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SOME RESULTS ON NORMAL DIGRAPHS

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Normal digraphs have been investigated in several papers by Petrić, Torgašev and the author. A digraph is called normal if its adjacency matrix is normal. Let $G^{-}_{x_1, x_2, \ldots, x_k}$ be a digraph obtained from the normal digraph G by deleting some vertices $x_1, x_2, \ldots, x_k \in V(G)$. In this paper, we consider necessary and sufficient conditions under which the digraph $G^{-}_{x_1, x_2, \ldots, x_k}$ is also normal. Besides, we prove some other properties concerning normal digraphs.

In this paper we consider connected or disconnected digraphs without loops or multiple edges. The vertex set of a digraph G is denoted by $V(G)$, and its order (number of vertices) by $| G |$.

If $x, y \in V(G)$, then the edge xy is called simple if x is adjacent to y and y is not adjacent to x. If x is adjacent to y and y is adjacent to x, then the edge xy is called double. If xy is simple, then we write $x \rightarrow y$; if xy is double we write $x \leftrightarrow y$.

Let $A = A(G) = [a_{ij}]$ be $0 - 1$ adjacency matrix of G, where $a_{xy} = 1$ if x is adjacent to y, and $a_{xy} = 0$ otherwise. A digraph G is called normal if its adjacency matrix $A(G)$ is normal, i.e., if $AA' = A'A$, where A' is the transpose of A. G is called symmetric if its adjacency matrix is symmetric, i.e. if $A' = A$.

In view of an obvious one-to-one correspondence between symmetric digraphs and graphs, in this paper we consider only normal non-symmetric digraphs (proper normal digraphs, in short).

A graph theoretical interpretation of the normality condition is as follows.

Let x and y be any two (not necessarily distinct) vertices of a digraph G. A vertex $z \in V(G)$ is called common successor (shortly suc) of x and y, if z is adjacent to both x and y. Similarly, a vertex $z \in V(G)$ is called common predecessor

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(shortly prc) of x and y, if both x and y are adjacent to z. Let $\mathrm{suc}_G(x, y) = \mathrm{suc}(x, y)$ be the number of all common successors of x and y, and $\mathrm{prc}_G(x, y) = \mathrm{prc}(x, y)$ be the number of all common predecessors of x and y.

The following normality condition can be very useful in checking the normality of particular digraphs.

Proposition 1 (Petrić[6], Torgašev[7]) — A digraph G is normal if and only if $\mathrm{suc}(x, y) = \mathrm{prc}(x, y)$, for any two vertices $x, y \in V(G)$.

Let G be a graph of order n and let $A(G) = [a_{ij}]$ be its adjacency matrix. Denote by

$$a_i = (a_{i1}, a_{i2}, \ldots, a_{in}), \quad a_i' = (a_{1i}, a_{2i}, \ldots, a_{ni})$$

the corresponding ith row and ith column of the matrix $A(G)$ ($i = 1, 2, \ldots, n$). The normality condition of G exactly means that

$$\langle a_i, a_j \rangle = \langle a_i', a_j' \rangle \quad (i, j = 1, 2, \ldots, n) \qquad \ldots (1)$$

where $\langle x, y \rangle$ is the scalar product in an orthonormal basis in the space R^n.

Now, assume that G is a normal digraph of order n. Denote by $G^-_{x_1, x_2, \ldots, x_k}$ ($k < n$) induced subgraph of G which is obtained by deleting some vertices $x_1, x_2, \ldots, x_k \in V(G)$. It is natural to ask under which conditions digraph $G^-_{x_1, x_2, \ldots, x_k}$ is also normal.

Let $A = A(G)$ be the adjacency matrix of the normal digraph G in the form

$$\left[\begin{array}{cccc|cccc}
a_{11} & a_{12} & \ldots & a_{1k} & a_{1,k+1} & a_{1,k+2} & \ldots & a_{1n} \\
a_{21} & a_{22} & \ldots & a_{2k} & a_{2,k+1} & a_{2,k+2} & \ldots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
a_{k1} & a_{k2} & \ldots & a_{kk} & a_{k,k+1} & a_{k,k+2} & \ldots & a_{kn} \\
\hline
a_{k+1,1} & a_{k+1,2} & \ldots & a_{k+1,k} & b_{11} & b_{12} & \ldots & b_{1r} \\
a_{k+2,1} & a_{k+2,2} & \ldots & a_{k+2,k} & b_{21} & b_{22} & \ldots & b_{2r} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \ldots & a_{nk} & b_{r1} & b_{r2} & \ldots & b_{rr}
\end{array}\right] \qquad \ldots (2)$$

where $r = n - k$ and $b_{ij} = a_{k+i,k+j}$ ($i, j = 1, 2, \ldots, r$). Without loss of generality we can assume that $x_i = i$ ($i = 1, 2, \ldots, k$) and $V(G) = \{1, 2, \ldots, n\}$.

Denote by a_i the corresponding ith row (vector) of the matrix (2) and define the following n-dimensional vectors :

$$A_i = (a_{i1}, a_{i2}, ..., a_{ik}, \underbrace{0, 0, ..., 0}_{n-k}) \quad (i = 1, 2, ..., n),$$

$$B_i = (\underbrace{0, 0, ..., 0}_{k}, b_{i1}, b_{i2}, ..., b_{ir}) \quad (i = 1, 2, ..., r).$$

Assume that digraph $\overleftarrow{G}_{x_1, x_2, ..., x_k}$ is normal. By normality condition of the digraph G we find that

$$\langle a_i, a_j \rangle = \langle a_i', a_j' \rangle \qquad (i, j = k + 1, k + 2, ..., n).$$

Having in mind the forms of vectors A_i, B_i and the previous relation, we have that $\langle A_i + B_i, A_j + B_j \rangle = \langle A_i' + B_i', A_j' + B_j' \rangle$. The last relation can be written in the form

$$\langle A_i, A_j \rangle + \langle B_i, B_j \rangle = \langle A_i', A_j' \rangle + \langle B_i', B_j' \rangle. \qquad \ldots (3)$$

Since $\overleftarrow{G}_k = \overleftarrow{G}_{x_1, x_2, ..., x_k}$ is obtained by deleting vertices $x_1, x_2, ..., x_k \in V(G)$, it has adjacency matrix $A(\overleftarrow{G}_k) = [b_{ij}]_{r \times r}$ $(i, j = 1, 2, ..., r)$. Relation (3) and the normality condition (1) for $\overleftarrow{G}_k$ imply

$$\langle A_i, A_j \rangle = \langle A_i', A_j' \rangle \qquad (i, j = k + 1, k + 2, ..., n). \qquad \ldots (4)$$

As an immediate consequence of relation (4) we get the next result.

Proposition 2 — Let G be a normal digraph and assume that the adjacency matrix of G has the form (2). Then $\overleftarrow{G}_{x_1, x_2, ..., x_k}$ is a normal diagrah if and only if relation (4) is satified.

Proposition 3 — Let G be a normal digraph and $x_1 \in V(G)$. Then $\overleftarrow{G}_1 = \overleftarrow{G}_{x_1}$ is a normal digraph if and only if all edges from the vertex x_1 are double.

PROOF : Assume that $\overleftarrow{G}_1$ is normal. Suppose the adjacency matrix of G has the form (2). For the digraph $\overleftarrow{G}_1$ vectors A_i, A_i' have the forms

$$A_2 = (a_{21}), \quad A_2' = (a_{12}),$$

$$A_3 = (a_{31}), \quad A_3' = (a_{13}),$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$A_n = (a_{n1}), \quad A_n' = (a_{1n}).$$

By normality condition $\langle A_i, A_j \rangle = \langle A_i', A_j' \rangle$ $(i, j = 2, 3, ..., n)$ we find that $a_{i1} a_{j1} = a_{1i} a_{1j}$. Putting $i = j$ we have that $a_{i1}^2 = a_{1i}^2$, wherefrom we get $a_{i1} = a_{1i}$ $(i = 2, 3, ..., n)$.

Conversely, assume that all edges from the vertex x_1 are double, then $A_i = A_i'$ (i = 2, 3, ..., n). Using the normality condition (1) for G and relation (3) we easily find that digraph G_1^- is normal. □

By Proposition 3 we immediately get the next result.

Corollary 1 — Let G be a normal digraph. If all edges from the vertices $x_1, x_2, ..., x_k \in V(G)$ ($k < n$) are double, then $G^-_{x_1, x_2, ..., x_k}$ is a normal digraph.

For a digraph G let G_0 be the graph with the same vertex set as G and two vertices $x, y \in V(G)$ adjacent in G_0 if and only if x is adjacent to y or y is adjacent to x in G. G_0 is called the basic graph of G and G is called a digraph over G_0. G_0 is obviously connected if G is.

Proposition 4 — Let G be a normal digraph and vertices $x_1, x_2 \in V(G)$. If $G_2^- = G^-_{x_1, x_2}$ is normal and x_1, x_2 do not have the same neighbours in the basic graph G_0 into the set $V(G) \setminus \{x_1, x_2\}$, then all edges from x_1 and x_2 are double.

PROOF : Suppose the adjacency matrix of the digraph G has the form (2). For the digraph G_2^- vectors A_i, A_i' will have the forms

$$\begin{aligned} &A_3 = (a_{31}, a_{32}), \quad A_3' = (a_{13}, a_{23}), \\ &A_4 = (a_{41}, a_{42}), \quad A_4' = (a_{14}, a_{24}), \\ &\vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\quad \vdots \\ &A_n = (a_{n1}, a_{n2}), \quad A_n' = (a_{1n}, a_{2n}). \end{aligned}$$

Since by assumption x_1, x_2 do not have the same neighbours in the set $V(G) \setminus \{x_1, x_2\}$, it follows that there exists at least one vertex $x_i = x_{i_0}$ in the induced normal digraph $G_2^- = G^-_{x_1, x_2}$ such that x_1 is adjacent to x_{i_0} and x_2 is not adjacent to x_{i_0}. The corresponding vectors (i_0th row and i_0th column) for the vertex x_{i_0} are $A_{i_0} = (a_{i_0 1}, 0)$, $A_{i_0}' = (a_{1 i_0}, 0)$. Using relation (4) we have that $\langle A_{i_0}, A_{i_0} \rangle = \langle A_{i_0}', A_{i_0}' \rangle$, whence we easily find that $a_{i_0 1} = a_{1 i_0} = 1$, that is $A_{i_0} = A_{i_0}' = (1, 0)$.

Now, we assume that vertex x_1 contains at least one simple edge, say $x_1 x_{j_0}$ or $x_{j_0} x_1$. Without loss of generality we can suppose that the edge $x_1 x_{j_0}$ is present in G. Consider the case when $x_{j_0} \neq x_2$, i.e. $x_{j_0} \in V(G_2^-) = V(G) \setminus \{x_1, x_2\}$. Corresponding vectors for the vertex x_{j_0} are $A_{j_0} = (a_{j_0 1}, a_{j_0 2}) = (0, a_{j_0 2})$ and $A_{j_0}' = (a_{1 j_0}, a_{2 j_0}) = (1, a_{2 j_0})$.

By relation $\langle A_{j_0}, A_{j_0} \rangle = \langle A_{j_0}', A_{j_0}' \rangle$ we get that $0 + a_{j_0 2} = 1 + a_{2 j_0}$. From the last relation we find as the unique solution $a_{j_0 2} = 1$ and $a_{2 j_0} = 0$, that is $A_{j_0} = (0, 1)$ and $A_{j_0}' = (1, 0)$. By the scalar product of vectors A_{i_0}, A_{j_0} we have

$$\langle A_{i_0}, A_{j_0} \rangle = \langle A'_{i_0}, A'_{j_0} \rangle,$$

wherefrom $\langle (1, 0), (0, 1) \rangle = \langle (1, 0), (1, 0) \rangle$, i.e. $0 = 1$, which is a contradiction. Consequently, all edges $x_1 x_{j_0} (x_{j_0} \neq x_2)$ are double in G.

Finally, assume that vertex x_1 contains a simple edge $x_1 x_2$. The corresponding x_1th row and x_1th column of vertex x_1 will be $a_1 = (0, 1, a_{13}, a_{14}, ..., a_{1n})$ and $a'_1 = (0, 0, a_{31}, a_{41}, ..., a_{n1})$, respectively. Since x_1 has all double edges in the set $V(\bar{G_2})$, we get that $a_{1i} = a_{i1}$ $(i = 3, 4, ..., n)$. Whence we immediately get that $\langle a_1, a_1 \rangle \neq \langle a'_1, a'_1 \rangle$, what is a contradiction to normality of G.

Therefore, all edges corresponding to x_1 are double. In a similar way we conclude that all edges corresponding to x_2 are also double. □

Next, a vertex $x \in V(G)$ is called simple if x contains only double edges in G; otherwise x is called mixed. Let G be a normal digraph (connected or disconnected). Denote by $V^0(G) \subseteq V(G)$ the set of all simple vertices in G. Let $G^+_{x_1, x_2, ..., x_k}$ be an induced overgraph of G obtained by adding new vertices $x_1, x_2, ..., x_k$ and joining these to some vertices of the set $V^0(G)$ by double edge. Finally, denote by G^+ a digraph which is obtained from G by adding only double edges between simple vertices. Similarly, denote by G^- a digraph which is obtained from G by deleting only double edges between simple vertices.

Proposition 5 — Let G be a normal digraph and $A = [a_{ij}]$ be the corresponding adjacency matrix of G. Then G^+, G^- and $G^+_{x_1, x_2, ..., x_k}$ are also normal digraphs.

PROOF : First, we shall prove that G^+ and G^- are normal. Denote by $A^0 = [a^0_{ij}]_{n_0 \times n_0}$ the adjacency matrix of induced subgraph of G obtained by deleting all simple vertices from G. Besides, denote by $C = [c_{ij}]_{l \times l}$ the adjacency matrix of induced subgraph of G obtained by deleting all mixed vertices from G. It is obvious that matrix $C = [c_{ij}]$ is symmetric and $|G| = n_0 + l$. By a suitable numeration of vertices of G, the corresponding adjacency matrices A and A' will take the forms

$$\begin{bmatrix} A^0_{n_0 \times n_0} & B_{n_0 \times l} \\ B'_{l \times n_0} & C_{l \times l} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} (A^0_{n_0 \times n_0})' & B_{n_0 \times l} \\ B'_{l \times n_0} & C_{l \times l} \end{bmatrix},$$

respectively.

Using the normality condition $AA' = A'A$ for G, we immediately get the following relation

$$\begin{bmatrix} A(A^0)' + BB' & A^0B + BC \\ B'(A^0)' + CB' & B'B + CC \end{bmatrix} = \begin{bmatrix} (A^0)'A^0 + BB' & (A^0)'B + BC \\ B'A^0 + CB' & B'B + CC \end{bmatrix}.$$

The last matrix equation yields the relations

$$A^0 (A^0)' = (A^0)' A^0 \qquad \ldots (5)$$

$$A^0 B = (A^0)' B. \qquad \ldots (6)$$

It is also easy to see that G is a normal if and only if relations (5) and (6) are satisfied.

Since G^+ and G^- have the same corresponding matrices A^0 and B as G, by (5) and (6) we easily see that G^+ and G^- are also normal.

Now, we shall prove that $G_k^+ = G^+_{x_1, x_2, \ldots, x_k}$ is also a normal digraph. For digraph G_k^+, we can define matrices A_k^0 and C_k in the same way, as matrices A^0 and C (respectively) for the normal digraph G in the previous case.

It is clear that the corresponding matrix A_k^0 of G_k^+ coincides to the matrix A^0 of G. Since G is a normal, it follows that relation (5) is obeyed for the digraph G_k^+. In order to prove the normality of G_k^+ it is sufficient to show that relation (6) is also fulfilled.

Indeed, the corresponding matrices B_k and C_k take the forms $[B_{n_0 \times l}, O_{n_0 \times k}]$ and $[c_{ij}^k]_{(l+k)\times(l+k)}$ respectively, where $O_{n_0 \times k}$ is all-zero matrix of order $n_0 \times k$. The relation $A_k^0 B_k = (A_k^0)' B_k$ is equivalent to the following matrix equation

$$[(A^0 B)_{n_0 \times l},\ O_{n_0 \times k}] = [((A^0)' B)_{n_0 \times l}, O_{n_0 \times k}],$$

wherefrom we get the statement. □

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