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NONSMOOTH MULTIOBJECTIVE PROGRAMS WITH V-ρ-INVEXITY[1]

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In this paper, we define V-ρ-invexity for vector-valued functions, and prove the generalized Karush-Kuhn-Tucker sufficient optimality theorem and duality theorems for nonsmooth multiobjective programs involving locally Lipschitz functions. Also, weak vector saddle-point theorems are obtained under V-ρ-invexity conditions.

Key Words : Nonsmooth Multiobjective Programs; V-ρ-Invexity; Duality; Weak Vector Saddle-Point Theorems

1. INTRODUCTION AND PRELIMINARIES

Jeyakumar and Mond[1] defined generalized V-invexity concepts of differentiable multiobjective programming problems which preserve the sufficient optimality conditions and duality results as in the scalar case, and avoid the major difficulty of verifying that the inequality holds for the same function $\eta(\cdot, \cdot)$ for invex functions. Recently, Mishra and Mukherjee[2] and Liu[3] extended the results of Jeyakumar and Mond to nonsmooth multiobjective programming problems.

On the other hand, Jeyakumar[4] introduced ρ-invexity for differentiable scalar-valued functions, and investigated the sufficiency of the Karush-Kuhn-Tucker conditions and the duality theorems for scalar nonlinear programming problem. Also, Jeyakumar[5] defined ρ-invexity for nonsmooth scalar-valued functions, studied duality theorems for nonsmooth optimization problems, and gave relationships between saddle-points and optima. Bector[6] defined the (B, η)-invex functions.

In this paper, we define V-ρ-invexity for vector-valued functions, which is a generalization of the concept of V-invexity and (B, η)-invexity, and establish sufficient optimality conditions and duality results for nonsmooth multiobjective programs under the V-ρ-invexity assumptions. Further weak vector saddle-point theorems are obtained.

Let R^n be the n dimensional Euclidean space and R^n_+ be its nonnegative orthant. Throughout this paper, the following conventions for vectors in R^n will be used :

$$x > y \quad \text{if and only if} \quad x_i > y_i,\ i = 1, \ldots, n,$$

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$$x \geqq y \text{ if and only if } x_i \geqq y_i,\ i = 1, \ldots, n$$

and $x \not> y$ is the negation of $x > y$.

A real-valued function $f: R^n \to R$ is said to be locally Lipschitz if for any $z \in R^n$ there exists a positive constant K and a neighbourhood N of z such that, for each $x, y \in N$,

$$| f(x) - f(y) | \leqq K \| x - y \| .$$

The Clarke generalized directional derivative of a locally Lipschitz function f at x in the direction d denoted by $f^0(x; d)$ is :

$$f^0(x; d) = \limsup_{\substack{y \to x \\ t \downarrow 0}} t^{-1} (f(y + td) - f(y)).$$

The Clarke generalized subgradient of f at x is denoted by

$$\partial f(x) = \{\xi : f^0(x; d) \geqq \xi^t d, \quad \forall\, d \in R^n\}.$$

In this paper, we consider a nonsmooth multiobjective program (P) :

(P) Minimize $(f_1(x), \ldots, f_p(x))$

$$\text{subject to } x \in X = \{x \in R^n : g_j(x) \leqq 0,\ j = 1, \ldots, m\},$$

where $f_i : R^n \to R$, $i = 1, \ldots, p$ and $g_j : R^n \to R$, $j = 1, \ldots, m$ are locally Lipschitz functions.

We define V-ρ-invexity for locally Lipschitz functions as follows.

Definition 1.1 — Let $f_i : R^n \to R$ and $g_j : R^n \to R$ be locally Lipschitz functions for $i = 1, \ldots, p$ and $j = 1, \ldots, m$, respectively.

(1) $f = (f_1, \ldots, f_p)$ is V-ρ-invex with respect to functions η and $\theta : R^n \times R^n \to R^n$ if there exists $\alpha_i : R^n \times R^n \to R_+ \setminus \{0\}$ and $\rho_i \in R$, $i = 1, \ldots, p$ such that for any $x, u \in R^n$ and any $\xi_i \in \partial f_i(u)$,

$$\alpha_i(x, u)\,[f_i(x) - f_i(u)] \geqq \xi_i\, \eta(x, u) + \rho_i \| \theta(x, u) \|^2.$$

(2) $g = (g_1, \ldots, g_m)$ *is* V-σ-invex with respect to functions η and $\theta : R^n \times R^n \to R^n$ if there exists $\beta_j : R^n \times R^n \to R_+ \setminus \{0\}$ and $\sigma_i \in R$, $j = 1, \ldots, m$ such that for any $x, u \in R^n$ and any $\zeta_j \in \partial g_j(u)$,

$$\beta_j(x, u)\,[g_j(x) - g_j(u)] \geqq \zeta_j\, \eta(x, u) + \sigma_j \| \theta(x, u) \|^2.$$

Recall from Sawaragi *et al.*[7] that $u \in X$ is said to be a weak minimum of (P) if there exists no $x \in X$ such that $f_i(x) < f_i(u)$, $i = 1, \ldots, p$.

2. SUFFICIENCY AND DUALITY THEOREMS

In this section, we show that the generalized Karush-Kuhn-Tucker conditions are sufficient for a weak minimum of (*P*).

Theorem 2.1 — *Let* $(u, \tau, \lambda) \in R^n \times R^p \times R^m$ *satisfy the generalized Karush-Kuhn-Tucker conditions as follows* :

$$0 \in \sum_{i=1}^{p} \tau_i \partial f_i(u) + \sum_{j=1}^{m} \lambda_j \partial g_j(u),$$

$$g_j(u) \leqq 0 \text{ and } \lambda_j g_j(u) = 0, \; j = 1, \ldots, m,$$

$$\tau_i \geqq 0, \; i = 1, \ldots, p, \; \tau^t e > 0,$$

$$\lambda_j \geqq 0, \quad j = 1, \ldots, m.$$

If f *is* V-ρ-*invex and* g *is* V-σ-*invex with respect to the same functions* η *and* θ, *and* $\sum_{i=1}^{p} \tau_i \rho_i + \sum_{j=1}^{m} \lambda_j \sigma_j \geqq 0$, *then* u *is a weak minimum of* (*P*).

PROOF : Since $0 \in \sum_{i=1}^{p} \tau_i \partial f_i(u) + \sum_{j=1}^{m} \lambda_j \partial g_j(u)$, there exist $\xi_i \in \partial f_i(u)$ and $\zeta_j \in \partial g_j(u)$ such that

$$\sum_{i=1}^{p} \tau_i \xi_i + \sum_{j=1}^{m} \lambda_j \zeta_j = 0. \qquad \ldots (2.1)$$

Suppose that u is not a weak minimum of (*P*). Then there exists $x \in X$ such that $f_i(x) < f_i(u)$, $i = 1, \ldots, p$. Since $\alpha_i(x, u) > 0$, we have

$$\alpha_i(x, u) \, f_i(x) < \alpha_i(x, u) f_i(u), \quad i = 1, \ldots, p.$$

By the V-ρ-invexity of f, for all i,

$$\xi_i \eta(x, u) + \rho_i \| \theta(x, u) \|^2 < 0, \text{ for each } \xi_i \in \partial f_i(u).$$

Hence, we have

$$\sum_{i=1}^{p} \tau_i \xi_i \eta(x, u) + \sum_{i=1}^{p} \tau_i \rho_i \| \theta(x, u) \|^2 < 0.$$

Since $\sum_{i=1}^{p} \tau_i \rho_i + \sum_{j=1}^{m} \lambda_j \sigma_j \geqq 0$, it follows from (2.1) that

$$\sum_{j=1}^{m} \lambda_j \zeta_j \eta(x, u) + \sum_{j=1}^{m} \lambda_j \sigma_j \| \theta(x, u) \|^2 > 0.$$

Then, by the V-σ-invexity of g, we have

$$\sum_{j=1}^{m} \beta_j (x, u) [\lambda_j g_j(x) - \lambda_j g_j(u)] > 0.$$

Since $\lambda_j g_j(u) = 0$, $j = 1, \ldots, m$, we have

$$\sum_{j=1}^{m} \beta_j (x, u) \lambda_j g_j(x) > 0,$$

which contradicts the conditions $\beta_j(x, u) > 0$, $\lambda_j \geqq 0$ and $g_j(x) \leqq 0$. Thus u is a weak minimum of (P).

For the problem (P), consider a corresponding Mond-Weir[8] dual problem (D) :

(D) Maximize $(f_1(u), \ldots, f_p(u))$

$$\text{subject to } 0 \in \sum_{i=1}^{p} \tau_i \partial f_i(u) + \sum_{j=1}^{m} \lambda_j \partial g_j(u),$$

$$\lambda_j g_j(u) \geqq 0, \; j = 1, \ldots, m,$$

$$\tau_i \geqq 0, \; i = 1, \ldots, p, \; \tau^t e = 1,$$

$$\lambda_i \geqq 0, \; j = 1, \ldots, m,$$

where $e = (1, \ldots, 1)^t \in R^p$.

Now we establish duality theorems for (P) and (D).

Theorem 2.2 *(Weak duality) — Let x be a feasible for (P) and (u, τ, λ) a feasible for (D). Assume that* $\sum_{i=1}^{p} \tau_i \rho_i + \sum_{j=1}^{m} \lambda_j \rho_j \geqq 0$. *If f is V-ρ-invex and g is V-σ-invex with respect to the same functions η and θ, then $f(x) \not< f(u)$.*

PROOF : From the feasibility conditions and $\beta_j(x, u) > 0$, we have $\beta_j(x, u) \lambda_j g_j(x) \leqq \beta_j(x, u) \lambda_j g_j(u)$. Then, by the V-σ-invexity of g, we have

$$\lambda_j \zeta_j \eta(x, u) + \lambda_j \sigma_j \| \theta(x, u) \|^2 \leqq 0, \text{ for each } \zeta_j \in \partial g_j(u).$$

Hence, we have

$$\sum_{j=1}^{m} \lambda_j \zeta_j \eta(x, u) + \sum_{j=1}^{m} \lambda_j \sigma_j \| \theta(x, u) \|^2 \leqq 0, \text{ for each } \zeta_j \in \partial g_j(u).$$

Since $0 \in \sum_{i=1}^{p} \tau_i \partial f_i(u) + \sum_{j=1}^{m} \lambda_j \partial g_j(u)$, there exist $\xi_i \in \partial f_i(u)$ and $\zeta_j \in \partial g_j(u)$ such that

$$\sum_{i=1}^{p} \tau_i \xi_i + \sum_{j=1}^{m} \lambda_j \zeta_j = 0.$$

This implies that

$$\sum_{i=1}^{p} \tau_i \xi_i \, \eta(x, u) + \sum_{j=1}^{m} \lambda_j \zeta_j \, \eta(x, u) = 0.$$

Hence, from the assumption $\sum_{i=1}^{p} \tau_i \rho_i + \sum_{j=1}^{m} \lambda_j \rho_j \geq 0$, we have

$$\sum_{i=1}^{p} \tau_i \xi_i \, \eta(x, u) + \sum_{i=1}^{p} \tau_i \rho_i \| \theta(x, u) \|^2 \geq 0.$$

From the V-ρ-invexity of f, we have

$$\sum_{i=1}^{p} \alpha_i(x, u) [\tau_i f_i(x) - \tau_i f_i(u)] \geq 0.$$

Since $\alpha_i(x, u) > 0$, $\tau_i \geq 0$ and $\tau^t e = 1$, we have $f(x) \not< f(u)$.

Theorem 2.3 *(Strong duality) — Let $\bar{x}$ be a weak minimum of (P) at which a constraint qualification is satisfied. Then there exist $\bar{\tau} \in R^p$ and $\bar{\lambda} \in R^m$ such that $(\bar{x}, \bar{\tau}, \bar{\lambda})$ is feasible for (D). If, also, f is V-ρ-invex and g is V-σ-invex with respect to the same functions η and θ, then $(\bar{x}, \bar{\tau}, \bar{\lambda})$ is a weak maximum of (D).*

PROOF : Since $\bar{x}$ is a weak minimum of (P) and a constraint qualification is satisfied at $\bar{x}$, from the generalized Karush-Kuhn-Tucker theorem (see, for example, Theorem 6.1.3 of Clarke[9]), there exist $\tau \in R^p$ and $\lambda \in R^m$ such that

$$0 \in \sum_{i=1}^{p} \tau_i \partial f_i(\bar{x}) + \sum_{j=1}^{m} \lambda_j \partial g_j(\bar{x}),$$

$$\lambda_j g_j(\bar{x}) = 0, \; j = 1, \ldots, m,$$

$$\tau_i \geq 0, \; i = 1, \ldots, p, \; \tau^t e > 0,$$

$$\lambda_j \geq 0, \; j = 1, \ldots, m.$$

Since $\tau_i \geq 0$, $i = 1, \ldots, p$ and $\tau^t e > 0$, we can scale the τ_i's and λ_j's as

$$\bar{\tau}_i = \frac{\tau_i}{\sum_{i=1}^{p} \tau_i} \quad \text{and} \quad \bar{\lambda}_j = \frac{\lambda_j}{\sum_{i=1}^{p} \tau_i}$$

Then $(\bar{x}, \bar{\tau}, \bar{\lambda})$ is feasible for (D). Since $\bar{x}$ is feasible for (P), it follows from weak duality that $f(\bar{x}) \not< f(u)$ for any feasible u for (D). Hence, $(\bar{x}, \bar{\tau}, \bar{\lambda})$ is a weak maximum of (D).

3. Weak Vector Saddle-Point Theorems

In this section, we prove weak vector saddle-point theorems for the nonsmooth multiobjective program (P) in which functions are locally Lipschitz.

For the problem (P), a point (x, τ, λ) is said to be a critical point if, x is a feasible point for (P), and

$$0 \in \partial \left(\sum_{i=1}^{p} \tau_i f_i(x) + \sum_{j=1}^{m} \lambda_j g_j(x) \right),$$

$$\lambda_j g_j(x) = 0, \ \lambda_j \geq 0, \ j = 1, \ldots, m,$$

$$\tau_i \geq 0, \ i = 1, \ldots, p, \ \tau^t e = 1.$$

Note that

$$\partial \left(\sum_{i=1}^{p} \tau_i f_i(x) + \sum_{j=1}^{m} \lambda_j g_j(x) \right) = \sum_{i=1}^{p} \tau_i \partial \left(f_i(x) + \sum_{j=1}^{m} \lambda_j g_j(x) \right).$$

Let $L(x, \lambda) = f(x) + \lambda^t g(x)e$, where $x \in R^n$ and $\lambda \in R^m_+$. Then, a point $(\bar{x}, \bar{\lambda}) \in R^n \times R^m_+$ is said to be a weak vector saddle-point if

$$L(\bar{x}, \lambda) \not> L(\bar{x}, \bar{\lambda}) \not< L(x, \bar{\lambda}), \text{ for all } x \in R^n \text{ and } \lambda \in R^m_+.$$

Theorem 3.1 *(Saddle-point conditions) — Let $(\bar{x}, \bar{\tau}, \bar{\lambda})$ be a critical point of (P). Assume that $f(\cdot) + \bar{\lambda}^t g(\cdot)e$ is V-ρ-invex with respect to functions η and θ and $\sum_{i=1}^{p} \bar{\tau}\rho_i \geq 0$. Then $(\bar{x}, \bar{\lambda})$ is a weak vector saddle-point of (P).*

PROOF : Since $(\bar{x}, \bar{\tau}, \bar{\lambda})$ is a critical point for (P), there exist $\xi_i \in \partial \left(f_j(\bar{x}) + \sum_{j=1}^{m} \bar{\lambda}_j g_i(\bar{x}) \right)$ such that $\sum_{i=1}^{p} \bar{\tau}_i \xi_i = 0$. Since $\sum_{i=1}^{p} \bar{\tau}_i \rho_i \geq 0$,

$$\sum_{i=1}^{p} \tau_i \xi_i \eta(x, \bar{x}) + \sum_{i=1}^{p} \bar{\tau}_i \rho_i \| \theta(x, \bar{x}) \|^2 \geqq 0.$$

Then, by the V-ρ-invexity of $f(\cdot) + \bar{\lambda}^t g(\cdot)e$, we have

$$\sum_{i=1}^{p} \alpha_i(x, \bar{x}) \bar{\tau}_i [f_i(x) - f_i(\bar{x}) + \bar{\lambda}^t g(x) - \bar{\lambda}^t g(\bar{x})] \geqq 0, \text{ for any } x \in R^n.$$

Since $\alpha_i(x, \bar{x}) > 0$, $\bar{\tau}_i \geqq 0$ and $\bar{\tau}^t e = 1$,

$$f(\bar{x}) + \bar{\lambda}^t g(x)e \not< f(x) + \bar{\lambda}^t g(x)e, \text{ for any } x \in R^n,$$

i.e.,

$$L(\bar{x}, \bar{\lambda}) \not< L(x, \bar{\lambda}), \text{ for any } x \in R^n.$$

Now, since $\lambda^t g(\bar{x}) \leqq 0$, for any $\lambda \in R_+^m$,

$$\bar{\lambda}^t g(x) - \lambda^t g(\bar{x}) \geqq 0, \text{ for any } \lambda \in R_+^m.$$

Thus, $f(\bar{x}) + \bar{\lambda}^t g(\bar{x})e - (f(x) + \lambda^t g(\bar{x})e) \in R_+^p$ and hence

$$L(\bar{x}, \lambda) \not< L(\bar{x}, \bar{\lambda}), \text{ for any } \lambda \in R_+^m.$$

Therefore, $(\bar{x}, \bar{\lambda})$ is a weak vector saddle-point of (P).

Theorem 3.2 — *If there exists* $\bar{\lambda} \in R_+^m$ *such that* $(\bar{x}, \bar{\lambda})$ *is a weak vector saddle-point, then* $\bar{x}$ *is a weak minimum of* (P).

PROOF : Assume that $(\bar{x}, \bar{\lambda})$ is a weak vector saddle-point. From the left inequality of saddle-point conditions,

$$f(\bar{x}) + \lambda^t g(\bar{x})e \not> (f(\bar{x}) + \bar{\lambda}^t g(\bar{x})e, \text{ for any } \lambda \in R_+^m.$$

Thus $\lambda^t g(\bar{x})e \not< \bar{\lambda}^t g(\bar{x})e$, for any $\lambda \in R_+^m$, and hence we have

$$\lambda^t g(\bar{x}) \leqq \bar{\lambda}^t g(\bar{x}), \text{ for any } \lambda \in R_+^m. \qquad \ldots (3.1)$$

Since λ can be taken arbitrary large, $g(\bar{x}) \leqq 0$. Hence $\bar{\lambda}^t g(\bar{x}) \leqq 0$. Letting, $\lambda = 0$ in (3.1), $\bar{\lambda}^t g(\bar{x}) \geqq 0$. Therefore, $\bar{\lambda}^t g(\bar{x}) = 0$. Now, from the right inequality of saddle-point conditions and $\bar{\lambda}^t g(\bar{x}) = 0$, we have, for any feasible x for (P),

$$f(\bar{x}) \not< f(x).$$

Hence, $\bar{x}$ is a weak minimum of (P).

Remark : Jeyakumar and Mond[1] defined V-invex multiobjective programs and V-invex multiobjective fractional programs; they showed that the V-invex multiobjective programs can be

regarded as the V-invex multiobjective programs. Of course, we can formulate multiobjective fractional programs with V-ρ-invexity; but we can easily check that multiobjective fractional programs with V-ρ-invexity can not be regarded as multiobjective programs with V-ρ-invexity. So, in the near future, we will study multiobjective fractional programs with V-ρ-invexity.

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