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ON THE METRIC DIMENSION OF A GRAPH

B. SOORYANARAYANA

Department of Mathematics, Ambedkar Institute of Technology, Bangalore 560 056

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A recent result on metric dimension (Theorem 3.2, page 223, *Disc. Appl. Math.* 70 (1996), 217-29) is refined herein to obtain: "A graph G with $\beta(G) = k$ cannot have $K_{2^k+1} - (2^{k-1} - 1)e$ as a subgraph."

Key Words : Vertex Set; Distance between Vertices; Metric Dimension; Landmark

INTRODUCTION

Recently in 1996; Samir Khuller, Balaji Raghavachari and Azriel Rosenfeld[1] have discussed various aspects of the metric dimension of a graph introduced by Harary (see[2]; p.144 for details) and have obtained numerous results. However, in some cases the results have not been amplified to get a closer view of those graphs with small metric dimension.

In this paper, we include some results on graphs with metric dimension k which give a glimpse into the landmarks dealt by them in a very simple and elegant manner. We obtain an improvement of the main result Theorem 3.2, page 223[1] proving that a graph with metric dimension k cannot have $K_{2^k+1} - (2^{k-1} - 1)e$ as a subgraph.

Throughout this article G denote a finite simple connected undirected graph. We recall that the metric dimension of G, denoted by $\beta(G)$ is defined as the cardinality of a minimal subset S of V having the property that for each pair of vertices u, v in V, there exists a w in S such that $d(w, u) \neq d(w, v)$. The coordinate of each v of $V(G)$ with respect to each landmark b_i belonging to S is defined as usual with i^{th} component as $d(v, b_i)$, for each i. For the terms not defined here we refer to [1] and [2].

The main result is obtained *via* a sequence of elementary results.

Lemma 1 — For any graph G with $\beta(G) = 2$, all the vertices of a triangle of G cannot have either the same first or the same second coordinate.

PROOF : Suppose v_1, v_2 and v_3 are the vertices of a triangle of a graph G with $\beta(G) = 2$, having the same first coordinate a. Let the second coordinate of v_1 be b. Then the second coordinate of v_2 and v_3 both must be either $b + 1$ or $b - 1$ only, by hypothesis. Since these are adjacent, this is not admissible. Hence the lemma.

Remark 2 : The above result can be extended on the same lines for the graph with $\beta(G) = k$ as:

For any graph G with $\beta(G) = k$, all the vertices of a triangle of G cannot have the same set of ordered $k - 1$ coordinates.

As a consequence of Lemma 1 and Theorem 3.2 of (2) we observe :

Lemma 3 — Any graph G with $\beta(G) = k$, cannot have a complete sub-graph $K_{2^{k-1}+1}$ whose vertices are all equidistant from any of the landmark of G.

Lemma 4 — Let G any graph with $\beta(G) = k$. Let G' be a complete subgraph K_{2^k} of G. Then for every i, $1 \le i \le k$, the vertices of G' admits a partition into two subsets V_1 and V_2 with $|V_1| = |V_2|$ and such that every vertex lying in $V_j (j = 1, 2)$ has the same ith component.

PROOF : Since each vertex of G' is adjacent to every other vertex of G', the ith coordinate of each vertex can be chosen from either $\{a_i, a_i + 1\}$ or $\{a_i, a_i - 1\}$, and this can be done in ${}^2C_1 = 2$ ways. Using Lemma 3, we observe that the ith coordinate is the same a_i for exactly 2^{k-1} verices, and is the same $a_i + 1$ (or $a_i - 1$) for the remaing 2^{k-1} vertices. These sets of vertices yield the desired partition.

Repeated application of the above result leads to the following:

Theorem 5 — *Let G be any graph with $\beta(G) = k$. Let G' be a complete subgraph K_{2^k} of G. Then for every i, $1 \le i \le k$, the vertex set of G' can be partitioned into 2^i subsets each of cardinality 2^{k-1} and such that all vertices lying in the same partition have the same ordered set of i coordinates.*

Lemma 6 — Let G be any graph with $\beta(G) = k$. Let G' be a subgraph of G satisfying the following stipulations:

(1) $|V(G')| = 2^k$

(2) G' contains atleast 2^{k-1} vertices of degree 2^{k-1}. Then an optimal assignment of i^{th} coordinate for every vertex of G' can only be done by either of the sets $\{a_i, a_i + 1\}$ or $\{a_i, a_i - 1\}$.

PROOF : Let G^* be the complete subgraph $K_{2^{k-1}}$ of G'. Let i^{th} coordinate of one of the vertices of G^* be a_i. Then since each vertex of G' is of degree $2^k - 1$, a_i is adjacent to every vertex of $G' - G^*$ and so can be chosen from one of the sets $\{a_i, a_i + 1\}$ or $\{a_i, a_i - 1\}$ which can be done in ${}^2C_1 = 2$ ways. Hence a distinct coordinate of each vertex of G' can be chosen in 2^k ways. This is optimal since $|V(G')| = 2^k$.

We now prove our main result which refines Theorem 3.2 of [2]:

Theorem 7 — *A graph G with $\beta(G) = k$, cannot contain $K_{2^k+1} - (2^{k-1} - 1)e$ as a subgraph of G.*

PROOF : The proof is by induction on k.

If $k = 1$, the result is trivial since G is a path (cf. Theorem 3.1 of {1}). Let us assume that the result holds for $\beta(G) \le k - 1$ $(k \ge 2)$ and let G be any graph with $\beta(G) = k$. If possible, let G contain $K_{2^{k-1}+1} - (2^{k-1} - 1)e$ as a subgraph G'. Since only $2^{k-1} - 1$ edges are removed from

K_{2^k+1}, there exist atleast 2^{k-1} vertices of G' each with degree equal to 2^k (if there exist $2^{k-1}+1$ vertices of degree less than 2^k, then $|E(G')| = (1/2)\sum_{v \in G'} \deg v \leq (1/2)\ \{2^k(2^{k-1}+1) + (2^{k-1}-1)2^k\} = 2^{2k-1} < (2^{2k-1}) + 1 = {}^{k(2+1)}C_2 - (2^{k-1}-1) = |E(G')|$, leading to a contradiction). Also, since the degree of each vertex is greater than $2^k - (2^{k-1}-1) = 2^{k-1}+1$, it follows that 2^{k-1} vertices of degree 2^k together with any other vertex of G' forms a subgraph $G'' = K_{2^{k-1}+1} - (2^{k-2}-1)e$ of G'. Hence by inductive hypothesis $\beta(G'') > k-1$, This implies that $\beta(G') \geq k$. Suppose $\beta(G') = k$. Let v be a vertex of G' such that deg v = Min deg $v' \theta' \in G'$. Then by the above Lemma 6, G-v is optimal with respect to the coordinates. Since degree of every vertex of G'' is 2^k, it follows that v is adjacent to each vertex of G''. Now by Lemma 3, the vertices of G'' cannot have the same i^{th} coordinate. Thus v must have the same ith coordinate as that of atleast one vertex of $G' - v$, for every i, which is a contradiction to the optimality of coordinates of $G' - v$. Hence, $\beta(G') > k$, implies that $\beta(G) > k$, thereby proving the theorem.

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