k_i} - 1\}$ be simple system for $A_{\lambda_i}$ and for $j = 1, 2, ..., s$ let $J^{(2)}_{l_j} = \{\alpha_{l_{j-1}+1}, \alpha_{l_{j-1}+2}, ..., \alpha_{l_j} - 1, 2e_{l_j}\}$ be simple system for $C_{\mu_j}$.*

*Let $J = J^{(1)} + J^{(2)}$ be a simple system for $\Psi = \sum_{i=1}^{r} A_{\lambda_i} + \sum_{j=1}^{s} C_{\mu_j}$, where $J^{(1)} = \sum_{i=1}^{r} J^{(1)}_{k_i}$ and $J^{(2)} = \sum_{j=1}^{s} J^{(2)}_{l_j}$.*

*Let $\Psi'$ be the subsystem of $\Phi$ with simple system*

$$J' = J'^{(1)} + J'^{(2)} = \sum_{v=1}^{r-1} \left\{ D^{(1)}_{k_v, \lambda_v + 1} + \sum_{t=1}^{\lambda_v} D'_{k_{v,t}} \right\} + \sum_{t=1}^{\lambda_r + 1} D'_{k_{r,t}} + \sum_{u=1}^{s-1} D^{(2)}_{l_u, \mu_u}.$$

*Then $\bar{J} = \{(J^{(1)}; J'^{(1)}), (J^{(2)}; J'^{(2)})\}$ is a perfect system in $\Phi$.*

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