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FIXED POINT THEOREMS FOR GENERALIZED UNIFORMLY LIPSCHITZIAN SEMIGROUPS

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Fixed point theorems for generalized uniformly Lipschitzian semigroups are proved in p-uniformly convex Banach space. As applications, their corollaries are given in a Hilbert space, in L^p spaces, in Hardy space H^p, and in Sobolev spaces $H^{k,p}$, for $1 < p < \infty$ and $k \geq 0$.

Key Words : *p*-uniformly Convex Banach Space; Generalized Uniformly Lipschitzian Semigroup; Left Invariant Mean

1. INTRODUCTION

Let K be a nonempty subset of a Banach space E and T a mapping of K into itself. T is said to be uniformly k-Lipschitzian if

$$\| T^n x - T^n y \| \leq k \| x - y \|$$

for all x, y in K and $n = 1, 2, \ldots$. These mappings were first studied by Goebel and Kirk[1]. They proved that such T has a fixed point if K is a bounded closed convex subset of a uniformly convex Banach space E and $k < k^*$ (k^* is the unique solution of the equation $\left(1 - \delta_E\left(\frac{1}{k^*}\right)\right)k^* = 1$ with $\delta_E(\cdot)$ being the modulus of convexity of E) Lifschitz[2] proved in Hilbert space that a uniformly k-Lipschitzian mapping with $k < \sqrt{2}$ has a fixed point. Lim[3, 4] and Lim, Xu and Xu[5] extended Goebel-Kirk's theorem in L^p spaces. Downing and Ray[6] and Ishihara and Takahashi[7] verified that the Lifschitz's theorem is valid for uniformly Lipschitzian semigroups in Hilbert spaces.

Recently, Xu[8] proved some fixed point theorems for a uniformly k-Lipschitzian semigroup in uniformly convex Banach spaces which is left reversible or for which the space of right uniformly continuous functions on the semigroup has a left invariant mean.

Now, we consider the following class of mappings which we call generalized uniformly Lipschitzian mapping, whose nth iterate T^n satisfies the following condition :

$$\| T^n x - T^n y \| \le a \| x - y \| + b(\| x - T^n x \| + \| y - T^n y \|) + c(\| x - T^n y \| + \| y - T^n x \|)$$

for each $x, y \in K$ and $n \ge 1$, where a, b, c are the nonnegative constants such that $b + c < 1$.

This class of generalized uniformly Lipschitzian mappings are more general than nonexpansive and uniformly k-Lipschitzian mappings and it can be seen by taking $b = c = 0$.

In this paper, we prove fixed point theorems for generalized uniformly Lipschitzian semigroups in p-uniformly convex Banach space. Next we give their corollaries in a Hilbert space, in L^p spaces, in Hardy space H^p, and in Sobolev spaces $H^{k,p}$, for $1 < p < \infty$ and $k \ge 0$. Our results[3, 4] improve and extend results of Downing and Ray[6], Lau[9], Lim[3], Ishihara and Takahashi[7], and Xu[8].

2. PRELIMINARIES

Let G be a semitopological semigroup, i.e., G is a semigroup with a Hausdorff topology such that for each $a \in G$ the mapping $s \to a \cdot s$ and $s \to s \cdot a$ from G to G are continuous. Denote by $C(G)$ the Banach space of bounded continuous real valued functions on G. Then for $f \in C(G)$ and $a \in G$, define $(l_a f)(s) = f(as)$ and $(r_a f)(s) = f(sa)$ for all $s \in G$. If E is a closed subspace of $C(G)$ containing constants and $l_a(E) \subseteq E$ for all $a \in G$, then $m \in E^*$ is called a left invariant mean if $\| m \| = m(1) = 1$ and $m(l_a f) = m(f)$ for all $a \in G$ and $f \in G$. Let $RUC(G)$ be the space of bounded right uniformly continuous functions on G, i.e., all $f \in C(G)$ such that the function $a \to r_a f$ is continuous when $C(G)$ has the norm topology. Then $RUC(G)$ is closed translation invariant int-algebra of $C(G)$ containing constants[10]. Let $\{x_s : s \in G\}$ be a bounded family of elements of E. Then as in the proof of Lau[9] (Lemma 3.4), we have for each $p > 1$ and x in E that the function $h(s) = \| x_s - x \|^p$ is in $RUC(G)$. For a mean m on $RUC(G)$, denote by $m_s \| x_s - x \|^p$ the value of m at the function h.

Let K be a nonempty subset of a Banach space E. Then a family $S = \{T_t : t \in G\}$ of mappings from K into itself is said to be a generalized uniformly Lipschitzian semigroup on K if T satisfies the following :

(*i*) $T_{ts}(x) = T_t T_s(x)$ for $t, s \in G$ and $x \in K$;

(*ii*) the mapping $(s, x) \to T_s(x)$ from $G \times K$ into K is continuous when $G \times K$ has the product topology; and

(*iii*) for each $s \in G$

$$\| T_s x - T_s y \| \le a \| x - y \| + b(\| x - T_s x \| + \| y - T_s y \|) + c(\| x - T_s y \| + \| y - T_s x \|),$$

for $x, y \in K$, where a, b, $c > 0$ such that $b + c < 1$.

Let $p > 1$ and denote by λ the number in [0, 1] and by $W_p(\lambda)$ the function $\lambda \cdot (1 - \lambda)^p + \lambda^p \cdot (1 - \lambda)$.

The functional $\| \cdot \|^p$ is said to be uniformly convex (cf. Zalinescu[11]) on the Banach space E if there exists a positive constant c_p such that for all $\lambda \in [0, 1]$ and $x, y \in E$ the following inequality holds :

$$\| \lambda x + (1-\lambda)y \|^p \le \lambda \| x \|^p + (1-\lambda) \| y \|^p - W_p(\lambda) \cdot c_p \cdot \| x - y \|^p. \quad \ldots (1)$$

Xu[12] proved that the functional $\| \cdot \|^p$ is uniformly convex on the whole Banach space E if and only if E is p-uniformly convex, i.e., there exists a constant $c_p > 0$ such that the moduli of convexity[13] $\delta_E(\varepsilon) \ge c_p \cdot \varepsilon^p$ for all $0 \le \varepsilon \le 2$.

The following Lemma is useful to prove our results.

Lemma 1[8] — Let K be a closed convex subset of a p-uniformly convex Banach space E, m a left invariant mean on $RUC(G)$, and $\{x_s : x \in G\}$ a bounded family of elements of E. Then there exists a unique element $z \in K$ such that

$$m_s \| x_s - z \|^p \le m_s \| x_s - x \|^p - c_p \cdot \| x - z \|^p \quad \ldots (2)$$

for all $x \in K$, where the constant c_p is an in (1).

3. Main Results

Now we are in position to give our result.

***Theorem* 1** — *Let K be a nonempty closed convex subset of a p-uniformly convex Banach space E and let $S = \{T_s : s \in G\}$ be a generalized Lipschitzian semigroup on K with* $\left(\dfrac{2^{p-1}\alpha^p - 1}{c_p - 2^{p-1}\beta^p} \right) \cdot \dfrac{2^{p-1}(\alpha^p + \beta^p)}{1 + c_p} < 1$ *where* $\alpha = \dfrac{a+b+c}{1-b-c}$ *and* $\beta = \dfrac{2b+2c}{1-b-c}$ *and c_p being as in (1). Suppose that $\{T_s x_0 : s \in G\}$ is bounded for some $x_0 \in K$ and that $RUC(G)$ has a left invariant mean. Then there exists a $z \in K$ such that $T_s z = z$ for all $s \in G$.*

PROOF : Let m be a left invariant mean on $RUC(G)$. Then by Lemma 1, we can inductively define a sequence $\{x_n\}_{n \ge 1}$ in K such that

$$m_t \| T_t x_{n-1} - x_n \|^p \le m_t \| T_t x_{n-1} - x \|^p - c_p \cdot \| x - x_n \|^p \quad \ldots (3)$$

By using (iii), after a simple calculation, we have

$$\| T_s T_t x - T_s y \| \le \alpha \| T_t x - y \| + \beta \| y - T_s y \| \quad \ldots (4)$$

where $\alpha = \dfrac{a+b+c}{1-b-c}$ and $\beta = \dfrac{2b+2c}{1-b-c}$. Putting $x = T_s x_n$ into (3), we have

$$c_p \| T_s x_n - x_n \|^p \le m_t \| T_t x_{n-1} - T_s x_n \|^p - m_t \| T_t x_{n-1} - x_n \|^p$$

$$= m_t \| T_{st} x_{n-1} - T_s x_n \|^p - m_t \| T_t x_{n-1} - x_n \|^p$$

$$\le m_t \{ \alpha \| T_t x_{n-1} - x_n \| + \beta \| x_n - T_s x_n \| \}^p - m_t \| T_t x_{n-1} - x_n \|^p$$

or

$$\| T_s x_n - x_n \|^p \leq \left(\frac{2^{p-1}\alpha^p - 1}{c_p - 2^{p-1}\beta^p} \right) \cdot m_t \| T_t x_{n-1} - x_n \|^p. \quad \ldots (5)$$

Now, inserting $x = T_s x_{n-1}$ into (3), we have

$$m_t \| T_t x_{n-1} - x_n \|^p \leq \frac{2^{p-1}(\alpha^p + \beta^p)}{1 + c_p} \cdot m_t \| T_t x_{n-1} - x_{n-1} \|^p. \quad \ldots (6)$$

Combining (5) and (6), we have

$$\begin{aligned} m_t \| T_t x_n - x_n \|^p &\leq \left(\frac{2^{p-1}\alpha^p - 1}{c_p - 2^{p-1}\beta^p} \right) \cdot \frac{2^{p-1}(\alpha^p + \beta^p)}{1 + c_p} \cdot m_t \| T_t x_{n-1} - x_{n-1} \|^p \\ &\leq A^n \cdot m_t \| T_t x_0 - x_0 \|^p, \end{aligned}$$

where

$$A = \left(\frac{2^{p-1}\alpha^p - 1}{c_p - 2^{p-1}\beta^p} \right) \cdot \frac{2^{p-1}(\alpha^p + \beta^p)}{1 + c_p} < 1$$

by the assumption of Theorem. It follows that

$$\begin{aligned} \| x_n - x_{n-1} \|^p &\leq m_t (\| x_n - T_t x_{n-1} \| + \| T_t x_{n-1} - x_{n-1} \|)^p \\ &\leq 2^{p-1} m_t \| x_n - T_t x_{n-1} \|^p + 2^{p-1} m_t \| T_t x_{n-1} - x_{n-1} \|^p \\ &\leq 2^{p-1} \left[\left\{ \frac{2^{p-1}(\alpha^p + \beta^p)}{1 + c_p} \right\} + 1 \right] \cdot m_t \| T_t x_{n-1} - x_{n-1} \|^p \\ &\leq 2^{p-1} \left[\left\{ 1 + \frac{2^{p-1}(\alpha^p + \beta^p)}{1 + c_p} \right\} \right] \cdot A^{n-1} m_t \cdot \| T_t x_0 - x_0 \|^p \end{aligned}$$

which shows that $\{x_n\}$ is a Cauchy sequence. Let $z = \lim_{n \to \infty} x_n$. Then for each $s \in G$, we have

$$\begin{aligned} &\| z - T_s z \|^p \\ &\leq (\| z - x_n \| + \| x_n - T_s x_n \| + \| T_s x_n - T_s z \|)^p \\ &\leq \left[\left(\frac{1 + a + 2c}{1 - b - c} \right) \cdot \| z - x_n \| + \left(\frac{1 + b + c}{1 - b - c} \right) \cdot \| x_n - T_s x_n \| \right]^p \\ &\leq 2^{p-1} \left[\left(\frac{1 + a + 2c}{1 - b - c} \right)^p \cdot \| z - x_n \|^p + \left(\frac{1 + b + c}{1 - b - c} \right)^p \cdot \| x_n - T_s x_n \|^p \right] \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Therefore, $T_s z = z$ for all $s \in G$. The proof is complete.

A semitopological semigroup G is said to be left reversible if any two closed right ideals have nonvoid intersection. In this case, $(G, \ )$ is a directed system when the binary relation " " on G is defined by $a \ \ b$ if and only if $\{a\} \cup \overline{aG} \supseteq \{b\} \cup \overline{bG}$, where $\overline{D}$ is the closure of a set D.

Let $\{x_a : a \in G\}$ be a bounded net in a uniformly convex Banach space E and K a nonempty closed convex subset of E. For a fixed $p > 1$, let us set

$$r(x) = \inf_{b \in S} \sup_{a \ \ b} \| x_a - x \|^p$$

and

$$r = \inf \{r(x) : x \in K\}.$$

Then we have a unique point $z \in K$ (called the asymptotic centre of the net $\{x_a\}$ in K) such that $r(z) = r$.

We need following lemma to prove our next theorem.

Lemma 2^8 — Let E be a p-uniformly convex Banach space for some $p > 1$. Then we have

$$r(z) \leq r(x) - c_p \cdot \| x - z \|^p$$

for all $x \in K$, where the constant c_p as in (1).

***Theorem* 2** — *Let* E be a p-uniformly convex Banach space for some $p > 1$, K a nonempty closed convex subset of E, and $S = \{T_s : s \in G\}$ a generalized uniformly Lipschitzian semigroup on K with $\left(\dfrac{2^{p-1}\alpha^p - 1}{c_p - 2^{p-1}\beta^p} \right) \cdot \dfrac{2^{p-1}(\alpha^p + \beta^p)}{1 + c_p} < 1$ where $\alpha = \dfrac{a+b+c}{1-b-c}$, $\beta = \dfrac{2b+2c}{1-b-c}$ and c_p as in (1). Suppose that G is left reversible and $\{T_s x_0 : s \in G\}$ is bounded for some x_0 in K. Then there exists an element $z \in K$ such that $T_s z = z$ for every $s \in G$.

PROOF : Define a sequence $\{x_n\} \subset K$ in the following manner : x_{n+1} is the asymptotic centre of the net $\{T_s x_n\}$ in K. Then by Lemma 2, we have for $x \in K$ and $n = 1, 2, \ldots,$

$$c_p \| x_{n+1} - x \|^p \leq \inf_s \sup_t \| T_t x_n - x \|^p - \inf_s \sup_t \| T_t x_n - x_{n+1} \|^p. \qquad \ldots (7)$$

Noting the inequality

$$\inf_s \sup_t \| T_t y - x \|^p \leq \inf_s \sup_t \| T_{at} y - x \|^p$$

is valid for all $x, y \in K$ and every $a \in G$, we get from (6)

$$c_p \cdot \| x_{n+1} - T_a x_{n+1} \|^p$$

$$\leq \inf_s \sup_t \| T_{at} x_n - T_a x_{n+1} \|^p - \inf_s \sup_t \| T_t x_n - x_{n+1} \|^p$$

$$\leq 2^{p-1} \left\{ \alpha^p \inf_s \sup_t \| T_t x_n - x_{n+1} \|^p + \beta^p \inf_s \sup_t \| x_{n+1} - T_a x_{n+1} \|^p \right\}$$

$$- \inf_s \sup_t \| T_t x_n - x_{n+1} \|^p$$

implies

$$\| x_{n+1} - T_a x_{n+1} \|^p \leq \left(\frac{2^{p-1} \alpha^p - 1}{c_p - 2^{p-1} \beta^p} \right) \inf_s \sup_t \| T_t x_n - x_{n+1} \|^p$$

$$\leq \left(\frac{2^{p-1} \alpha^p - 1}{c_p - 2^{p-1} \beta^p} \right) \inf_s \sup_t \| T_t x_n - x_n \|^p.$$

Then as in the proof of Theorem 1, it follows that the sequence $\{x_n\}$ converges in norm to some element $z \in K$ for which $T_s(z) = z$ for each $s \in G$. This completes the proof.

Since a Hilbert space H is 2-uniformly convex and the following identity holds :

$$\| \lambda x + (1-\lambda) y \|^2 = \lambda \| x \|^2 + (1-\lambda) \| y \|^2 - \lambda(1-\lambda) \| x - y \|^2 \quad \ldots (8)$$

for all x, y in H and $\lambda \in [0, 1]$.

By Theorem 1 and Theorem 2 and by inequality (8), we have following results :

Corollary 1 — Let K be a nonempty closed convex subset of a Hilbert space H and let $\mathcal{S} = \{T_s : s \in G\}$ be a generalized uniformly Lipschitzian semigroup on K with $\frac{(2\alpha^2 - 1)(\alpha^2 + \beta^2)}{1 - 2\beta^2} < 1$, where $\alpha = \frac{a+b+c}{1-b-c}$ and $\beta = \frac{2b+2c}{1-b-c}$. Suppose that there is $x_0 \in K$ such that $\{T_s x_0\}$ is bounded and that either G is left reversible or the space $RUC(G)$ has a left invariant mean. Then there is some $z \in K$ such that $T_s(z) = z$ for every $s \in G$.

If $1 < p \leq 2$, then we have for all x, y in L^p and $\lambda \in [0, 1]$

$$\| \lambda x + (1-\lambda) y \|^2 \leq \lambda \| x \|^2 + (1-\lambda) \| y \|^2 - \lambda(1-\lambda)(p-1) \| x - y \|^2 \quad \ldots (9)$$

(The inequality (9) is contained in Lim *et al.*[5] and Smarzewski[14].)

Assume that $2 < p < \infty$ and t_p is the unique zero of the function $g(x) = -x^{p-1} + (p-1)x + p - 2$ in the interval $(1, \infty)$. Let

$$c_p = (p-1)(1+t_p)^{2-p} = \frac{1 + t_p^{p-1}}{(1+t_p)^{p-1}}.$$

Then we have the following inequality

$$\| \lambda x + (1-\lambda) y \|^p \leq \lambda \| x \|^p + (1-\lambda) \| y \|^p - c_p \cdot W_p(\lambda) \cdot \| x - y \|^p \quad \ldots (10)$$

As a consequence of Theorem 1 and Theorem 2 with inequality (9) and (10) we have the following.

Corollary 2 — Let K be a nonempty closed subset of L^p, $1 < p < \infty$, G a semitopological semigroup which is left reversible or for which $RUC(G)$ has a left invariant mean, and $S = \{T_s : s \in G\}$ a generalized uniformly Lipschitzian semigroup on K with

$$\left(\frac{2\alpha^2 - 1}{p - 1 - 2\beta^2} \right) \cdot \frac{2(\alpha^2 + \beta^2)}{p} < 1 \qquad \text{for } 1 < p \leq 2$$

and

$$\left(\frac{2^{p-1}\alpha^p - 1}{c_p - 2^{p-1}\beta^p} \right) \cdot \frac{2^{p-1}(\alpha^p + \beta^p)}{1 + c_p} < 1 \qquad \text{for } 2 < p < \infty.$$

Suppose that there is some $x_0 \in K$ such that $\{T_s x_0 : x \in G\}$ is bounded. Then there exists an element $z \in K$ such that $T_s(z) = z$ for each $s \in G$.

Let H^p, $1 < p < \infty$, denote the Hardy space[16] of all functions x analytic in unit disc $|z| < 1$ of the complex plane and such that

$$\| x \| = \lim_{r \to 1^-} \left(\frac{1}{2\pi} \int_0^{2\pi} |x(re^{i\theta})|^p \, d\theta \right)^{\frac{1}{p}} < \infty.$$

Now, let Ω be an open subset of $\mathbb{R}^n$. Denote by $H^{k,p}(\Omega)$, $k \geq 0$, $1 < p < \infty$, the Sobolev space [17, p. 149] of distribution x such that $D^\alpha x \in L^p(\Omega)$ for all $|\alpha| = \alpha_1 + \ldots + \alpha_n \leq k$ equipped with the norm

$$\| x \| = \left(\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha x(\omega)|^p \, d\omega \right)^{\frac{1}{p}}.$$

Let $(\Omega_\alpha, \sum_\alpha, \mu_\alpha)$, $\alpha \in \Lambda$, be a sequence of positive measure spaces, where index set Λ is finite or countable. Given a sequence of linear subspaces X_α in $L^p(\Omega_\alpha, \sum_\alpha, \mu_\alpha)$, $\alpha \in \Lambda$, we denote by $L_{q,p}$, $1 < p < \infty$ and $q = \max\{2, p\}$[18], the linear space of all sequences $x = \{x_\alpha \in X_\alpha : \alpha \in \Lambda\}$ equipped with the norm

$$\| x \| = \left(\sum_{\alpha \in \Lambda} (\| x_\alpha \|_{p,\alpha})^q \right)^{\frac{1}{q}},$$

where $\| \cdot \|_{p,\alpha}$ denote the norm in $L^p(\Omega_\alpha, \sum_\alpha, \mu_\alpha)$.

Finally, let $L_p = L^p(S_1, \sum_1, \mu_1)$ and $L_q = L^q(S_2, \sum_2, \mu_2)$ where $1 < p < \infty$, $q = \max\{2, p\}$ and $(S_i, \sum_i, \mu_i)$ are positive measure spaces. Denote by $L_q(L_p)$ the Banach spaces [19, III. 2.10] of all measurable L_p-value function x on S_2 such that

$$\| x \| = \left(\int_{S_2} (\| x(s) \|_p)^q \mu_2(ds) \right)^{\frac{1}{q}}.$$

There spaces are q-uniformly convex with $q = \max\{2, p\}$ [14, 20] and the norm in these spaces satisfies

$$\| \lambda x + (1-\lambda) y \|^q \leq \lambda \| x \|^q + (1-\lambda) \cdot \| y \|^q - d \cdot W_q(\lambda) \cdot \| x - y \|^q$$

with a constant

$$d = d_p = \begin{cases} \dfrac{p-1}{8} & \text{for } 1 < p \leq 2 \\ \dfrac{1}{p \cdot 2^p} & \text{for } 2 < p < \infty. \end{cases}$$

Now, from Theorems 1 and 2, we have the following result.

Corollary 3 — Let K be a nonempty subset of the space E, where $E = H^p$, or $E = H^{k,p}(\Omega)$, or $E = L_{q,p}$ or $E = L_q(L_p)$, and $1 < p < \infty$, $q = \max\{2, p\}$, $k \geq 0$. Let G be a semitopological semigroup which is left reversible or for which $RUC(G)$ has a left invariant mean and $S = \{T_s : s \in G\}$ a generalized uniformly Lipschitzian semigroup on K with $\left(\dfrac{2^{q-1}\alpha^p - 1}{d - 2^{q-1}\beta^q} \right) \cdot \dfrac{2^{q-1}(\alpha^q + \beta^q)}{1+d} < 1$ where $\alpha = \dfrac{a+b+c}{1-b-c}$ and $\beta = \dfrac{2b+2c}{1-b-c}$. Suppose that there is a some $x_0 \in K$ such that $\{T_s x_0 : s \in G\}$ is bounded. Then there is an element $z \in K$ such that $T_s(z) = z$ for each $s \in G$.

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