Indian J. pure appl. Math., 28(3) : 311-317, March 1997

ZARISKI TOPOLOGY AND SEPARATING IDEALS IN COMMUTATIVE BANACH ALGEBRAS

RAMESH V. GARIMELLA

Department of Mathematics, Tennessee Technological University,
Cookeville, TN 38505, *USA*

(*Received* 28 *December* 1995; *after revision* 24 *September* 1996;
accepted 4 *October* 1996)

Let B be a commutative unital Banach algebra, *Spec* B be the prime spectrum of B, and ψ be the set of all minimal prime ideals of B. We prove that if the subspace $(Spec\ B) \setminus \Psi$ is separable under the Zariski topology, then any separating ideal of B is a nil ideal.

1. INTRODUCTION

Runde[10] showed that if every non-maximal prime ideal is closed in a commutative unital semiprime Banach algebra B, then any derivation on B and any epimorphism from a commutative Banach algebra onto B are continuous. In an earlier paper, the author[6] has shown that if the set of all non-closed prime ideals in a commutative unital semiprime Banach algebra B is at most countable, then every derivation on B is continuous. A similar result can be obtained for epimorphisms onto B. Illoussamen[8] showed that in an integral domain B if the space of all non-zero prime ideals is separable under the Zariski topology, then any derivation on B and any epimorphism onto B are continuous. In this paper we extend some of the results of Illoussamen[8] by giving sufficient conditions for a separating ideal to be a nil ideal. For further history and related results we also refer to references [1, 4, 7 and 9].

Throughout the following we suppose that B is a commutative unital Banach algebra. A closed ideal $\mathcal{J}$ of B is said to be a separating ideal of B if for every sequence $\{x_n\}$ in B, there is a positive integer N such that $\overline{x_1x_2 \ldots x_n \mathcal{J}} = \overline{x_1 x_2 \ldots x_N \mathcal{J}}$ for each $n \geq N$. For any derivation D on B, let $\mathcal{J}(D) := \{x \in B : \exists\, x_n \to 0 \text{ with } D(x_n) \to x\}$. Similarly, for any epimorphism h from a Banach algebra A onto B, let $\mathcal{J}(h) := \{y \in B : \exists\, x_n \to 0 \text{ in } A \text{ s.t. } h(x_n) \to y \text{ in } B\}$. It can be shown that both $\mathcal{J}(D)$ and $\mathcal{J}(h)$ are separating ideals of B. For a proof, we refer to Lemma 1.6 of Sinclair[11]. By the closed graph theorem D is continuous on B if and only if $\mathcal{J}(D) = \{0\}$. Similarly h is continuous from A onto B if and only if $\mathcal{J}(h) = \{0\}$.

The intersection of all maximal ideals in a commutative unital Banach algebra B is known as the Jacobson radical of B. Since each maximal ideal of B is closed, the Jacobson radical is a closed ideal. An ideal P of B is said to be a prime ideal if $a \in B \setminus P, b \in B \setminus P$, then $ab \in B \setminus P$. A Banach algebra is said to be an integral domain if the zero ideal is a prime ideal. The intersection of all prime ideals of B is said to be the nil radical of B. The nil radical of B consists of all the nilpotent elements of B and is also known as the prime radical of B. An ideal I of B is said to be a nil ideal if I is contained in the nil radical of B. For any commutative unital Banach algebra B, let *Spec B* denote the set of all prime ideals of B. *Spec B* is known as the prime spectrum of B. For $a \in B$, let $V(a) := \{P \in Spec\, B \setminus a \notin P\}$. The collection $\{V(a) \mid a \in B\}$ forms a basis for the open sets of *Spec B*. The resulting topology on *Spec B* is known as the Zariski topology.

2. MAIN RESULT

Before we get to the main result, we shall need the following lemmas.

Lemma 1 — Let P be a prime ideal of a commutative unital Banach algebra B. If P is the intersection of all closed ideals properly containing it, then P must contain every separating ideal of B.

PROOF : Refer to the proof of Lemma 1.1 of Curtis[2].

Lemma 2 — Let x be a non-nilpotent element of B contained in the Jacobson radical. Then there exists a prime ideal Q of B such that $x \notin Q$ and $\bigcap_{n=1}^{\infty} (x^n)$ is contained in Q where (x^n) is the principal ideal generated by x^n.

PROOF : Since x is not nilpotent, the set $M = \{x, x^2, x^3, \ldots\}$ is a multiplicatively closed set not containing the zero vector. Let Q be a maximal element in the set of all ideals not intersecting M under the set theoretic inclusion. It is straightforward to verify that Q is a prime ideal of B. Suppose $\bigcap_{n=1}^{\infty} (x^n)$ is not contained in Q. Then Q is properly contained in $Q + \bigcap_{n=1}^{\infty} (x^n)$. By the maximality of Q, $\left(Q + \bigcap_{n=1}^{\infty} (x^n) \right) \bigcap M \neq \phi$. Hence there exists a positive integer m such that $x^m \in Q + \bigcap_{n=1}^{\infty} (x^n)$. This implies that $x^m (1 - cx) \in Q$ for some c in B. Since x is in the Jacobson radical, $1 - cx$ is invertible. Therefore $x^m \in Q$. This is a contradition. ☐

Since the following lemma is a direct consequence of the Mittag-Leffler theorem (refer to Esterle[5]), we shall state it without the proof.

Lemma 3 — Let B be a commutative Banach algebra, and $S \subset B$ be a non-zero closed ideal. If there exists a sequence $\{x_n\}$ of non-zero elements such that $\overline{x_1 x_2 \dots x_n S} = S$ for all positive integers, n, then the ideal $J = \bigcap_{n=1}^{\infty} x_n S$ is dense in S.

Lemma 4 — Let P be a closed prime ideal of B, and let $\{I_n\}$ be a sequence of ideals such that I_n is not contained in P for each n. If $\bigcap_{n=1}^{\infty} I_n$ is contained in P, then P must contain any separating ideal of B.

PROOF : Let $\mathcal{J}$ be a separating ideal of B. For each n, choose $y_n \in (I_1 \cap I_2 \cap \dots \cap I_n) \setminus P$. Then there exists an integer m such that $\overline{y_1 y_2 \dots y_n \mathcal{J}} = \overline{y_1 y_2 \dots y_m \mathcal{J}}$ for all $n \geq m$. Set $S = \overline{y_1 y_2 \dots y_m \mathcal{J}}$ and $x_i = y_{m+i}$ for each $i \geq 1$. Clearly $\overline{x_1 x_2 \dots x_k S} = S$ for $k \geq 1$. Hence by Lemma 3, $\bigcap_{n=1}^{\infty} x_n S$ is dense in S. But $\bigcap_{n=1}^{\infty} x_n S \subseteq \bigcap_{n=1}^{\infty} I_n \subseteq P$. Since P is a closed ideal and $\bigcap_{n=1}^{\infty} x_n S$ is dense in S, S is contained in P (i.e. $\overline{y_1 y_2 \dots y_m \mathcal{J}} \subseteq P$). However P is a prime ideal not containing $y_1, \dots, y_m$. Therefore $\mathcal{J}$ is contained in P.

Even though the following result is straightforward from the definition of the Zariski topology and is implicit in Illoussamen[18], we state and prove it as a separate lemma.

Lemma 5 — Let B be a commutative unital Banach algebra, and *Spec B* be the prime spectrum of B equipped with the Zariski topology. Suppose $\mathcal{R}$ is a dense subset of a subspace $\mathcal{N}$ of *Spec B*.

Then $$\bigcap_{P \in \mathcal{R}} P = \bigcap_{Q \in \mathcal{N}} Q$$

PROOF : Since $\mathcal{R}$ is contained in $\mathcal{N}$, clearly $\bigcap_{Q \in \mathcal{N}} Q$ is contained in $\bigcap_{P \in \mathcal{R}} P$.

Let $t \in \bigcap_{P \in \mathcal{R}} P$. Suppose there is a prime ideal Q of B in $\mathcal{N}$ not containing t. Let $V(t) = \{P \in \mathit{Spec}\, B \mid t \notin P\}$. Then $V(t)$ is a non-empty basic open subset of *Spec B* such that $V(t) \cap \mathcal{N} \neq \phi$. Since $\mathcal{R}$ is a dense subset of $\mathcal{N}$, there must exist a prime ideal P in $\mathcal{R}$ not containing t. This is a contradiction. □

Now we are ready to state and prove the main result.

Theorem — Let B be a commutative unital Banach algebra and let *Spec B* be the prime spectrum of B. Let Ψ be a subset of *Spec B* containing every minimal prime ideal P of B and at most countably many prime ideals containing P. If the subspace (*Spec* B)$\setminus\Psi$ is separable under the topology induced by the Zariski topology on *Spec* B, then any separating ideal contained in the Jacobson radical of B is a nil ideal.

PROOF : Suppose the result is false. Then there exists a separating ideal $\mathcal{J}$ contained in the Jacobson radical of the algebra which is not a nil ideal. By Cusack[3] there are finitely many minimal prime ideals $P_1, P_2, \ldots, P_k$ not containing $\mathcal{J}$ such that the set $\mathcal{J} \cap P_1 \cap P_2 \cap \ldots \cap P_k$ consists of all the nilpotent elements of $\mathcal{J}$. Further, it was noted in Cusack[3] that any minimal prime ideal not containing a separating ideal is closed. Hence P_i is closed for each i, $1 \leq i \leq k$. By our assumption each $P_i\,(1 \leq i \leq k)$ belongs to Ψ. Now for each $i\,(1 \leq i \leq k)$, let

$$\phi_{P_i} = \{Q \in \mathit{Spec}\, B \mid \mathcal{J} \not\subset Q,\ P_j \not\subset Q \text{ for } j \neq i, \text{ and } P_i \text{ is properly contained in } Q\}.$$

Claim 1 — For each $i\,(1 \leq i \leq k)$, ϕ_{P_i} is a non-empty set and $\bigcap_{Q \in \phi_{P_i}} Q = P_i$.

PROOF OF CLAIM 1 : Fix a positive integer i between 1 and k. First we show that ϕ_{P_i} is non-empty. Let $\hat{P}_i$ be the intersection of all closed ideals properly containing P_i. By Lemma 1, P_i is properly contained in $\hat{P}_i$. First notice that $\mathcal{J} \cap P_1 \cap \cdots \cap P_{i-1} \cap \hat{P}_i \cap P_{i+1} \cap \cdots \cap P_k$ is not contained in P_i. For, if $\mathcal{J} \cap P_1 \cap \cdots \cap P_{i-1} \cap \hat{P}_i \cap P_{i+1} \cap \cdots \cap P_k$ is contained in P_i, since P_i is a prime ideal containing neither $\mathcal{J}$ nor P_j for $j \neq i$, $\hat{P}_i = P_i$. Next, let $t \in (\mathcal{J} \cap P_1 \cap \cdots \cap P_{i-1} \cap \hat{P}_i \cap P_{i+1} \cap \cdots \cap P_k)\ P_i$. Then by Lemma 2, there exists a prime ideal Q such that $t \notin Q$ and $\bigcap_{n=1}^{\infty} (t^n)$ is contained in Q. Since Q contains neither $\mathcal{J}$ nor P_j for $j \neq i$, Q must contain P_i. Since P_i is a closed prime ideal and Q contains $\bigcap_{n=1}^{\infty} (t^n)$, Lemma 4 implies that Q must properly contain P_i. Hence $Q \in \phi_{P_i}$. This proves ϕ_{P_i} is non-empty.

Now we show that $\bigcap_{Q \in \phi_{P_i}} Q = P_i$ for each $i\,(1 \leq i \leq k)$. Fix an i. Let $\hat{P}_i$ be defined as in the above paragraph. If $\left(\left(\bigcap_{Q \in \phi_{P_i}} Q\right) \cap P_1 \cap \cdots \cap P_{i-1} \cap \hat{P}_i \cap P_{i+1} \cap \cdots \cap P_k\right)$ is not contained in P_i, then for any element t in $\left(\left(\bigcap_{Q \in \phi_{P_i}} Q\right) \cap P_1 \cap \cdots \cap P_{i-1} \cap \hat{P}_i \cap P_{i+1} \cap \cdots \cap P_k\right) \setminus P_i$, Lemma 2 implies that $\bigcap_{n=1}^{\infty} (t^n)$ is contained in P_i. Since the separating ideal $\mathcal{J}$ is not contained in P_i, by Lemma 4 it follows that $t \in P_i$. This is absurd. Therefore $\left(\bigcap_{Q \in \phi_{P_i}} Q\right) \cap P_1 \cap \cdots \cap P_{i-1} \cap \hat{P}_i \cap P_{i+1} \cap \cdots \cap P_k$ is contained in P_i. Since P_i is a prime ideal containing neither $\hat{P}_i$ nor P_j for $j \neq i$, P_i must contain $\bigcap_{Q \in \phi_{P_i}} Q$. But obviously P_i is contained in $\bigcap_{Q \in \phi_{P_i}} Q$. Therefore $\bigcap_{Q \in \phi_{P_i}} Q = P_i$. This completes the proof of the claim.

Notice that in view of Lemma 4 the set ϕ_{P_i} is uncountable for each $i\,(1 \leq i \leq k)$.

Claim 2 — If $\mathcal{N}$ is a countable subset of $\phi_{P_i}\,(1 \leq i \leq k)$, then $\bigcap_{Q \in \phi_{P_i} \setminus \mathcal{N}} Q = P_i$.

PROOF OF CLAIM 2 : Note that $\left(\bigcap_{Q \in \phi_{P_i} \setminus \mathcal{N}} Q\right) \cap \left(\bigcap_{Q \in \mathcal{N}} Q\right) = \left(\bigcap_{Q \in \phi_{P_i}} Q\right) = P_i$. The last equality is true because of Claim 1. Since P_i is a prime ideal, either $\left(\bigcap_{Q \in \phi_{P_i} \setminus \mathcal{N}} Q\right)$ or $\left(\bigcap_{Q \in \mathcal{N}} Q\right)$ is contained in P_i. Since $\mathcal{N}$ is a countable set, Lemma 4 implies that $\left(\bigcap_{Q \in \mathcal{N}} Q\right)$ is not contained in P_i. Hence $\left(\bigcap_{Q \in \phi_{P_i} \setminus \mathcal{N}} Q\right)$ is contained in P_i. This establishes the claim.

Now we are ready to conclude the proof of the theorem. Once again fix an $i\,(1 \leq i \leq k)$. Let $\{Q_n\}$ be a dense subset of $(Spec\ B) \setminus \Psi$ under the Zariski topology. By Lemma 5, $\left(\bigcap_{n=1}^{\infty} Q_n\right) = \left(\bigcap_{Q \in Spec\,B \setminus \Psi} Q\right)$. Further

$$\left(\bigcap_{Q \in Spec\,B \setminus \Psi} Q\right) \subseteq \left(\bigcap_{Q \in \phi_{P_i} \setminus \Psi} Q\right) = P_i.$$

The last equality in the above is true because from the hypothesis Ψ can contain at most countably many members of ϕ_{P_i}. Therefore $\bigcap_{n=1}^{\infty} Q_n$ is contained in P_i. Hence by Lemma 4, P_i must contain a Q_k for some k. Since P_i is a minimal prime ideal, $Q_k = P_i$. This is a contradiction which completes the proof of the theorem. □

Remark : Since the image of a derivation is contained in the Jacobson radical of the algebra[12], the separating ideal of any derivation must be contained in the Jacobson radical. Also it is easy to note that the separating ideal of any epimorphism onto B is contained in the Jacobson radical of the algebra. Hence in the above theorem it was not too restrictive to assume that the separating ideal is contained in the Jacobson radical of the algebra.

Corollary 1 — Let B be a commutative unital semiprime Banach algebra. Let *Spec* B be the prime spectrum of B, and Ψ be the set of all minimal prime ideals of B. If the subspace $(Spec\ B)\backslash\Psi$ is separable under the Zariski topology, then any derivation on B and any epimorphism onto B are continuous.

The following result is in Illoussamen[8].

Corollary 2 — Let B be a commutative unital Banach algebra which is an integral domain. If $(Spec\ B)\backslash\{0\}$ is separable under the Zariski topology, then any derivation on B and any epimorphism onto B are continuous.

Finally we conclude the paper by giving an example of a commutative Banach algebra B that is not an integral domain yet satisfies the hypothesis of the theorem given above. Let A be a commutative unital Banach algebra which is an integral domain. Further assume that there is a sequence $\{Q_n, n \geq 1\}$ of non-zero prime ideals in A intersecting to the zero ideal of A such that $(Spec\ A)\backslash\{Q_n,\ n \geq 1\}$ contains a non-zero prime ideal. For example, one could take A to be a weighted power series algebra or a disc algebra or a semisimple weighted L^1-space with the identity adjoined. Let M be a Banach A-module. Let

$$B := A \oplus M = \{(a,\ m) \mid a \in A,\ \ m \in M\}.$$

On B, we define the norm, vector addition, scalar multiplication, and multiplication of vectors as follows :

For all $a_1, a_2 \in A$; $m_1, m_2 \in M$; and for any complex number c,

(i) $\|(a_1, m_1)\| = \|a_1\| + \|m_1\|$

(ii) $(a_1, m_1) + (a_2, m_2) = (a_1 + a_2, m_1 + m_2)$

(iii) $c(a_1, m_1) = (ca_1, cm_1)$

(iv) $(a_1, m_1) \cdot (a_2, m_2) = (a_1\, a_2, a_1 m_1 + a_2 m_2)$.

Under the above operations B becomes a commutative Banach algebra with (1, 0) as the identity element where 1 is the identity element of A and 0 is the zero vector in M. Since A is an integral domain, $P_0 := \{(\theta, m) \mid m \in M\}$ (where θ is the zero vector in A) is a prime ideal of B. Since $(\theta, m) \cdot (\theta, m) = (\theta, 0)$ for any $m \in M$, every prime ideal of B contains P_0. Thus P_0 is the smallest prime ideal of B. Now for each $n \geq 1$, let $P_n := \{(a,\ m)\ :\ a \in Q_n,\ m \in M\}$. It is easy to see P_n is a prime ideal of B. As a matter of fact for any prime ideal Q of A, $P := \{(a,\ m)\ :\ a \in Q,\ m \in M\}$ is a prime ideal of B. Note that $\bigcap_{n \geq 1} P_n = P_0$. This is true because

$\bigcap_{n \geq 1} Q_n = \{\theta\}$. Since $(Spec\ A) \backslash \{Q_n; n \geq 1\}$ contains a non-zero prime ideal of A, $(Spec\ B) \backslash \{P_n : n \geq 1\}$ contains a prime ideal of B other than P_0. Now let Ψ be any countable subset of $(Spec\ B) \backslash \{P_n : n \geq 1\}$ containing P_0. Since $(Spec\ B) \backslash \Psi$ contains $\{P_n : n \geq 1\}$, $(Spec\ B) \backslash \Psi$ is separable under the Zariski topology. Thus B is a commutative unital Banach algebra satisfying the hypothesis of the main theorem. However, clearly B is not an integral domain.

ACKNOWLEDGEMENT

The author is grateful to the referee for valuable comments and suggestions. He would like to thank Vickie Mayberry for typing the paper. Also, he would like to acknowledge the support of the Faculty Research Committee of his university in conducting the above research.

REFERENCES

1. W. G. Bade and P. C. Curtis Jr., *J. Funct. Anal.* **29** (1978), 88-103.
2. P. C. Curtis, Jr., Derivations on commutative Banach algebras, *Proc. of Long Beach 1981, Lecture Notes in Math*, Vol. 975, Springer-Verlag, 1983, pp. 328-33.
3. J. M. Cusack, *J. London Math. Soc.* **16** (1977), 493-500.
4. H. G. Dales, *Bull. London Math. Soc.* **10** (1978), 129-83.
5. J. Esterle, Elements for a classification of commutative radical Banach algebras, *Proc. Long Beach 1981, Lecture Notes in Math.*, Vol. 975, Springer-Verlag, 1983, 4-65.
6. R. V. Garimella, *Proc. Am. Math. Soc.* **117** (1993), 167-74.
7. R. V. Garimella, On separating ideals of commutative Banach algebras, *Interaction between Functional Analysis, Harmonic Analysis and Probability*, Vol. 175, Marcel Dekker, 1996, New York, pp. 181-85.
8. E. H. Illoussamen, Continuité des dérivations et des épimorphismes dans certaines algèbres de Banach, *Rend. circ. Mat. Palermo* (to appear),
9. M. Mathieu, Where to Find the Image of a Derivation, Banach Center Publications, Polish Academy of Sciences, Vol. 30, Warszawa, 1994, pp. 237-49.
10. V. Runde, *Pacific J. Math.* **147**, (1991), 365-74.
11. A. M. Sinclair, Automatic Continuity of Linear Operators, London Math. Soc. Lecture Notes Ser., Vol. 21, Cambridge Univ. Press, Cambridge, 1976.
12. M. P. Thomas, *Ann. Math.* (2), **128** (1988), 435-60.