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SOME THEOREMS IN THE GENERALIZED THEORY OF THERMOELASTICITY FOR PRESTRESSED BODIES

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A generalized theory of thermoelasticity for prestressed bodies is deduced from the results of a previous paper (Dhaliwal and Wang[6]). A uniqueness theorem and a reciprocity relation have been established for the theory considered.

1. INTRODUCTION

During the last three decades, many papers[1-9] have been devoted to the generalized theory of thermoelasticity, which avoids the paradox of propagation of heat with an infinite speed. This paradox is produced by the diffusion-type equation governing the temperature field in the classical theories. Lord and Shulman[1] employing a modified Fourier's law, developed a generalized theory of thermoelasticity whose governing system of equations is entirely hyperbolic and hence predicts a finite speed for heat propagation. Green and Lindsay[2], employing an entropy production inequality proposed by Green and Laws[3], developed a temperature-rate dependent thermoelasticity that includes the temperature-rate among constitutive variables and also predicts a finite speed for heat propagation. The detailed reference regarding the development of this subject can be found in a nice review paper by Chandrasekharaiah[8].

In a previous paper Dhaliwal and Wang[6] introduced a generalized theory of thermoelasticity for prestressed bodies with microstructures. The purpose of the present paper is to deduce a generalized theory for prestressed thermoelastic bodies from the results obtained in Dhaliwal and Wang[6] and prove some general theorems such as uniqueness and reciprocity theorem.

In section 2, we state some results obtained in Dhaliwal and Wang[6] and, by letting the coupling constants equal to zero, deduce a generalized theory for prestressed thermoelastic bodies without microstructure. In section 3, we define a class of initial boundary value problems and prove a uniqueness theorem. In section 4, the reciprocity relation is established.

2. The Field Equations

In this section, we summarize the main constitutive and balance equations derived in Dhaliwal and Wang[6] and deduce the field equations for general prestressed heat conducting elastic solids. Consider a thermoelastic body, whose configuration changes continuously under external mechanical actions and heating, from a reference state Ω_0 to a deformed state Ω. Let $\partial\Omega_0$ and $\partial\Omega$ be the surfaces of Ω_0 and Ω, respectively. Let x_i denote the position of the particle X in the reference state, and y_i the position in the deformed state, where $y_i = y_i(x_1, x_2, x_3, t)$ and t is the time. We assume that

$$J \equiv \det(\partial y_i/\partial x_j) > 0.$$

The field equations derived in section 4 of Dhaliwal and Wang[6] are the following :

$$\sigma_{ji,j} + \rho_0 F_i = \rho_0 \ddot{u}_i \qquad \ldots (1)$$

$$M_{ijk,k} - N_{ji} + \rho_0 F_{ji} = \rho_0 Y_{ki} \ddot{u}_{jk} \qquad \ldots (2)$$

$$\rho_0 \theta_0 \dot{\eta} = -q_{i,i} + \rho_0 S \qquad \ldots (3)$$

and the constitutive relations are

$$\begin{aligned}\sigma_{ij} = {} & u_{j,k} P_{ki} + u_{jk} S_{ik} + u_{jk,m} T_{mik} - \beta_{ij}\theta + D_{ijmn} v_{mn} + D_{ijmnk} v_{mnk} + C_{ijmn} e_{mn} \\ & + x_{jk}(-B_{k*i}\theta + D_{kimn} e_{mn} + G_{kimnl} v_{mnl} + C'_{kimn} v_{mn}) \\ & + x_{jk,m}(-B_{i*km}\theta + D_{nlikm} e_{nl} + G_{nlikm} v_{nl} + C_{ikmrst} v_{rst}) \qquad \ldots (4)\end{aligned}$$

$$N_{ij} = -B_{j*i}\theta + D_{jimn} e_{mn} + G_{jimnl} v_{mnl} + G'_{jimn} v_{mn} \qquad \ldots (5)$$

$$M_{ijk} = u_{j,m} T_{imk} - B_{j*ki}\theta + D_{mnjki} e_{mn} + G_{mnjki} v_{mn} + C_{jkilmn} v_{lmn} \qquad \ldots (6)$$

$$\rho_0 \eta = d\theta + \beta_{ij} e_{ij} + B_{i*j} v_{ij} + B_{i*jk} v_{ijk} \qquad \ldots (7)$$

$$q_i + \tau \dot{q}_i = -a_i \theta - k_{ij}\theta_{,j} - h_{ijk} e_{jk} - h'_{ijk} v_{jk} - h_{ijkl} v_{jkl} \qquad \ldots (8)$$

where

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \qquad \ldots (9)$$

$$v_{ij} = u_{ij} + x_{ki} u_{k,j} \qquad \ldots (10)$$

$$v_{ijk} = x_{mi,k} u_{m,j} + u_{ji,k}. \qquad \ldots (11)$$

If we consider a thermoelastic body without microstructure, the initial generalized stress tensors $S_{ik} = 0$ and $T_{mik} = 0$. We let the coupling constants between dipolar

displacement and conventional displacement and dipolar displacement and temperature field approach zero, i.e.

$$B_{k*i} \to 0, B_{i*km} \to 0, C'_{ijkl} \to, C_{ijklmn} \to 0, D_{ijkl} \to 0, D_{ijklm} \to 0,$$

$$G_{ijklm} \to 0, h'_{ijk} \to 0, h_{ijkl} \to 0,$$

then the eqns. (1) and (3) are separated from eqn. (2) and give rise to the following generalized equations of thermoelasticity for general heat-conducting elastic bodies

$$\rho_0 \ddot{u}_i = \sigma_{ji,j} + \rho_0 F_i \qquad \ldots (12)$$

$$\rho_0 \theta_0 \dot{\eta} = -q_{i,i} + \rho_0 S \qquad \ldots (13)$$

with the constitutive relations

$$\sigma_{ij} = C_{ijmn} e_{mn} + e_{jk} P_{ki} - \beta_{ij} \theta \qquad \ldots (14)$$

$$\rho_0 \eta = d\theta + \beta_{ij} e_{ij} \qquad \ldots (15)$$

$$q_i + \tau q_i = -a_i \theta - k_{ij} \theta_{,j} - h_{ijk} e_{jk} \qquad \ldots (16)$$

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \qquad \ldots (17)$$

The coefficients C_{ijmn}, β_{ij}, k_{ij}, h_{ijk}, a_i and d are functions of the position coordinates. C_{ijmn} and β_{ij} have the usual symmetry about their indices. The presence of initial heat flux vector is determined by the nonuniformity of the temperature θ_0. This fact now implies the appearance of the coefficients h_{ijk} and a_i in the expression for q_i. If θ_0 is constant, then $h_{ijk} = 0$ and $a_i = 0$.

Substituting from eqns. (14)-(16) into eqns. (12) and (13), we get the system of governing equations for the linear generalized thermoelasticity for prestressed bodies as

$$(d_{ijmn} e_{mn} - \beta_{ij} \theta)_{,j} + \rho_0 F_i = \rho_0 \ddot{u}_i \qquad \ldots (18)$$

$$\theta_0 \left(1 + \tau \frac{\partial}{\partial t}\right)(\beta_{ij} \dot{e}_{ij} + d\dot{\theta}) = (k_{ij} \theta_{,j} + h_{ijk} e_{jk} + a_i \theta)_{,j} + \rho_0 \left(1 + \tau \frac{\partial}{\partial t}\right) S \qquad \ldots (19)$$

where

$$d_{ijmn} = C_{ijmn} + \delta_{jn} P_{mi}$$

δ_{jn} is the usual Kronecker delta.

It is noticeable that eqns. (18) and (19) are both hyperbolic type equations if the coefficients have proper signs. As a consequence, heat will propagate at a finite speed.

If $P_{ij} = 0$ and θ_0 is a constant, eqns. (18) and (19) reduce to

$$(C_{ijmn}\, e_{mn} - \beta_{ij}\, \theta)_{,j} + \rho_0 F_i = \rho_0 \ddot{u}_i \qquad \ldots (20)$$

$$\theta_0 \left(1 + \tau \frac{\partial}{\partial t} \right) (\beta_{ij}\, \dot{e}_{ij} + d\dot{\theta}) = (k_{ij}\, \theta_{,j})_{,i} + \rho_0 \left(1 + \tau \frac{\partial}{\partial t} \right) S \qquad \ldots (21)$$

which are the same set of equations as obtained by Sherief and Dhaliwal[4] and agree with those derived by Lebon[5].

3. A Uniqueness Theorem

In this section we first define an initial boundary value problem associated with the thermoelastic eqns. (18) and (19) for prestressed bodies and then prove a uniqueness theorem.

In the context of the theory considered, an initial boundary value problem consists in determining the functions $u_i(x, t)$ and $\theta(x, t)$ for $x \in \Omega$ and $t > 0$, by solving eqns. (18) and (19) with $F_i(x, t)$ and $S(x, t)$ as prescribed functions for $x \in \Omega$ and $t > 0$, P_{mi} and θ_0 as prescribed functions for $x \in \Omega$, under the initial conditions

$$u_i(x, 0) = 0,\ \dot{u}_i(x, 0) = 0,\ \theta(x, 0) = 0,\ q_i(x, 0) = 0,\ x \in \Omega \qquad \ldots (22)$$

and the boundary conditions

$$u_i = \bar{u}_i,\ x \in \partial\Omega_1,\ t > 0;\ \sigma_{ij}\, n_j = \hat{\sigma}_i,\ x \in \partial\Omega_1^c,\ t > 0 \qquad \ldots (23)$$

$$\theta = \bar{\theta},\ x \in \partial\Omega_2,\ t > 0;\ q_i\, n_i = \hat{q},\ x \in \partial\Omega_2^c,\ t > 0 \qquad \ldots (24)$$

where $\bar{u}_i, \bar{\theta}, \hat{q}$ and $\hat{\sigma}_i$ are prescribed functions in the domains of their definition, $\partial\Omega_i + \partial\Omega_i^c = \partial\Omega$ $(i = 1, 2)$ and n_i is the outward normal vector to $\partial\Omega$.

To establish the uniqueness theorem, we need the following assumption :

(1) mass density $\rho_0(x, t)$ in the reference configuration and the specific heat $d(x, t)$ are strictly positive, that is, there exist positive constants r and d_0 with the same dimension as of $\rho_0(x, t)$ and $d(x, t)$ respectively, such that

$$\rho_0(x, t) \geq r > 0, \text{ and } d(x, t) \geq d_0 > 0; \qquad \ldots (25)$$

(2) d_{ijkl} is positive definite, that is, there exists a positive constant α such that

$$d_{ijkl}\, e_{ij}\, e_{kl} \geq \alpha\, e_{ij}\, e_{ij} \qquad \ldots (26)$$

for all tensors e_{ij};

(3) the heat conductivity tensor k_{ij} is symmetric and positive definite in the sense that there exists a positive constant k such that

$$k_{ij}\, \xi_i\, \xi_j \geq k\, \xi_i\, \xi_i \qquad \ldots (27)$$

for all vectors ξ_i.

It is obvious that the assumption (3) implies that there exists a symmetric, positive definite second order tensor b_{ij} such that $b_{ij} = k_{ij}^{-1}$ and

$$b_{ij}\,\xi_i\,\xi_j \geq b\,\xi_i\,\xi_i \qquad \ldots (28)$$

for all vectors ξ_i, where $b > 0$ is a constant.

We now establish a lemma which leads to the uniqueness theorem

Lemma 1 — Suppose $(u_i\,\theta)\,(x, t)$ is a solution of the initial boundary value problem described above with $F_i\,(x, t) = 0$, $S\,(x, t) = 0$, $\bar{u}_i\,(x, t) = 0$, $\bar{\theta}\,(x, t) = 0$, $\hat{\sigma}_i = 0$ and $\hat{q} = 0$. Keeping the initial temperature $\theta_0 > 0$ positive constant, let

$$D\,(t) = \frac{1}{2}\int_\Omega \left[\rho_0\,\dot{u}_i\,\dot{u}_i + d_{ijkl}\,e_{ij}\,e_{kl} + d\theta^2 + \frac{\tau}{\theta_0}\,b_{ij}\,q_i\,q_j\right] d\Omega \qquad \ldots (29)$$

then, we have

$$\dot{D}\,(t) \leq 0 \qquad \ldots (30)$$

PROOF : Taking into account the symmetry of d_{ijkl} and b_{ij} from eqn. (29), we find that

$$\dot{D}\,(t) = \int_\Omega \left[\rho_0\,\ddot{u}_i\,\dot{u}_i + d_{ijkl}\,\dot{e}_{ij}\,e_{kl} + d\,\dot{\theta}\,\theta + \frac{\tau}{\theta_0}\,b_{ij}\,q_i\,\dot{q}_j\right] d\Omega. \qquad \ldots (31)$$

Using eqns. (12)-(16) and integration by parts, we find that

$$\dot{D}\,(t) = \int_\Omega \sigma_{ij,\,j}\,\dot{u}_i\,d\Omega + \int_\Omega d_{ijkl}\,\dot{e}_{ij}\,e_{kl}\,d\Omega + \int_\Omega \theta\,(\rho_0\,\dot{\eta} - \beta_{ij}\,\dot{e}_{ij})\,d\Omega$$

$$+ \int_\Omega \frac{\tau}{\theta_0}\,b_{ij}\,q_i\,\dot{q}_j\,d\Omega$$

$$= \int_\Omega (-\sigma_{ij} + d_{ijkl}\,e_{kl} - \beta_{ij}\,\theta)\,\dot{e}_{ij}\,d\Omega + \int_\Omega \left[\frac{\tau}{\theta_0}\,b_{ij}\,q_i\,\dot{q}_j - \frac{1}{\theta_0}\,\theta\,q_{i,\,i}\right] d\Omega$$

$$= \int_\Omega \left[\frac{\tau}{\theta_0}\,b_{ij}\,q_i\,\dot{q}_j + (\theta/\theta_0)_{,i}\,q_i\right] d\Omega$$

$$= \int_\Omega \left[\frac{\tau}{\theta_0}\,b_{ij}\,q_i\,\dot{q}_j + \frac{1}{\theta_0}\,\theta_{,i}\,q_i\right] d\Omega, \qquad \ldots (32)$$

where we have used the fact that $F_i = 0, S = 0$ and the homogeneous boundary conditions. From the assumption (3) and eqn. (16), we find, for constant initial temperature θ_0, that

$$\theta_{,i} = -\,b_{ij}\,(q_j + \tau\dot{q}_j). \qquad \ldots (33)$$

Substituting from eqn. (33) into eqn. (32), we get

$$\dot{D}(t) = -\int_{\Omega} \frac{1}{\theta_0} b_{ij} q_i q_j \, d\Omega. \qquad \ldots (34)$$

Taking into account eqn. (28) in eqn. (34), we find that $\dot{D}(t) \leq 0$, and complete the proof.

Theorem 1 (uniqueness) — If assumptions (1)-(3) hold and $\theta_0 > 0$, then there exists at most one solution to the initial boundary value problem defined by eqns. (18), (19) and (22)-(24).

PROOF : It suffices to show that for $F_i = 0, S = 0$ and homogeneous boundary conditions, the solution is trivial.

From Lemma 1, solving the differential inequality (30) we find that

$$D(t) \leq D(0). \qquad \ldots (35)$$

Taking into account the initial conditions, we find that D (0) = 0 and hence

$$D(t) \leq 0. \qquad \ldots (36)$$

Assumptions (1)-(3) and the inequality (36) imply that

$$\int_{\Omega} \left[r\dot{u}_i \dot{u}_i + \alpha e_{ij} e_{ij} + d_0 \theta^2 + \frac{\tau}{\theta_0} b q_i q_i \right] d\Omega = 0. \qquad \ldots (37)$$

Equation (37) readily yields the trivial solution :

$$u_i(x, t) = 0 \quad \text{and} \quad \theta(x, t) = 0, \quad \text{for} \quad x \in \Omega, t \geq 0.$$

4. A RECIPROCITY THEOREM

Throughout this section we assume that the initial temperature θ_0 is a constant. We shall prove a reciprocity theorem utilizing the properties of convolution. The parallel result in the context of the linear coupled theory of thermoelasticity for prestressed bodies is due to Iesan[10].

Let $w(x, t)$ and $v(x, t)$ be functions defined on $\Omega \times [0, t_0)$. For $0 \leq t \leq t_0$, the convolution of w and v is defined by

$$w * v = \int_0^t w(x, t - \tau) \, v(x, \tau) \, d\tau. \qquad \ldots (38)$$

The notation

$$1 * v = \int_0^t v(x, \tau) \, d\tau \qquad \ldots (39)$$

will also be used in this section. Let

$$R^r \equiv \{u_i^r, \theta^r, \sigma_{ij}^r, q_i^r, e_{ij}^r\}, \; r = 1, 2, \qquad \ldots (40)$$

be the solution of the initial boundary value problem defined by eqns. (18), (19) and

(22)-(24), corresponding to the external data system

$$V^r \equiv \{F_i^r, S^r, \bar{u}_i^r, \bar{\theta}^r, \hat{\sigma}_i^r, \hat{q}^r\}, \; r = 1, 2; \qquad \ldots (41)$$

then we will prove the following lemma and the reciprocity theorem:

Lemma 2 — If the assumption (3) of the last section is true and for $t_1, t_2 \in [0, \infty)$, $r, s \in 1, 2$, let

$$A_{rs}(t_1, t_2) = \int_{\partial\Omega} \left(\sigma_{ij}^r(t_1)\, \dot{u}_i^s(t_2) - \frac{1}{\theta_0}\, \theta^r(t_1)\, q_j^s(t_2) \right) n_j\, dS$$

$$- \int_\Omega \left(\rho_0\, \dot{u}_i^r(t_1)\, \dot{u}_i^s(t_2) + d_{ijkl}\, \dot{e}_{ij}^s(t_2)\, e_{kl}^r(t_1) + d\theta^r(t_1)\, \dot{\theta}^s(t_2) \right) d\Omega$$

$$+ \int_\Omega \left(F_i^r(t_1)\, u_i^s(t_2) + \frac{1}{\theta_0}\, \rho_0\, \theta^r(t_1)\, S^s(t_2) \right) d\Omega$$

$$- \frac{\tau}{\theta_0} \int_\Omega b_{ij}\, \dot{q}_i^r(t_1)\, q_j^s(t_2)\, d\Omega, \qquad \ldots (42)$$

then

$$A_{rs}(t_1, t_2) = A_{sr}(t_2, t_1). \qquad \ldots (43)$$

PROOF : Using the divergence theorem, we find that eqn. (42) may be rewritten as

$$A_{rs}(t_1, t_2) = \int_\Omega \left(\sigma_{ij,j}^r(t_1) + F_i^r(t_1) - \rho_0\, \dot{u}_i^r(t_1) \right) \dot{u}_i^s(t_2)\, d\Omega$$

$$+ \int_\Omega \sigma_{ij}^r(t_1)\, \dot{e}_{ij}^s(t_2)\, d\Omega - \int_\Omega \frac{1}{\theta_0}\, \theta_{,j}^r(t_1)\, q_j^s(t_2)\, d\Omega$$

$$+ \int_\Omega \frac{1}{\theta_0} \left[- q_{j,j}^s(t_2) + \rho_0\, S^s(t_2) \right] \theta^r(t_1)\, d\Omega$$

$$- \frac{\tau}{\theta_0} \int_\Omega b_{ij}\, \dot{q}_i^r(t_1)\, q_j^s(t_2)\, d\Omega$$

$$- \int_\Omega \left(d_{ijkl}\, \dot{e}_{ij}^s(t_2)\, e_{kl}^r(t_1) + d\theta^r(t_1)\, \dot{\theta}^s(t_2) \right) d\Omega. \qquad \ldots (44)$$

Using eqns. (12) and (13) in the above result, we find that

$$A_{rs}(t_1, t_2) = \int_\Omega \left(\sigma_{ij}^r(t_1) - d_{ijkl}\, e_{kl}^r(t_1) \right) \dot{e}_{ij}^s(t_2)\, d\Omega + \int_\Omega \rho_0\, \dot{\eta}^s(t_2)\, \theta^r(t_1)\, d\Omega$$

$$- \int_\Omega \frac{1}{\theta_0}\, \theta_{,j}^r(t_1)\, q_j^s(t_2)\, d\Omega - \frac{\tau}{\theta_0} \int_\Omega b_{ij}\, \dot{q}_i^r(t_1)\, q_j^s(t_2)\, d\Omega$$

$$- \int_\Omega d\theta^r(t_1)\, \dot{\theta}^s(t_2)\, d\Omega. \qquad \ldots (45)$$

Using eqn. (14) and (15) in equation (45), we find that

$$A_{rs}(t_1, t_2) = -\int_\Omega \frac{1}{\theta_0} \theta_j^r(t_1)\, q_j^s(t_2)\, d\Omega - \frac{\tau}{\theta_0}\int_\Omega b_{ij}\, \dot{q}_i^r(t_1)\, q_j^s(t_2)\, d\Omega. \quad \ldots (46)$$

From eqns. (33) and (46), we find that

$$A_{rs}(t_1, t_2) = \frac{1}{\theta_0}\int_\Omega b_{ij}\, q_i^r(t_1)\, q_j^s(t_2)\, d\Omega$$

$$= A_{sr}(t_2, t_1), \quad \ldots (47)$$

which completes the proof.

Theorem 2 (Reciprocity) — If the assumption (3) holds and R^r be the solution of the initial boundary value problem corresponding to the external data system V^r, $r = 1, 2$, then

$$\int_{\partial\Omega} \sigma_{ij}^1 * u_i^2\, n_j\, dS - 1 * \int_{\partial\Omega} \frac{1}{\theta_0}\theta^1 * q_j^2\, n_j\, dS + \int_\Omega F_i^1 * u_i^2\, d\Omega$$

$$+ 1 * \int_\Omega \frac{1}{\theta_0}\rho_0\, \theta^1 * S^2\, d\Omega$$

$$= \int_{\partial\Omega} \sigma_{ij}^2 * u_i^1\, n_j\, dS - 1 * \int_{\partial\Omega} \frac{1}{\theta_0}\theta^2 * q_j^1\, n_j\, dS + \int_\Omega F_i^2 * u_i^1\, d\Omega$$

$$+ 1 * \int_\Omega \frac{1}{\theta_0}\rho_0\, \theta^2 * S^1\, d\Omega. \quad \ldots (48)$$

PROOF : Taking $t_1 = t - \tau$, $t_2 = \tau$ in eqn. (43) and integrating with respect to τ, from 0 to t, we get

$$\int_{\partial\Omega}\left(\sigma_{ij}^1 * \dot{u}_i^2 - \frac{1}{\theta_0}\theta^1 * q_j^2\right) n_j\, dS - \int_\Omega \rho_0\, \dot{u}_i^1 * \dot{u}_i^2\, d\Omega$$

$$+ \int_\Omega\left(F_i^1 * \dot{u}_i^2 + \frac{1}{\theta_0}\rho_0\, \theta^1 * S^2\right) d\Omega$$

$$- \frac{\tau}{\theta_0}\int_\Omega b_{ij}\, \dot{q}_i^1 * q_j^2\, d\Omega - \int_\Omega\left(d_{ijkl}\, e_{kl}^1 * \dot{e}_{ij}^2 + d\theta^1 * \dot{\theta}^2\right) d\Omega$$

$$= \int_{\partial\Omega}\left(\sigma_{ij}^2 * \dot{u}_i^1 - \frac{1}{\theta_0}\theta^2 * q_j^1\right) n_j\, dS - \int_\Omega \rho_0\, \dot{u}_i^2 * \dot{u}_i^1\, d\Omega$$

$$+ \int_\Omega\left(F_i^2 * \dot{u}_i^1 + \frac{1}{\theta_0}\rho_0\, \theta^2 * S^1\right) d\Omega$$

$$- \frac{\tau}{\theta_0}\int_\Omega b_{ij}\, \dot{q}_i^2 * q_j^1\, d\Omega - \int_\Omega\left(d_{ijkl}\, e_{kl}^2 * \dot{e}_{ij}^1 + d\theta^2 * \dot{\theta}^1\right) d\Omega. \quad \ldots (49)$$

Since

$$w * v = v * w, \quad w + \dot{v} = \dot{w} * v \qquad \ldots (50)$$

for all $w(x, t)$ and $v(x, t)$ with $w(x, 0) = 0$ and $v(x, 0) = 0$, we find that

$$\int_\Omega \rho_0 \dot{u}_i^1 * \dot{u}_i^2 \, d\Omega = \int_\Omega \rho_0 \dot{u}_i^2 * \dot{u}_i^1 \, d\Omega \qquad \ldots (51)$$

$$\int_\Omega b_{ij} \dot{q}_i^1 * q_j^2 \, d\Omega = \int_\Omega b_{ij} \dot{q}_i^2 * q_j^1 \, d\Omega \qquad \ldots (52)$$

and

$$\int_\Omega \left(d_{ijkl} e_{kl}^1 * \dot{e}_{ij}^2 + d\theta^1 * \dot{\theta}^2 \right) d\Omega = -\int_\Omega \left(d_{ijkl} e_{kl}^2 * \dot{e}_{ij}^1 + d\theta^2 * \dot{\theta}^1 \right) d\Omega. \qquad \ldots (53)$$

Substituting from eqns. (51)-(53) into eqn. (49), we find that

$$\int_{\partial\Omega} \left(\sigma_{ij}^1 * \dot{u}_i^2 - \frac{1}{\theta_0} \theta^1 * q_j^2 \right) n_j \, dS + \int_\Omega \left(F_i^1 * \dot{u}_i^2 + \frac{1}{\theta_0} \rho_0 \theta^1 * S^2 \right) d\Omega$$

$$= \int_{\partial\Omega} \left(\sigma_{ij}^2 * \dot{u}_i^1 - \frac{1}{\theta_0} \theta^2 * q_j^1 \right) n_j \, dS + \int_\Omega \left(F_i^2 * \dot{u}_i^1 + \frac{1}{\theta_0} \rho_0 \theta^2 * S^1 \right) d\Omega. \qquad \ldots (54)$$

In eqn. (54), replacing variable t by τ, and integrating from 0 to t with respect to τ, we get the desired result.

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