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z-EMBEDDING AND FIBERED PRODUCT

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The present paper deals with the study concerning z-embedding of the fibered product $X \times_Z Y$ of Tychonoff spaces X and Y in $\beta X \times_{\beta Z} \beta Y$, where Z is a Tychonoff space and βS denotes the Stone-Čech compactification of a Tychonoff space S.

INTRODUCTION

By a space we mean a Hausdorff space and a map means a continuous map. The letters X, Y and Z are used for spaces.

For a set A in a space X, the closure (interior) of A in X will be denoted by $\mathrm{Cl}_X A$ ($\mathrm{Int}_X A$) or simply by Cl A (Int A) if it is clear from the context. The Stone-Čech compactification of a Tychonoff space X is denoted by βX and for a continuous map f, βf denotes its Stone extension.

The real line is denoted by R and N stands for the set of natural numbers.

Blair[2] introduced the notion of z-embedding which has further been studied by him[3-5]. Later Blair and Hager[6], dealt with the z-embedding of the product of $X \times Y$ of Tychonoff spaces X and Y in $\beta X \times \beta Y$. The present paper continues this study in the context of fibered product. Specifically, in section 2 of this paper, we study the conditions for z-embedding of the fibered product $X \times_Z Y$ of Tychonoff spaces X and Y with respect to the Tychonoff space Z in the fibered product $\beta X \times_{\beta Z} \beta Y$. In section 3, it is proved that the z-embedding of $X \times_Z Y$ in $\beta X \times_{\beta Z} \beta Y$ is equivalent to the equality coz $(X \times_Z Y)$ = coz $X \otimes_Z$ coz Y, where coz is a functor from the category of Tychonoff spaces and continuous maps to the Alexandroff's category of cozero-fields and A-morphisms. That the above results are generalizations of the corresponding results due to Blair and Hager, follows immediately by taking Z the singleton space.

For a real-valued function f on a set X, the zero set of f denoted by Zf is defined to be the set $\{x \in X \mid f(x) = 0\}$ and the cozero set of f is the complement of Zf which is denoted by coz f. A zero set (cozero set) in X is the zero set (cozero set) of some real-valued function on X. The family of zero sets in X is denoted by ZX and that of cozero sets in X by coz X.

Denote by $C(X)$, the ring of all real-valued continuous functions on a space X. A subspace S of a space X is said to be z-embedded in X if given a zero set P in S, there exists a zero set Z in X such that $P = Z \cap S$. A space X is called an Oz-space if each open set in X is z-embedded in X (Blair[4]). A set A in a space X is said to be an F_σ-set, if A is a union of countably many closed sets in X. Each cozero set is an F_σ-set. A set A in X is called regular open if Int ClA = A and A is called regular closed if Cl Int A = A.

For spaces X, Y and Z and maps f from X to Z and g from Y to Z, the fibered product of X and Y with respect to Z, f and g is the subspace $\{(x, y) \in X \times Y \mid f(x) = g(y)\}$ of $X \times Y$ which is denoted by $X \times_Z Y$. If $A \subset X$ and $B \subset Y$, then $A \times_Z B$ stands for the set $(A \times B) \cap (X \times_Z Y)$. Notice that the fibered product $X \times_Z Y$ is the pullback of f and g in the category of topological spaces and continuous maps: Recall that, for objects X, Y and Z and morphisms f and g from X to Z and from Y to Z, respectively, in a category C, the pullback of f and g is an object P together with morphisms p_1 and p_2 from P to X and from P to Y, respectively, such that (i) $f \circ p_1 = g \circ p_2$ and (ii) given an object W and morphisms l and k from W to X and from W to Y, respectively, with $f \circ l = g \circ k$, there exists a unique morphism h from W to P such that $p_1 \circ h = l$ and $p_2 \circ h = k$.

For relevant literature and terms not explained here, we refer to Gillman and Jerison[7], Hager[11], Porter and Woods[12] and Walker[13].

2. z-EMBEDDING OF $X \times_Z Y$ IN $\beta X \times_{\beta Z} \beta Y$

In this section, following a technique due to Blair and Hager[6], we obtain necessary and sufficient conditions for z-embedding of $X \times_Z Y$ in $\beta X \times_{\beta Z} \beta Y$. We begin with some preliminary results and definitions.

Note 2.1 — (a) Suppose that $A \subset X \times_Z Y$. Since $X \times_Z Y$ is closed in $X \times Y$, $\mathrm{Cl}_{X \times_Z Y} A = \mathrm{Cl}_{X \times Y} A$: We shall write ClA for this set. Further, if $X \times_Z Y$ is also open in $X \times Y$, then $\mathrm{Int}_{X \times_Z Y} A = \mathrm{Int}_{X \times Y} A$: We shall write Int A for this set,

(b) If A and B are subsets of a space X, where B is clopen (closed and open), then $\mathrm{Cl}(A \cap B) = \mathrm{Cl}A \cap B$.

Example 2.2 — Let S^1 be the unit circle in the Euclidean plane R^2. Define $f: I \to S^1$ by $f(t) = (\cos 2\pi t, \sin 2\pi t)$ and $g: I \to S^1$ by $g(t) = (1, 0)$, $t \in I$, where I is the closed unit interval of R. Suppose that $A = [0, 1)$ and $B = (0, 1/2)$. Then $A \times_{S^1} B = \{0\} \times (0, 1/2)$ and Cl $(A \times_{S^1} B) = \{0\} \times [0, 1/2]$, while Cl $A \times_{S^1}$ Cl $B = \{0, 1\} \times [0, 1/2] \neq$ Cl $(A \times_{S^1} B)$. Also, it is clear that Int $(A \times_{S^1} B) \neq$ Int $A \times_{S^1}$ Int B. Thus, we infer that the operation of closure and that of interior need not distribute over a fibered product.

Proposition 2.3 — If $A \subset X, B \subset Y$ and $X \times_Z Y$ is open in $X \times Y$, then (i) Cl $(A \times_Z B) = \mathrm{Cl}A \times_Z \mathrm{Cl}B$ and (ii) Int $(A \times_Z B) = \mathrm{Int}A \times_Z \mathrm{Int}B$.

PROOF : Note that $\mathrm{Cl}\,(A \times_Z B) = \mathrm{Cl}\,((A \times B) \cap (X \times_Z Y)) = \mathrm{Cl}\,(A \times B) \cap (X \times_Z Y)$ $= (\mathrm{Cl}\,A \times \mathrm{Cl}\,B) \cap (X \times_A Y) = \mathrm{Cl}\,A \times_Z \mathrm{Cl}\,B$.

For the second part, using (i) it can be easily shown that $(X \times_Z Y) - \mathrm{Int}\,(A \times_Z B) = (X \times_Z Y) - (\mathrm{Int}\,A \times_Z \mathrm{Int}\,B)$.

Definitions 2.4 — A cozero-rectangle in $X \times Y$ is a set of the form $P \times Q$, where $P \in \mathrm{coz}\,X$ and $Q \in \mathrm{coz}\,Y$. By a cozero (open) rectangle in $X \times_Z Y$, we shall mean a set of the form $P \times_Z Q$, where, P and Q are cozero (open) sets in X and Y, respectively. If X and Y are Tychonoff spaces, then the family of cozero rectangles in $X \times_Z Y$ forms a base for open sets in $X \times_Z Y$.

For $f \in C(X)$ and a nonempty set A in X, the oscillation of f on A is denoted by $\mathrm{osc}\,(f, A)$ and $\mathrm{osc}\,(f, A) \overset{\text{def}}{=} \sup\,\{|\,f(x) - f(y)\,|\,|\,x, y \in A\}$.

Theorem 2.5 — Let S, T and Z be Tychonoff spaces and X and Y be sets in S and T, respectively. Then, the following statements are equivalent.

(a) If X is z-embedded in S and Y is z-embedded in T, then $X \times_Z Y$ is z-embedded in $S \times_Z T$.

(b) $X \times_Z Y$ is z-embedded in $\beta X \times_{\beta Z} \beta Y$.

(c) Each cozero set is $X \times_Z Y$ is a countable union of cozero rectangles in $X \times_Z Y$.

(d) Given $f \in C\,(X \times_Z Y)$ and $\varepsilon > 0$, there exists a countable cover $\{A_n \mid n \in N\}$ of $X \times_Z Y$, where each A_n is a cozero rectangle in $X \times_Z Y$ with $\mathrm{osc}\,(f, A_n) < \varepsilon$ for each $n \in N$.

PROOF : (a) $\Rightarrow$ (b) : Since $X \times_Z Y = X \times_{\beta Z} Y$, it follows by noting that X is z-embedded in βX and Y is z-embedded in βY, because X and Y are C^*-embedded in βX and βY, respectively.

(b) $\Rightarrow$ (c) : Suppose that $P \in \mathrm{coz}\,[X \times_Z Y]$. Then, there exists $Q \in \mathrm{coz}\,[\beta X \times_{\beta Z} \beta Y]$ such that $P = Q \cap (X \times_Z Y)$. Being a cozero set, Q is an F_σ-set in the compact space $\beta X \times_{\beta Z} \beta Y$. Hence, Q is Lindelöf. Now, for each $q \in Q$, there exists a cozero rectangle Aq in $\beta X \times_{\beta Z} \beta Y$ such that $q \in Aq \subset Q$. As the family $\{Aq \mid q \in Q\}$ is an open cover of Q, we get a countable subset K of Q such that $Q = \bigcup_{q \in K} Aq$. Consequently, $P = \bigcup_{q \in k} (Aq \cap (X \times_Z Y))$. Since for each $q \in K$, $Aq \cap (X \times_Z Y)$ is a cozero rectangle in $X \times_Z Y$, the result follows.

(c) $\Rightarrow$ (d) : Let $f \in C\,(X \times_Z Y)$ and $\varepsilon > 0$. Cover R by a countable family $\{I_i \mid i \in N\}$ of open intervals in R, each of length $\varepsilon/2$. As $f^{-1}(I_i) \in \mathrm{coz}\,[X \times_Z Y]$, there exists a countable family $\{A_{in} \mid n \in N\}$ of cozero rectangles in $X \times_Z Y$ whose union is $f^{-1}(I_i)$. Clearly, $\{A_{in} \mid i, n \in N\}$ is a countable cover of $X \times_Z Y$ with $\mathrm{osc}\,(f, A_{in}) < \varepsilon$ for each $i, n \in N$.

(d) $\Rightarrow$ (c) : Let $P \in \text{coz}\ [X \times_Z Y]$. Then $P = \text{coz}\ f$ for some $f \in C\ (X \times_Z Y)$. By hypothesis, for $m \in N$, there exists a cover $\{A_{mn} \mid n \in N\}$ of $X \times_Z Y$ such that each A_{mn} is a cozero rectangle in $X \times_Z Y$ and osc $(f, A_{mn}) < 1/m$, for each $n \in N$. Let $M = \{(m, n) \in N \times N \mid A_{mn} \subset \text{coz}\ f\}$. We claim that $\text{coz}\ f = \bigcup_{(m,n) \in M} A_{mn}$. It suffices to show that $\text{coz}\ f \subset \bigcup_{(m,n) \in M} A_{mn}$. For $(x, y) \in \text{coz}\ f$, choose $m \in N$ such that $| f(x, y) | > 1/m$. Next, choose $n \in N$ such that $(x, y) \in A_{mn}$. Since, for $(x_1, y_1) \in A_{mn}$, $| | f(x, y) | - | f(x_1, y_1) | | \leq | f(x, y) - f(x_1, y_1) | < 1/m$, it follows that $f(x_1, y_1) \neq 0$, for otherwise $| f(x, y) | < 1/m$, a contradiction. Thus $(x_1, y_1) \in \text{coz}\ f$, so that $A_{mn} \subset \text{coz}\ f$. This gives that $(m, n) \in M$. Hence, the result.

(c) $\Rightarrow$ (a) : Let $P \in \text{coz}\ (X \times_Z Y)$. Then $P = \bigcup_{n \in N} A_n \times_Z B_n$, where $A_n \in \text{coz}\ X$ and $B_n \in \text{coz}\ Y$. Since X and Y are z-embedded in S and T respectively, there exist cozero sets P_n and Q_n in S and T, respectively, such that $A_n = P_n \cap X$ and $B_n = Q_n \cap Y$. As $Q = \bigcup_{n \in N} P_n \times_Z Q_n \in \text{coz}\ (S \times_Z T)$ and $P = Q \cap (X \times_Z Y)$, the result follows.

Lemma 2.6 — If $\{U_\alpha \mid \alpha \in \mathcal{A}$, $\mathcal{A}$ being a nonempty index set$\}$ is an open cover of a space X such that osc $(f, U_\alpha) < \varepsilon/3$ for each $\alpha \in \mathcal{A}$, where $f \in C(X)$, then osc $(f, \text{Cl}\ U_\alpha) \leq \varepsilon$ for each $\alpha \in \mathcal{A}$.

PROOF : For $x, y \in \text{Cl}\ U_\alpha$, $\alpha \in \mathcal{A}$, there exist $\beta, \gamma \in \mathcal{A}$ such that $x \in U_\beta$ and $y \in U_\gamma$. Choose, $a \in U_\alpha \cap U_\beta$ and $b \in U_\alpha \cap U_\gamma$. Then, $| f(x) - f(y) | \leq | f(x) - f(a) | + | f(a) - f(b) | + | f(b) - f(y) | < \varepsilon$. This gives the result.

Theorem 2.7 — Suppose that X and Y are Tychonoff Oz-spaces and Z is a Tychonoff space. If $X \times_Z Y$ is open in $X \times Y$, then the following conditions are equivalent to each of the four conditions in Theorem 2.5.

(e) Each cozero set in $X \times_Z Y$ is a countable union of open rectangles in $X \times_Z Y$.

(f) Given $f \in C(X \times_Z Y)$ and $\varepsilon > 0$, there exists a countable cover $\{A_n \mid n \in N\}$ of $X \times_Z Y$, where each A_n is an open rectangle in $X \times_Z Y$ such that osc $(f, U_n) < \varepsilon$ for each $n \in N$.

PROOF : (c) $\Rightarrow$ (e) : It is trivial.

(e) $\Rightarrow$ (f) : It is similar to the proof of (c) $\Rightarrow$ (d).

(f) $\Rightarrow$ (d) : Let $f \in C\ (X \times_Z Y)$ and let $\varepsilon > 0$. Then, by (*f*), there exists a countable cover $\{U_n \times_Z V_n \mid n \in N\}$ of $X \times_Z Y$ by open rectangles in $X \times_Z Y$ such that osc $(f, U_n \times_Z V_n) < \varepsilon/6$ for each $n \in N$. By Lemma 2.6, osc $(f, \text{Cl}(U_n \times_Z V_n)) \leq \varepsilon/2$. Using Proposition 2.3, we get that osc $(f, \text{Int Cl}\ U_n \times_Z \text{Int Cl}\ V_n) \leq \varepsilon/2$. Result, now follows by noting that (i) being regular open sets in Oz-spaces X and Y, respectively, Int Cl U_n and Int Cl V_n are cozero sets (Blair[4]) and (ii) $\{\text{Int Cl}\ U_n \times_Z \text{Int Cl}\ V_n \mid n \in N\}$ is a cover of $X \times_Z Y$.

Corollary 2.8 — If X, Y and Z are Tychonoff spaces such that $X \times_Z Y$ is Lindelöf, then $X \times_Z Y$ is z-embedded in $\beta X \times_{\beta Z} \beta Y$.

PROOF : Suppose that $P \in coz\,[X \times_Z Y]$. Then, $P = \bigcup_{\alpha \in \mathcal{A}} A_\alpha$, where for each $\alpha \in \mathcal{A}$, A_α is a cozero rectangle in $X \times_Z Y$. Since, P is an F_σ-set in the Lindelöf space $X \times_Z Y$, P is Lindelöf. Thus, there exists a countable subfamily $\{A_{\alpha_n} \mid n \in N\}$ of the family $\{A_\alpha \mid \alpha \in \mathcal{A}\}$ such that $P = \bigcup_{n \in N} A_{\alpha_n}$. The result is now immediate from Theorem 2.5.

3. z-EMBEDDING AND THE FUNCTOR coz

In this section we establish an equivalence for z-embedding of $X \times_Z Y$ in $\beta X \times_{\beta Z} \beta Y$ in terms of the functor coz.

Definitions 3.1 — A separated cozero field on a set X is a family $\mathcal{A}$ of subsets of X with the following properties : $\varphi, X \in \mathcal{A}$, Given two distinct points of X, some set in $\mathcal{A}$ contains one and not the other; $\mathcal{A}$ is closed under finite intersection and countable union; disjoint sets with complements in $\mathcal{A}$ are contained in disjoints sets in $\mathcal{A}$; each member of $\mathcal{A}$ is the union of a sequence of sets with complements in $\mathcal{A}$. The system $\langle X, \mathcal{A} \rangle$ is called a separated Alexandroff space or an A-space.

A function $f : \langle X, \mathcal{A}_1 \rangle \to \langle Y, \mathcal{A}_2 \rangle$ is called an A-morphism if for each $A \in \mathcal{A}_2$, $f^{-1}(A) \in \mathcal{A}_1$.

A-spaces with the morphisms described above form a category, say Alex. If Tych is the category of Tychonoff spaces and continuous maps, then the assignment coz : Tych $\to$ Alex taking X in Tych to $\langle X, \mathrm{coz}\, X \rangle$ in Alex is a functor.

Remark 3.2 : The concept of A-spaces and that of zero set spaces are dual to each other in the sense that given a separated cozero field $\mathcal{A}$ on a set X, the family of complements of members in $\mathcal{A}$ forms a zero-set structure on X and vice-versa.

For literature concerning A-spaces and zero-set spaces, we refer to Alexandroff[1], Gilmour[8], Gordon[9] and Hager[10, 11].

Proposition 3.3 — Suppose that $\langle X, \mathcal{A} \rangle$ and $\langle Y, \mathcal{B} \rangle$ are A-spaces and Z is a set. Let $\mathcal{F}_0 = \{p^{-1}(A) \mid A \in \mathcal{A}\} \cup \{q^{-1}(B) \mid B \in \mathcal{B}\}$ and $\mathcal{F}$ be the family of countable unions of finite intersections of members of $\mathcal{F}_0$, where p and q are restrictions of the projection maps $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$, respectively to $X \times_Z Y$. Then, $\mathcal{F}$ is a separated cozero field on $X \times_Z Y$.

PROOF : It is easily deduced from 2.5 and 3.5 of Gordon[9].

The separated cozero field $\mathcal{F}$ on $X \times_Z Y$ shall be denoted by $\mathcal{A} \otimes_Z \mathcal{B}$.

Proposition 3.4 — Let $\langle X, \mathcal{A} \rangle$, $\langle Y, \mathcal{B} \rangle$ and $\langle Z, \mathcal{G} \rangle$ be A-spaces and let $f : \langle X, \mathcal{A} \rangle \to \langle Z, \mathcal{G} \rangle$ and $g : \langle Y, \mathcal{B} \rangle \to \langle Z, \mathcal{G} \rangle$ be A-morphisms. Then $\langle X \times_Z Y, \mathcal{A} \otimes_Z \mathcal{B} \rangle$ together with the maps $p : X \times_Z Y \to X$ and $q : X \times_Z Y \to Y$ is the pullback of f and g in the category Alex, where p and q are the restrictions to $X \times_Z Y$ of the projection maps from $X \times Y$ to X and from $X \times Y$ to Y, respectively.

PROOF : It is simple to prove.

Considering the A-spaces, $\langle X, \text{coz } X \rangle$ and $\langle Y, \text{coz } Y \rangle$, where X and Y are Tychonoff spaces in Proposition 3.3, we see that coz $X \otimes_Z$ coz Y consists of countable unions of cozero rectangles in $X \times_Z Y$. Thus, by Theorem 2.5, we have the following result.

Theorem 3.5 — For Tychonoff spaces, X, Y and Z, $X \times_Z Y$ in z-embedded in $\beta X \times_{\beta Z} \beta Y$ if and only if coz $(X \times_Z Y)$ = coz $X \otimes_Z$ coz Y.

Remark 3.6 : In view of Proposition 3.4, the equality coz $(X \times_Z Y)$ = coz $X \otimes_Z$ coz Y means that the functor coz transforms the pullback $X \times_Z Y$ of the morphisms $f : X \to Z$ and $g : Y \to Z$ in the c ategory Tych to the pullback $\langle X \times_Z Y,$ coz $X \otimes_Z$ coz $Y \rangle$ of the morphisms $f : \langle X, \text{coz } X \rangle \to \langle Z, \text{coz } Z \rangle$ and $g : \langle Y, \text{coz } Y \rangle \to \langle Z, \text{coz } Z \rangle$ in the category Alex.

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