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ON THE ASYMPTOTIC BEHAVIOUR OF NONHOMOGENEOUS LINEAR DIFFERENCE EQUATIONS

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For the linear difference equation

$$\Delta x_n = \sum_{i=0}^{r} a_n^i x_{n+i} + b_n, \qquad n \in \mathbb{N}$$

sufficient conditions for the existence of an asymptotically constant solution are presented.

It is known that any linear difference equation

$$c_n^r y_{n+r} + \ldots + c_n^1 y_{n+1} + c_n^0 y_n = d_n, \qquad n \in \mathbb{N}, \qquad \ldots (E_1)$$

can be transformed to a number of different forms. One such transformation (which we show below after Theorem 1) leads to the equation :

$$\Delta x_n = \sum_{i=0}^{r} a_n^i x_{n+i} + b_n, \qquad n \in \mathbb{N}. \qquad \ldots (E_2)$$

Here by $\mathbb{N}$, $\mathbb{R}$ we denote the set of positive integers and reals respectively. For any function $y : \mathbb{N} \to \mathbb{R}$ the forward difference operators are defined as follows :

$$\Delta y_n = y_{n+1} - y_n, \qquad n \in \mathbb{N}$$

$$\Delta^i y_n = \Delta(\Delta^{i-1} y_n), \quad \text{for } i > 1, \qquad n \in \mathbb{N}.$$

In what follows we use the convention

$$\sum_{j=n}^{n-k} y_j := 0; \quad \prod_{j=n}^{n-k} y_j := 1 \quad \text{for any } k, n \in \mathbb{N}.$$

Instead of $\lim_{n\to\infty} x_n = C$ we shall write $x_n = C + o(1)$, and similarly if $\lim_{n\to\infty} \frac{x_n}{y_n} = C$ then $x_n = y_n\,(C + o(1))$. If there exists some constant C such that $\left|\frac{x_n}{y_n}\right| \le C$ for all $n \in \mathbb{N}$ then we write $x_n = O(y_n)$.

Theorem 1 — Let a^i, b : $\mathbb{N} \to \mathbb{R}$, $a_n^0 \neq -1$ for every $n \in \mathbb{N}$, and $\sup_{n \ge m}\left[\max_i |a_n^i|\right] > 0$ for all $m \in \mathbb{N}$,

$$\sum_{j=1}^{\infty} |a_j^i| < \infty, \qquad i = 0, 1, \ldots, r, \qquad \ldots (1)$$

$$\sum_{j=1}^{\infty} b_j \text{ converges (perhaps conditionally).} \qquad \ldots (2)$$

Then for any arbitrary constant $C \neq 0$, $C \in \mathbb{R}$ there exists a solution x of (E_2) such that

$$x_n = C + o(1). \qquad \ldots (3)$$

PROOF : We shall prove the theorem for $C > 0$. The proof in the case when $C < 0$ is similar.

Since $C > 0$ so there exists a positive constant ε such that $C - \varepsilon > 0$. Let us denote

$$C_1 = C + \varepsilon, \quad I = [C - \varepsilon,\ C + \varepsilon].$$

$$\beta_n = \left|\sum_{j=n}^{\infty} b_j\right|, \qquad \ldots (4)$$

and

$$\alpha_n = \left(C_1 \sum_{i=0}^{r} \sum_{j=n}^{\infty} |a_j^i|\right) + \beta_n.$$

From (1) and (2) it follows that there exists $n_1 \in \mathbb{N}$ such that $\alpha_n \le \varepsilon$ for all $n \ge n_1$. Let l_∞ denote the Banach space of bounded sequences $x = \{\xi_n\}_{n=1}^{\infty}$ with the norm $\|x\| = \sup_{n \ge 1} |\xi_n|$.

Moreover, let $T \subset l_\infty$ be the set such that $x = \{\xi_n\}_{n=1}^{\infty} \in T$ if

$$\left.\begin{aligned} &\xi_n = C && \text{for } n = 1, 2, \ldots, n_1 - 1 \\ &\xi_n \in I_n && \text{for } n \geq n_1 \end{aligned}\right\}$$

where $I_n = [C - \alpha_n, \ C + \alpha_n]$. Note that $I_n \subseteq I$, for all $n \geq n_1$. It can be shown that T is a closed, convex, and compact subset of l_∞.

Define now an operator $\mathcal{A}$ by the formula

$$\mathcal{A}x = y = \left\{ \eta_n \right\}_{n=1}^{\infty}$$

where

$$\eta_n = \begin{cases} C & \text{for } n = 1, 2, \ldots, n_1 - 1 \\ C - \sum_{i=0}^{r} \sum_{j=n}^{\infty} a_j^i \xi_{j+i} - \sum_{j=n}^{\infty} b_j & \text{for } n \geq n_1. \end{cases}$$

Conditions (1) and (2) yield that the operator $\mathcal{A}$ is well defined on the whole space l_∞.

We shall prove that $\mathcal{A}$ maps the set T into T. Take any $x \in T$, so we get

$$\begin{aligned} |\eta_n - C| &\leq \sum_{i=0}^{r} \sum_{j=n}^{\infty} |a_j^i| \, |\xi_{j+i}| + \left| \sum_{j=n}^{\infty} b_j \right| \\ &\leq C_1 \sum_{i=0}^{r} \sum_{j=n}^{\infty} |a_j^i| + \left| \sum_{j=n}^{\infty} b_j \right| \end{aligned}$$

because $\xi_{j+i} \in I_{j+i} \subset I$, for all $j \geq n_1$, $i \in \{0, 1, \ldots, r\}$. Hence by (4)

$$C - \alpha_n \leq \eta_n \leq C + \alpha_n$$

and therefore $\eta_n \in I_n$ for all $n \geq n_1$. That is, $\mathcal{A}$ maps the set T into T.

We now prove that $\mathcal{A}$ is continuous on T. Take $\varepsilon_1 > 0$ and $\delta_1 = \varepsilon_1/\alpha$ where

$$\alpha = \sum_{i=0}^{r} \sum_{j=n_1}^{\infty} |a_j^i| .$$

Let $x = \left\{ \xi_n \right\}_{n=1}^{\infty}$ and $z = \left\{ \zeta_n \right\}_{n=1}^{\infty}$ be any two elements of the set T such that $\| x - z \| < \delta_1$. Then the absolute convergence of the series

$$\sum_{i=0}^{r} \sum_{j=n_1}^{\infty} a_j^i \xi_{j+i} , \quad \sum_{i=0}^{r} \sum_{j=n_1}^{\infty} a_j^i \zeta_{j+i}$$

yields

$$\| \mathcal{A}x - \mathcal{A}z \| = \sup_{n \ge n_1} \left| \left\{ C - \sum_{i=0}^{r} \sum_{j=n}^{\infty} a_j^i \xi_{j+i} - \sum_{j=n}^{\infty} b_j \right\} - \left\{ C - \sum_{i=0}^{r} \sum_{j=n}^{\infty} a_j^i \zeta_{j+i} - \sum_{j=n}^{\infty} b_j \right\} \right|$$

$$\le \sup_{n \ge n_1} \sum_{i=0}^{r} \sum_{j=n}^{\infty} | a_j^i | | \xi_{j+i} - \zeta_{j+i} |$$

$$\le \sup_{n \ge n_1} \sum_{i=0}^{r} \sum_{j=n}^{\infty} | a_j^i | \| x - z \| \le \delta_1 \sup_{n \ge n_1} \sum_{i=0}^{r} \sum_{j=n}^{\infty} | a_j^i | = \varepsilon_1 .$$

Therefore the operator $\mathcal{A}$ is continuous on T. So by the Schauder fixed point theorem there exists in the set T a solution of the equation $x = \mathcal{A}x$.

Let $w = \{ \omega_n \}_{n=1}^{\infty}$ be this solution mentioned above.

Since $w \in T$, it can be written as follows :

$$w = \{ C, ..., C, \omega_{n_1}, \omega_{n_1+1}, ..., \omega_n, \dots \}$$

and

$$\mathcal{A}w = \left\{ C, ..., C, C - \sum_{i=0}^{r} \sum_{j=n_1}^{\infty} a_j^i \omega_{j+i} - \sum_{j=n_1}^{\infty} b_j , ..., \right.$$

$$\left. ..., C - \sum_{i=0}^{r} \sum_{j=n}^{\infty} a_j^i \omega_{j+i} - \sum_{j=n}^{\infty} b_j , \dots \right\}.$$

Therefore

$$\omega_n = C - \sum_{i=0}^{r} \sum_{j=n}^{\infty} a_j^i \omega_{j+i} - \sum_{j=n}^{\infty} b_j , \quad \text{for } n \ge n_1. \qquad \dots (5)$$

Applying operator Δ to (5) we obtain

$$\Delta\omega_n = \sum_{i=0}^{r} a_n^i \omega_{n+i} + b_n, \qquad n \ge n_1.$$

This means that the sequence $\{ \omega_n \}_{n=1}^{\infty}$ satisfies eqn. (E_2) for $n \ge n_1$.

Equation (E_2) can be transformed to the form

$$x_n = -(1 + a_n^0)^{-1} \left\{ (a_n^1 - 1)x_{n+1} + \sum_{i=2}^{r} a_n^i x_{n+i} + b_n \right\}. \quad \text{... (6)}$$

Substituting in (6) $n = n_1 - 1$, $x_j = \omega_j$ for $j \geq n_1$, we obtain $x_{n_1 - 1}$.

Proceeding this way we find $x_{n_1 - 2}, ..., x_1$ one by one. Consequently we get the sequence which satisfies (E_2) for all $n \in \mathbb{N}$. Moreover this sequence coincides with the sequence w for $n \geq n_1$ and so possesses the asymptotic property (3) because $\omega_n \in I_n$ and diam $I_n \to 0$ as $n \to \infty$. ■

A similar method and property for the difference equation

$$\Delta^2 x_n + a_n F(x_n) = 0$$

can be found in Drozdowicz and Popenda[1] and for the homogeneous equation (E_2) in Popenda and Schmeidel[2]. We now turn our attention to eqn. (E_1).

Suppose that $c^i, d : \mathbb{N} \to \mathbb{R}$, $i = 0, 1, ..., r$, $r \geq 2$, $c_m^0 \neq 0$, $c_m^1 \neq 0$ and

$$\sup_{n \geq m} \left[\max_{2 \leq i \leq r} |c_n^i| \right] > 0 \quad \text{for all } m \in \mathbb{N}. \quad \text{... (7)}$$

Dividing eqn. (E_1) by c_n^1 and then multiplying it by

$$(-1)^n \prod_{j=1}^{n} [c_j^1/c_j^0]$$

we get

$$(-1)^n \left\{ \prod_{j=1}^{n} [c_j^1/c_j^0] \right\} y_{n+1} - (-1)^{n-1} \left\{ \prod_{j=1}^{n-1} [c_j^1/c_j^0] \right\} y_n$$

$$= (-1)^{n+1} [c_n^2/c_n^1] \left\{ \prod_{j=1}^{n} [c_j^1/c_j^0] \right\} y_{n+2} + ... +$$

$$+ (-1)^{n+1} [c_n^r/c_n^1] \left\{ \prod_{j=1}^{n} [c_j^1/c_j^0] \right\} y_{n+r}$$

$$+ (-1)^n [d_n/c_n^1] \left\{ \prod_{j=1}^{n} [c_j^1/c_j^0] \right\}. \quad \text{... (8)}$$

Substituting

$$(-1)^{n-1} \left\{ \prod_{j=1}^{n-1} [c_j^1/c_j^0] \right\} y_n = x_n$$

into equality (8), and putting

$$a_n^i = (-1)^i\,[c_n^i/c_n^1]\left\{\prod_{j=n+1}^{n+i-1}[c_j^0/c_j^1]\right\}, \quad i = 2, 3, ..., r$$

$$b_n = (-1)^n\,[d_n/c_n^1]\left\{\prod_{j=1}^{n}[c_j^1/c_j^0]\right\}$$

we obtain

$$\Delta x_n = a_n^2 x_{n+2} + \ldots + a_n^r x_{n+r} + b_n, \quad n \in \mathbb{N}.$$

Therefore on the basis of Theorem 1 we get.

Corollary — Assume that the conditions (7) are satisfied. If the series

$$\sum_{j=1}^{\infty}\left\{[c_j^i/c_j^1]\sum_{k=j+1}^{j+i-1}[c_k^0/c_k^1]\right\}, \quad i = 2, ..., r \qquad \text{... (9)}$$

converge absolutely, and if

$$\sum_{j=1}^{\infty}\left\{(-1)^j\,[d_j/c_j^1]\sum_{k=1}^{j}[c_k^1/c_k^0]\right\} \qquad \text{... (10)}$$

converges (perhaps conditionally), then for an arbitrary constant $C \in \mathbb{R}$, $C \neq 0$, there exists a solution y of (E_1) such that

$$y_n = (-1)^{n-1}\left\{\prod_{j=1}^{n-1}[c_j^0/c_j^1]\right\}(C + o(1)), \quad n \in \mathbb{N}. \qquad \text{... (11)}$$

One of the most interesting and most investigated problems in the qualitative theory of both differential and difference equations is the study of the oscillatory behaviour of solutions. The formula (11) we have obtained in Corollary 1 allows us to formulate one sufficient condition for the existence of (strictly) oscillatory solutions of (E_1).

We call a sequence $\{z_n\}_{n=1}^{\infty}$ eventually strictly oscillatory if there exists an integer $m \in \mathbb{N}$ such that $y_n y_{n+1} < 0$ for all $n \geq m$.

Theorem 2 — Let $c^i, d : \mathbb{N} \to \mathbb{R}$, $i = 0, 1, ..., r$, $r \geq 2$,

$$c_m^0 \neq 0,\; c_m^1 \neq 0,\; \sup_{n \geq m}\left[\max_{2 \leq i \leq r} |c_n^i|\right] > 0 \text{ for all } m \in \mathbb{N}. \qquad \text{... (12)}$$

Suppose there exist a positive constant γ and an integer μ such that $c_n^0 = \gamma c_n^1$ for all $n \geq \mu$, and

and
$$\left.\begin{aligned}&\sum_{j=1}^{\infty} |c_j^i/c_j^1| < \infty, \quad i = 2, \ldots, r, \\ &\sum_{j=1}^{\infty} (-1)^j [d_j/c_j^1] \gamma^{-j} \text{ converges.}\end{aligned}\right] \quad \ldots (13)$$

Then eqn. (E_1) possesses a family of eventually strictly oscillatory solutions.

The proof of this theorem follows directly from formula (11).

Remark : In fact from formula (11) it follows that if conditions (7), (9), (10) are fulfilled, $r \geq 2$, $c_n^0 \neq 0$, $c_n^1 \neq 0$, and there exists an increasing sequence $\{n_k\}_{k=1}^{\infty}$ of positive integers such that

$$c_{n_k}^0 c_{n_k}^1 > 0 \text{ for all } k \in \mathbb{N},$$

then eqn. (E_1) possesses a family of oscillatory solutions.

PROOF : Let C be any nonzero constant, say $C < 0$ and y suitable solution of (E_1) given by (11).

The formula $C + o(1)$ means that there exists a sequence $\{\varphi_n\}_{n=1}^{\infty}$, $\varphi_n \to 0$ with $n \to \infty$, and so there exists an integer $m \in \mathbb{N}$ such that $C + o(1) = C + \varphi_n < 0$ for all $n \geq m$. Therefore from (11) it follows that

$$y_{n_k} y_{n_k+1} = (-1)^{n_k - 1} \left\{ \prod_{j=1}^{n_k - 1} [c_j^0/c_j^1] \right\} (C + \varphi_{n_k}) (-1)^{n_k}$$

$$\times \left\{ \prod_{j=1}^{n_k} [c_j^0/c_j^1] \right\} (C + \varphi_{n_k+1})$$

$$= -\left\{ \prod_{j=1}^{n_k - 1} [c_j^0/c_j^1] \right\}^2 [c_{n_k}^0/c_{n_k}^1] (C + \varphi_{n_k}) (C + \varphi_{n_k+1}) < 0$$

for all k such that $n_k \geq m$. This means that this solution is oscillatory.

Examples — Let us consider the equation

$$-\frac{n+2}{n(n+1)(n+3)} y_{n+2} + y_{n+1} + y_n = (-1/2)^n \frac{n^3 + 4n^2 + n - 4}{n(n+1)(n+3)}, \quad n \in \mathbb{N}. \quad \ldots (14)$$

Observe that conditions of Theorem 2 are satisfied where $c^0 \equiv c^1 \equiv 1$, $\gamma = 1$, and the series

$$\sum_{j=1}^{\infty} |c_j^2/c_j^1| = \sum_{j=1}^{\infty} \frac{j+2}{j(j+1)(j+3)}$$

and
$$\sum_{j=1}^{\infty} (-1)^j [d_j/c_j^1] \gamma^j = \sum_{j=1}^{\infty} 2^{-j} \frac{j^3+4j^2+j-4}{j(j+1)(j+3)}$$

converge. So, by Theorem 2 this equation has a family of oscillatory solutions. It is easy to check that the sequence

$$\left\{ (-1)^n \left[C\left(1+\frac{1}{n}\right)+2^{-n} \right] \right\}_{n=1}^{\infty},$$

where C is arbitrary, forms a family of strictly oscillatory solutions of eqn. (14).

It is evident that in eqn. (14) the term $(-1/2)^n \frac{n^3+4n^2+n-4}{n(n+1)(n+3)}$ tends to zero as n increases. If we assume that the solution y of (14) is bounded or even $y_n = o(n^2)$ then also the term $-\frac{n+2}{n(n+1)(n+3)} y_{n+2}$ tends to zero, so for large n eqn. (14) looks like $y_{n+1} + y_n = 0$ which obviously has oscillatory solutions. Therefore someone could suppose that if he has complicated equation it suffices to omit terms which tend to zero and from the behaviour of solutions of the reduced equations deduced similar behaviour for the solutions of the initial equation. Such reasoning may leads to quite wrong conclusions as the following example shows :

In the equation

$$y_{n+1} + \frac{1}{2} y_n = \frac{3n+1}{2n(n+1)}, \quad n \in \mathbb{N}$$

the term on the right hand side tends to zero. The reduced equation $y_{n+1} + \frac{1}{2} y_n = 0$ has the general solution $y_n = C\left(-\frac{1}{2}\right)^n$, $n \in \mathbb{N}$. For arbitrary constant $C \in \mathbb{R}$ we obtain a solution which oscillates and is bounded. On the other hand the general solution of the non-reduced equation is $y_n = C\left(-\frac{1}{2}\right)^n + \frac{1}{n}$. It is evident that there does not exist $C \in \mathbb{R}$ for which this solution would be oscillatory. The second example which show us that the above described reasoning can leads us to wrong conclusions is as follows.

The equation

$$y_{n+2} - 2y_{n+1} + y_n = -\frac{4n^2+10n+2}{2n^3+10n^2+13n+3} y_{n+1}$$
$$+ \frac{4n^3+18n^2+22n+6}{2n^4+14n^3+33n^2+29n+6} y_n$$

has the general solution given by the formula

$$y_n = C_1 \frac{n}{n+1} + C_n \frac{n+1}{n}, \quad n \in \mathbb{N}.$$

Therefore it is evident that every solution tends to some constant, say $y_n \to C$ as $n \to \infty$. Therefore

$$\lim_{n \to \infty} \left(- \frac{4n^2 + 10n + 2}{2n^3 + 10n^2 + 13n + 3} y_{n+1} + \frac{4n^3 + 18n^2 + 22n + 6}{2n^4 + 14n^3 + 33n^2 + 29n + 6} y_n \right) = 0.$$

On the other hand, the reduced equation

$$y_{n+2} - 2 y_{n+1} + y_n = 0$$

has (in generally) unbounded solutions $y_n = C_3 + C_4 n$, $n \in \mathbb{N}$.

References

1. A. Drozdowicz and J. Popenda, *Proc. Am. Math. Soc.* **99** (1987), 135-40.
2. J. Popenda and E. Schmeidel, *Publ. Mat.* **38** (1994), 3-9.