Indian J. pure appl. Math., 28(2) : 189-195, February 1997

PARTIAL SOLUTIONS TO THE OPEN PROBLEM OF UNITIFICATION OF A WEAKLY RICKART *-RING

N. K. THAKARE[1] AND B. N. WAPHARE[2]

[1]*Vice-Chancellor, North Maharashtra University, Jalgaon*
[2]*Department of Mathematics, University of Pune, Pune* 411 007

(*Received* 2 *August* 1995; *after revision* 4 *June* 1996; *accepted* 16 *August* 1996)

We have obtained two partial solutions to the open problem of unitification of a weakly Rickart *-ring posed by S. K. Berberian. One partial solution is as follows :

If the characteristic of a weakly Rickart *-ring is nonzero then it can be embedded in a Rickart *-ring with preservation of *RP*'s.

INTRODUCTION

A *-ring A is a ring equipped with an involution $x \to x^*$, that is, an additive anti-automorphism of period at most two. If A is also an algebra over a field with an involution $\lambda \to \lambda^*$, we assume further that $(\lambda x)^* = \lambda^* x^*$ and call A a *-algebra. A C^*-algebra is a Banach *-algebra whose norm satisfies the identity $\| x^* x \| = \| x \|^2$. An element e in a *-ring A is a projection if it is self-adjoint ($e = e^*$) and idempotent ($e^2 = e$). The set of all projections in A is denoted by $\tilde{A}$. This set of all projections in A forms a partially ordered set with natural partial order defined by $e \leq f$ iff $e = ef$.

A *-ring A is said to be a weakly Rickart *-ring if for $x \in A$ there exists a projection $e \in A$ such that (i) $xe = x$, (ii) $xy = 0 \iff ey = 0$.

It is clear that if such a projection exists then it is unique; it is denoted by $RP(x)$ and called the annihilating right projection of x. Similarly in a weakly Rickart *-ring we can define annihilating left projection $LP(x)$.

It is proved in Berberian[1] (p. 28) that a weakly Rickart *-ring is a Rickart *-ring if and only if it has identity. Naturally we feel that if we adjoin identity to a weakly Rickart *-ring we may get a Rickart *-ring. In this respect Berberian[1] (p. 33) raised the following open problem :

Open Problem — Can every weakly Rickart *-ring be embedded in a Rickart *-ring ? With preservation of *RP*'s ?

Berberian has given a partial solution as follows :

Theorem 1 — Let A be a weakly Rickart $*$-ring. If there exists an involutary integral domain K such that A is a $*$-algebra over K and it is torsion free K-module, then A can be embedded in a Rickart $*$-ring with preservation of RP's.

We give the following more general partial answer to the open problem.

Theorem 2 — A weakly Rickart $*$-ring A can be embedded in a Rickart $*$-ring provided there exists, a ring K such that

(i) K is an integer domain involution,

(ii) A is a $*$-algebra over K,

(iii) for any $\lambda \in K - \{0\}$ there exists a projection $e_\lambda \in A$ that is an upper bound for the left projections of the right annihilators of λ, that is

$$x \in A,\ \lambda x = 0 \ \Rightarrow\ LP(x) \leq e_\lambda.$$

PROOF : Define $*$-algebra A_1 over K as follows :

$A_1 = A \oplus K$ (the additive group direct sum), with the following operations :

$$(a, \lambda) + (b, \mu) = (a + b,\ \lambda + \mu),$$

$$(a, \lambda)\,(b, \mu) = (ab + \mu a + \lambda b,\ \lambda\mu),$$

$$\mu(a, \lambda) = (\mu a, \mu\lambda),$$

$$(a, \lambda)^* = (a^*, \lambda^*);\ a,\ b \in A \text{ and } \lambda, \mu \in K.$$

Evidently A_1 is a *-ring with unity element $(0, 1)$; moreover, A is a *-ideal in A_1. We write $(a + \lambda 1)$ in place of (a, λ), $a \in A$, $\lambda \in K$. We claim that for any self-adjoint element $a \in A$ and a nonzero scalar $\lambda \in K$ there exists a largest projection g in A such that $ag = \lambda g$. Indeed, let e_λ be the projection as given by (iii). Let $RP(a) = e'$. Define $e = e' \vee e_\lambda$. Clearly $e' \leq e$ and $e' = e'e = ee'$. We know that $ae' = a$, which gives $ae'e = ae$ and therefore $a = ae' = ae'e = ae$. Also $a^* = a$ implies that $a = ea = eae \in eAe$. Thus $a - \lambda e \in eAe$. Let $h = RP(a - \lambda e)$. It is easy to see that eAe is a Rickart $*$-ring with preservation of RP's. Therefore $h \in eAe$. Define $g = e - h$ which is a projection in eAe, and $(a - \lambda e)g = (a - \lambda e)e - (a - \lambda e)h = 0$. Therefore $ag = \lambda g$.

Let k be any projection in A such that $ak = \lambda k$. Observe that $\lambda(ek - k) = e\lambda k - \lambda k = eak - \lambda k = ak - \lambda k = 0$. Let $LP(ek - k) = f$. It follows by (iii) that $f \leq e_\lambda \leq e$. Therefore $ek - k = f(ek - k) = fe(ek - k) = f(ek - ek) = 0$.

Now consider $(a - \lambda e)k = ak - \lambda ek = ak - \lambda k = 0$. This implies that $RP(a - \lambda e)k = hk = 0$. Hence $kg = k(e - h) = ke - kh = k - 0 = k$. That is $k \leq g$. Thus we have shown that g is the largest projection such that $ag = \lambda g$.

It readily follows that involution of A_1 is proper, hence it is sufficient to prove that every self-adjoint element of A_1 has annihilating right projection. Let $x = a + \lambda 1$ be a self-adjoint element of A_1; that means a and λ are self-adjoint elements in A

and K respectively. Obviously if $\lambda = 0$ then nothing to prove. If $\lambda \neq 0$ then by our claim there exists a largest projection g such that $ag = -\lambda g$; that is $xg = 0$. Set $e^1 = 1 - g$; we show that e^1 is the annihilating right projection of x in A_1. Of course $xe^1 = x$.

Assuming $xy = 0$, for some $y = b + \mu 1 \in A_1$ we have $\mu = 0$ and $ab + \lambda b = 0$. If we put $f^1 = LP(b)$ then we have $(a + \lambda f^1)b = 0$. This shows that $(a + \lambda f^1)f^1 = 0$. That is $af^1 = (-\lambda)f^1$. Since g is the largest projection such that $ag = (-\lambda)g$, we must have $f^1 \leq g$, i.e. $f^1 e^1 = 0$. Then $e^1 y = e^1 b = (e^1 f^1)b = 0$. Q.E.D.

We hasten to add a few relevant observations.

Let A be a weakly Rickart $*$-ring, K a ring such that

(i) K is a commutative ring with involution (possibly the identity involution), and

(ii) A is a $*$-algebra over K.

For a subset S of A, write $\tilde{S}$ for the set of projections in S, that is,

$$\tilde{S} = \{e \in S : e^* = e = e^2\}.$$

Thus $\tilde{S} = S \cap \tilde{A}$. If $\lambda \in K$ and $a \in A$, we write

$$\{\lambda\}^r = \{x \in A : \lambda x = 0\},$$

$$\{a\}^r = \{x \in A : ax = 0\}$$

("r" for right-annihilator). Then

$$(\{\lambda\}^r)^{\sim} = \{\lambda\}^r \cap \tilde{A} = \{e \in \tilde{A} : \lambda e = 0\}$$

is the set of projections in A that right-annihilates λ and similarly

$$(\{a\}^r)^{\sim} = \{a\}^r \cap \tilde{A} = \{e \in \tilde{A} : ae = 0\}.$$

With these notations, we give some equivalent formulations of the condition (iii) of Theorem 2.

Proposition 3 — Let A be a weakly Rickart $*$-ring and let K be a ring satisfying conditions (i) and (ii). The following conditions on the pair A, K are equivalent :

(a) For every $\lambda \in K - \{0\}$ there exists a projection $e_\lambda \in A$ such that

$$(\{\lambda\}^r)^{\sim} \cap (\{e_\lambda\}^r)^{\sim} = \{0\};$$

(b) for every $\lambda \in K - \{0\}$ there exists a projection $e_\lambda \in A$ such that

$$\{\lambda\}^r \cap \{e_\lambda\}^r = \{0\};$$

(c) (the condition (iii) of Theorem 2) for every $\lambda \in K - \{0\}$ there exists a projection $e_\lambda \in A$ that is an upper bound for the left-projections of the right-annihilators of λ, that is,

$$x \in A,\ \lambda x = 0 \ \Rightarrow\ LP(x) \leq e_\lambda.$$

PROOF : (a) $\Rightarrow$ (b) : $\lambda \in K - \{0\}$ and choose e_λ as in (a); let us show that e_λ satisfies the condition (b). Assuming

$$x \in \{\lambda\}^r \cap \{e_\lambda\}^r,$$

that $\lambda x = 0$ and $e_\lambda x = 0$,

we are to show that $x = 0$. Writing $e = LP(x)$, it is equivalent to show that $e = 0$. From $e_\lambda x = 0$ we infer that $e_\lambda e = 0$, thus $e \in (\{e_\lambda\}^r)^\sim$. Also,

$$0 = \lambda x = \lambda(ex) = (\lambda e)x,$$

where $\lambda e \in A$; therefore $(\lambda e)e = 0$; whence $\lambda e = 0$, $e \in (\{e_\lambda\}^r)^\sim$.

Thus $e \in (\{\lambda\}^r)^\sim \cap (\{\lambda\}^r)^\sim = \{0\}$ (by the hypothesis (a)).

(b) $\Rightarrow$ (c) : Let $\lambda \in K - \{0\}$ and choose e_λ as in (b). Assuming $x \in A$, $\lambda x = 0$, we are to show that $LP(x) \leq e_\lambda$. Write $e = LP(x)$. From

$$e_\lambda (x - e_\lambda x) = e_\lambda x - e_\lambda x = 0$$

we have $x - e_\lambda x \in \{e_\lambda\}^r$. Also

$$\lambda(x - e_\lambda x) = \lambda x - e_\lambda \lambda x = 0 - e_\lambda 0 = 0,$$

so $x - e_\lambda x \in \{\lambda\}^r$. Thus $x - e_\lambda x \in \{\lambda\}^r \cap \{e_\lambda\}^r = \{0\}$

(by the hypothesis (b)). So $x - e_\lambda x = 0$. From

$$0 = x - e_\lambda x = ex - e_\lambda x = (e - e_\lambda)x$$

we infer that $(e - e_\lambda)e = 0$, whence $e = e_\lambda e$, that is, $e \leq e_\lambda$.

(c) $\Rightarrow$ (a) : Let $\lambda \in K - \{0\}$ and choose e_λ as in (c). Assuming $e \in (\{\lambda\}^r)^\sim \cap (\{e_\lambda\}^r)^\sim$, we are to show that $e = 0$. Indeed, $\lambda e = 0$ implies $e = LP(e) \leq e_\lambda$ by the assumption in (c); but $e_\lambda e = 0$ by assumption, so $e = e_\lambda e = 0$. Q.E.D.

Corollary 4 — A is torsion-free $\Leftrightarrow$ A satisfies (iii) and the projection 0 (hence any projection of A) can play the role of e_λ ($\lambda \in K - \{0\}$). Consequently Berberian's theorem is a special case of Theorem 2.

PROOF : $\Leftarrow$: If $\lambda \in K - \{0\}$ and (c) is satisfied with $e_\lambda = 0$, then

$$\lambda x = 0 \Rightarrow LP(x) \leq e_\lambda = 0 \ \Rightarrow LP(x) = 0 \ \Rightarrow x = 0;$$

thus $\{\lambda\}^r = \{0\}$ for every $\lambda \in K - \{0\}$. In other words A is torsion-free.

$\Rightarrow$: Let $\lambda \in K - \{0\}$. Since A is torsion-free,

$$\lambda x = 0 \Rightarrow x = 0 \ \Rightarrow LP(x) = 0,$$

thus (c) is satisfied by $e_\lambda = 0$. Q.E.D.

Corollary 5 — A has a unity element $u \Leftrightarrow$ the condition in (c) holds for $\lambda = 0$.

PROOF : $\Leftarrow$: Choose a projection $e_0 \in A$ satisfying the implication in (c) with $\lambda = 0$. Since $Ox = 0$ for all $x \in A$, we have $LP(x) \leq e_0$ for all $x \in A$, whence it is clear that e_0 is a unity element for A.

$\Rightarrow$: If A has a unity element u then, for every $\lambda \in K$ (including $\lambda = 0$), the implication

$$\lambda x = 0 \Rightarrow LP(x) \leq u$$

is trivially true, thus (c) holds for all λ (including $\lambda = 0$). Q.E.D.

Corollary 5 explains why attention is restricted to $\lambda \in K - \{0\}$ (A, a weakly Rickart $*$-ring that is not a Rickart $*$-ring).

We give a counterexample wherein we show that Theorem 2 is an improvement over Theorem 1.

Example 6 — Consider $M_2(\mathbb{Z}_3)$ which is a Rickart $*$-ring and the unitless commutative C^*-algebra $C_\infty(T)$, where T is a noncompact, locally compact, Hausdorff space and $C_\infty(T)$ denotes the algebra of all continuous, complex valued functions x on T that 'vanish at infinity' in the sense that

$$\{t \mid |x(t)| \geq \varepsilon\} \text{ is compact for every } \varepsilon > 0.$$

It is known that, $C_\infty(T)$ is a weakly Rickart C^*-algebra iff T satisfies following two conditions :

(i) The compact open sets in T are basic for the topology, and

(ii) if P_n is any sequence of compact-open sets, and $U = \bigcup_1^\infty P_n$ then $\overline{U}$ is compact-open; see Berberian[1].

Define $A = C_\infty(T) \oplus M_2(\mathbb{Z}_3)$ (external direct sum of the $*$-rings). This newly constructed ring is a weakly Rickart $*$-ring as $C_\infty(T)$ and $M_2(\mathbb{Z}_3)$ are weakly Rickart $*$-rings.

One then immediately notes that A has no unity element, on account of the absence of the same in $C_\infty(T)$. It is easy to see that there does not exist an integral domain K such that A is torsion free K-module. Hence it does not have unitification in the sense of Berberian. However, $C_\infty(T)$ is a torsion-free $\mathbb{Z}$-module, we have for any nonzero λ, $(\{\lambda\}^r)^\sim$ is a subset of $M_2(\mathbb{Z}_3)$. But then $(0, 1) = e_\lambda$ is the required projection which fulfils the condition (iii) of Theorem 2. Therefore $A_1 = A \oplus \mathbb{Z}$ is a Rickart $*$-ring containing A that preserves RP's and LP's.

We provide the second partial solution of the open problem of unitification of weakly Rickart $*$-ring in the form of following theorem.

Theorem 7 — A weakly Rickart $*$-ring A can be embedded in a Rickart $*$-ring provided the characteristic of A is non-zero.

PROOF : Let A be a weakly Rickart $*$-ring with non-zero characteristic n. Since involution of A is proper therefore n must be a square-free integer. Hence A can be decomposed as $A = A_1 \oplus A_2 \oplus \ldots \oplus A_k$, where each A_i is a $*$-algebra over a finite

field, using Theorem 1 we get that each weakly Rickart $*$-ring A_i can be embedded in a Rickart $*$-ring with preservation of RP's. Therefore the direct product A can be embedded in a Rickart $*$-ring with preservation of RP's. Q.E.D.

Recall that Theorem 2 in particular says that if for every nonzero integer n, $(\{n\}^r)^\sim$ is bounded then the problem can be solved positively. The following corollary tells us that Theorem 7 is nothing but a part of the other side of Theorem 2.

Corollary 8 — Let A be a weakly Rickart $*$-ring. If there exists a nonzero integer n such that the complement of the set $(\{n\}^r)^\sim$ in the set $\tilde{A}$ (i.e. $\tilde{A} - (\{n\}^r)^\sim$) is bounded then A can be embedded in a Rickart $*$-ring.

PROOF : Let t be a projection in A such that for every e in $A - (\{n\}^r)^\sim$ we have $e \leq t$. We divide the proof in two parts.

Case 1 : If t is an element of $(\{n\}^r)^\sim$ then $nt = 0$. This shows that for any element e in $\tilde{A} - (\{n\}^r)^\sim$, $ne = n(te) = (nt)\ e = 0$. That means e is an element of $(\{n\}^r)^\sim$. Thus the set $\tilde{A} - (\{n\}^r)^\sim$ is empty; i.e. $\tilde{A} = (\{n\}^r)^\sim$. Now for any element a in A we have $na = n(ka)$, where $k = LP(a) \Rightarrow (nk)a = 0$. Hence characteristic of A is nonzero. Applying Theorem 5 we have the required embedding.

Case 2 : Let t be an element of $\tilde{A} - (\{n\}^r)^\sim$. We know that every element of $\tilde{A} - (\{n\}^r)^\sim$ is smaller than t.

Let e be an element of $(\{n\}^r)^\sim$. We claim that for k in $\tilde{A} - (\{n\}^r)^\sim$, $e \vee k$ is an element of $\tilde{A} - (\{n\}^r)^\sim$. Suppose $e \vee k$ is in $\lambda(\{n\}^r)^\sim$. That means $n(e \vee k) = 0$. Multiplying by k we have $n(e \vee k)k = 0$. This shows that $nk = 0$, which is a contradiction, as k is not an element of $(\{n\}^r)^\sim$. Thus $e \vee k$ is an element of $\tilde{A} - (\{n\}^r)^\sim$. Since t is an upper bound for $\tilde{A} - (\{n\}^r)^\sim$, we have $e \leq e \vee k \leq t$, for every e in $(\{n\}^r)^\sim$.

Combining these statements we can say that t is the supremum of $\tilde{A}$. Now it can be easily verified that t is the identity element of A. Hence there is no need of embedding. Q.E.D.

Let us also note one more interesting immediate consequence.

Corollary 9 — Let A be a weakly Rickart $*$-ring, whose set of projections forms a chain. Then A can be embedded in a Rickart $*$-ring.

PROOF : *Case* 1 : If for every nonzero integer n, $(\{n\}^r)^\sim$ is bounded subset of A, then by Theorem 2 we have the result.

Case 2 : If for some nonzero integer n, $(\{n\}^r)^\sim$ is unbounded then we prove that the characteristic of A is nonzero. Let e be any projection in A. Since $(\{n\}^r)^\sim$ is unbounded there exists a projection t in $(\{n\}^r)^\sim$ such that t is not less than e. But the set of projections forms a chain, therefore $e \leq t$ (i.e. $e = et = te$). By the definition of $(\{n\}^r)^\sim$ we have $nt = 0$. This shows that $nte = ne = 0$. That means e is in $(\{n\}^r)^\sim$. This is true for any projection e in A. Therefore $(\{n\}^r)^\sim = \tilde{A}$.

Let a be any element of A with annihilating left projection e. Then $na = n(ea) = (ne)a = 0$. $a = 0$. Hence the characteristic of the ring A is nonzero. As a consequence of Theorem 7, we have the desired embedding. Q.E.D.

We exhibit an interesting example of a weakly Rickart $*$-ring without identity which has no unitification in the sense of Theorem 2.

Example 10 — Let B be a Boolean ring of compact open subsets of a noncompact, Hausdorff topological space T. It is clear that B is a weakly Rickart $*$-ring without unity. Also consider Rickart $*$-ring $\mathbb{Z}_6$ with identity involution.

Define $*$-ring A by the external direct product of B and $\mathbb{Z}_6$ i.e. $A = B \oplus \mathbb{Z}_6$ with componentwise addition and multiplication. Clearly A is a weakly Rickart $*$-ring without identity. Observe that the characteristic of A is 6 which is nonzero, hence in the sense of Theorem 7, A has embedding. But $(\{6\}')^{\sim} = \tilde{A}$ and A has no identity therefore it can be easily shown that A does not satisfy the requirement (iii) of Theorem 2. Hence A has no embedding in the sense of Theorem 2.

One should note that the $*$-ring $A = C_\infty(T) \oplus M_2(\mathbb{Z}_3)$, which we have discussed in Example 6 is an example of a weakly Rickart $*$-ring which has embedding in the sense of Theorem 2 but has no embedding in the sense of Theorem 7 by the simple reason that the characteristic of A is zero.

Now we provide an example of a weakly Rickart $*$-ring which does not satisfy the conditions of Theorem 2 as well as Theorem 7.

Example 11 — We know that there are infinitely many prime numbers. Let us denote these prime numbers by $\{p_1, p_2, p_3, \ldots\}$. Consider the $*$-ring $A' = \mathbb{Z}_{p_1} \oplus \mathbb{Z}_{p_1 p_2} \oplus \ldots \oplus \mathbb{Z}_{p_1 \ldots p_r} \oplus \ldots$. Define

$$A = \{(x_1, x_2, \ldots,) \in A' / x_i = 0, \quad \forall\ i \text{ except finitely many}\}.$$

A is the required $*$-ring. The $*$-ring A is weakly Rickart $*$-ring which can be embedded in a Rickart $*$-ring, namely A_1 with preservation of RP's.

ACKNOWLEDGEMENT

The authors sincerely thank the referee for his valuable suggestions in this paper and thoughts for a future paper.

REFERENCE

1. S. K. Berberian, *Baer *-rings*, Springer-Verlag, Berlin, 1972.