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CONVERGENCE OF A CLASS OF QUARTIC INTERPOLATORY SPLINES

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Quartic interpolatory spline whose area in each subinterval matches with that of a given function has been studied. Unique existence of the interpolatory splines has been established and convergence properties have been investigated.

1. INTRODUCTION

Cubic and quadratic splines have been studied extensively by Ahlberg *et al.*[1], Marsden[3], Meir and Sharma[4,5], Sharma and Tzimbalario[6] and others. Their interesting interpolatory properties and approximation powers have been investigated. Considering a non negative measure $d\mu$, Sharma and Tzimbalario[6] have introduced ingenious interpolatory spline whose integral with respect to measure $d\mu$ in each sub interval matches with that of a given function. In the case when $\mu x = x$ this mean averaging condition reduces to that of area matching condition. In the present paper we study quartic interpolatory spline whose area in each sub interval matches with that of a given function. We establish the unique existence of the quartic spline satisfying area matching condition. Further, Meir and Sharma[4], in their landmark paper have given a technique to obtain sharp error estimates. Following Meir and Sharma[4] we establish error estimates for the quartic interpolatory spline and show that the class of quartic interpolatory splines converges at a fast rate to the function to be interpolated.

2. EXISTENCE AND UNIQUENESS

Let $P = \{0 = x_0 < x_1 < \ldots < x_n = 1\}$ be a sequence of points in the interval [0, 1] such that $x_i - x_{i-1}$ = constant = h (say) for each i. Let $S(5, P)$ denote the space of quartic splines which is the space of piecewise polynomials of order 5 belonging to the class $C^3[0, 1]$. Let f be a 1-periodic locally integrable function defined on [0, 1]. We shall prove the following :

Theorem 2.1 — For a 1-periodic locally integrable function f defined on [0, 1] there exists a unique 1-periodic quartic spline $s \in S(5, p)$ satisfying the condition

$$\int_{x_{i-1}}^{x_i} s\,dx = \int_{x_{i-1}}^{x_i} f\,dx;\ i = 1, 2, \ldots, n. \qquad \ldots (2.1)$$

We notice that condition of periodicity of s leads to the assertion $s^{(j)}(0) = s^{(j)}(1)$; $j = 0, 1, 2, 3$.

PROOF OF THE THEOREM : We set for convenience

$$\int_{x_{i-1}}^{x_i} f\,dx = F_i,\ i = 1, 2, \ldots, n.$$

Let $s \in S(5, P)$ be represented in each sub interval $[x_{i-1}, x_i]$ as follows :

$$s(x) = -\frac{1}{24h}(x_i - x)^4 M_{i-1} + \frac{1}{24h}(x - x_{i-1})^4 M_i$$
$$+ \frac{1}{4} c_i [(x_i - x)^2 + (x - x_{i-1})^2] - \frac{1}{2} d_i [(x_i - x) - (x - x_{i-1})] + e_i \qquad \ldots (2.2)$$

where $M_i = s'''(x_i)$. We observe that $s'''(x_{i-1}) = M_{i-1}$, therefore continuity of $s'''(x)$ is trivial.

It is easy to see that the continuity requirements of $s''(x)$ and s' give that

$$c_{i+1} - c_i = M_i h, \qquad \ldots (2.3)$$

and

$$d_{i+1} - d_i = \frac{h}{2}(c_{i+1} + c_i) \qquad \ldots (2.4)$$

respectively.

Further, continuity of s leads to the following relation :

$$e_{i+1} - e_i = \frac{1}{12} h^3 M_i - \frac{h^2}{4}(c_{i+1} - c_i) + \frac{h}{2}(d_{i+1} + d_i). \qquad \ldots (2.5)$$

Since s satisfies the interpolatory condition (2.1) we get

$$F_i = \frac{h^4}{120}(M_i - M_{i-1}) + \frac{h^3}{6} c_i + e_i h. \qquad \ldots (2.6)$$

Eliminating e_i and d_i on using (2.4) – (2.6) we obtain following relation between unknowns M_i's :

$$\frac{h^4}{120}[M_{i+1} + 26M_i + 66M_{i-1} + 26M_{i-2} + M_{i-3}]$$
$$= F_{i+1} - 3F_i + 3F_{i-1} - F_{i-2} \qquad \ldots (2.7)$$

$i = 1, 2, \ldots, n$, where $F_{n+r} = F_r$ and $M_{n+r} = M_r$ by assumption of periodicity of f and s.

It is evident that (2.7) represents a system of n equations in n unknowns M_i's. Moreover excess of coefficient of M_{i-1} over the sum of coefficients of M_{i+1}, M_i, M_{i-2} and M_{i-3} is positive. Therefore the coefficient matrix of the system of equations (2.7) is diagonally dominant. Thus coefficient matrix of system of eqns. (2.7) is invertible and hence the system of equations admits a unique solution. This establishes the unique existence of the quartic spline satisfying interpolatory condition (2.1).We thus complete the proof of Theorem 21.

3. Convergence

In this section we aim to obtain error estimates for the 1-periodic quartic interpolatory spline of Theorem 2.1. We denote by e the error function $s - f$, so that $e^{(j)} = s^{(j)} - f^{(j)}$; $j = 0, 1, 2, \ldots$. We suppose the function f to be in class C^3 [0, 1]. For convenience we represent by g_j the value of a function g at point x_j. Thus substituting $e_i''' + f_i'''$ for M_i in (2.7) we get

$$e_{i+1}''' + 26e_i''' + 66e_{i-1}''' + 26e_{i-2}''' + e_{i-3}'''$$

$$= \frac{120}{h^4}(F_{i+1} - 3F_i + 3F_{i-1} - F_{i-2}) - (f_{i+1}''' - f_i''')$$

$$- 27(f_i''' - f_{i-1}''') + 27(f_{i-1}''' - f_{i-2}''')$$

$$+ (f_{i-2}''' - f_{i-3}''') - 120 f_{i-1}'''.$$

Using Taylor's formula we get

$$F_{i+1} - F_i = h^2 f_i' + \frac{h^2}{24}[f''(\xi_i) - f''(\eta_i)]$$

where $\xi_i \in (x_i, x_{i+1})$ and $\eta_i \in (x_{i-1}, x_i)$.

Thus

$$F_{i+1} - 3F_i + 3F_{i-1} - F_{i-2} = h^4 [x_{i-2}, x_{i-1}, x_i] f' + \frac{h^4}{24}[f'''(\xi_i) - f'''(\eta_i)$$

$$-2(f'''(\alpha_i) - f'''(\beta_i)) + f'''(\mu_i) - f'''(\nu_i)]$$

where $\alpha_i \in (x_{i-1}, x_i)$, $\beta_i, \mu_i \in (x_{i-2}, x_{i-1})$, $\nu_i \in (x_{i-3}, x_{i-2})$, and $[x_{i-2}, x_{i-1}, x_i]f'$ represents the second divided difference of f' at points x_{i-2}, x_{i-1} and x_i .

If we suppose that $|e_{i-1}'''| = \max |e_j'''|, j = 1, 2, \ldots, n$ then

$$66|e_{i-1}'''| \le |e_{i+1}'''| + 26|e_i'''| + 26|e_{i-2}'''| + |e_{i-3}'''| + 56\,\omega(f''', h)$$

$$+ 120\ \{|f'''(\theta) - f_{i-1}'''| + \frac{1}{6}\omega(f''', h)\}$$

or $$12\,|\,e'''_{i-1}\,|\leq 196\,\omega\,(f''', h),$$

where $\theta \in (x_{i-2}, x_i)$.

Hence

$$\|\, e_i''' \,\| = \frac{49}{3}\,\omega\,(f''', h).$$

We observe by (2.2) that $s'''(x)$ is linear in $[x_{i-1}, x_i]$ so that

$$s'''(x) = \{M_{i-1}(x_i - x) + M_i(x - x_{i-1})\}/h.$$

Therefore $$\|\, e''' \,\| = \frac{52}{3}\,\omega\,(f''', h).$$

We have thus proved the following :

Theorem 3.1 — If f is a locally integrable 1-periodic function in class C^3 [0, 1] and s be the unique interpolatory spline of Theorem 2.1 then

$$\|\, e_i''' \,\| = \frac{49}{3}\,\omega\,(f''', h),$$

$$\|\, e''' \,\| = \frac{52}{3}\,\omega\,(f''', h);$$

where $e = s - f$ represents the error function.

4. Discussion

Theorem 3.1 provides the error estimates for the quartic interpolatory splines of Theorem 2.1. Theorem 3.1 establishes that quartic interpolatory spline of Theorem 2.1 is a good approximant to a 1-periodic function $f \in C^3$ [0, 1]. Rate of convergence of s to f is good and error approaches zero at a fast rate as $h \to 0$.

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