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GENERALIZED VECTOR VARIATIONAL-LIKE INEQUALITIES ON LOCALLY CONVEX HAUSDORFF TOPOLOGICAL VECTOR SPACES

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In this paper, we prove some existence theorems of solutions for generalized vector variational-like inequalities (GVVLI) for multifunctions in the setting of locally convex Hausdorff topological vector spaces, by using a particular form of the generalized KKM theorem due to Park. Several special cases of our inequalities are also discussed.

1. INTRODUCTION AND PRELIMINARIES

In the last twenty years, variational inequalities for numerical functions, which was originated from Hartmann and Stampacchia[9], have made much greatly developments in the theory and applications.

Recently, Giannessi[8] firstly introduced the vector variational inequalities in a finite-dimensional Euclidean space. Since then, Chen *et al.*[3-7], Lee *et al.*[11-14], Lee *et al.*[15], Park *et al.*[19], Siddiqi *et al.*[21] and Yang[22] have investigated vector variational inequalities in abstract spaces. In particular, Lee *et al.*[11,14], Lee *et al.*[15] and Chen *et al.*[4] introduced and studied vector variational inequalities for multifunctions with vector values.

Very recently, Ansari[1] introduced and investigated vector variational-like inequalities for multifunctions with vector values in reflexive Banach spaces under the η-pseudomonotonicity and V-hemicontinuity of multifunctions.

Our motivation of this paper is to consider vector variational-like inequalities in locally convex Hausdorff topological vector spaces.

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In this paper, we establish some existence theorems of solutions for the following generalized vector variational-like inequality (GVVLI) in the setting of locally convex Hausdorff topological vector spaces. Several special cases of our inequality are also discussed. In section 2, we establish some existence theorems for (GVVLI) on locally convex Hausdorff topological vector spaces under coercivity conditions, by using the upper semicontinuity of the multifunction T. Also, we obtain the existence theorem for (GVVLI) on a semireflexive locally convex Hausdorff topological vector space, under some conditions which are slightly different from those of Ansari[1], by using the η-pseudomonotonicity and V-hemicontinuity of the multifunction T. Let X and Y be two locally convex Hausdorff topological vector spaces and D a nonempty closed and convex subset of X. Let $T: X \to 2^{L(X, Y)}$ be a multifunction, where $L(X, Y)$ is the space of all continuous linear operators from X into Y, and $\eta : D \times D \to X$ a function. Let $\{C(x) \mid x \in D\}$ be a family of convex cones in Y such that $\mathrm{Int}C(x) \neq \phi$, $\forall x \in D$, where Int denotes the interior.

Consider the following generalized vector variational-like inequality (GVVLI) for multifunctions with vector values, which was firstly considered by Ansari[1]; (GVVLI) Find $x_0 \in D$ such that for each $x \in D$, there exists an $s_0 \in T(x_0)$ such that

$$\langle s_0, \eta(x, x_0)\rangle \notin -\mathrm{Int}\, C(x_0),$$

where $\langle s_0, y\rangle$ denotes the evaluation of s_0 at y.

Special Cases

(1) If $\eta(x, x_0) = x - x_0$, then (GVVLI) reduces to the following generalized vector variational inequality (GVVI) for multifunctions with vector values, which was studied by Lee *et al.*[15];

(GVVI) Find $x_0 \in D$ such that for each $x \in D$, there exists an $s_0 \in T(x_0)$ such that

$$\langle s_0, x - x_0\rangle \notin -\mathrm{Int}\, C(x_0).$$

(2) If $\eta(x, x_0) = x - x_0$ and $C(x)$ is a constant cone C for any $x \in D$, then (GVVLI) becomes the following generalized vector variational inequality (GVVI)′ for multifunctions with vector values, which was studied by Lee *et al.*[11];

(GVVI)′ Find $x_0 \in D$ such that for each $x \in D$, there exists an $s_0 \in T(x_0)$ such that

$$\langle s_0, x - x_0\rangle \notin -\mathrm{Int}\, C.$$

(3) If T is single valued, then (GVVLI) reduces to the following vector variational-like inequality (VVLI) for vector-valued functions, which is more generalized than the vector variational-like inequality studied by Yang[22] and Lee *et al.*[13];

(VVLI) Find $x_0 \in D$ such that for any $x \in D$,

$$\langle T(x_0), \eta(x, x_0)\rangle \notin -\mathrm{Int}\, C(x_0).$$

(4) In $\eta(x, x_0) = x - x_0$, then (VVLI) becomes the following vector variational inequality (VVI) for vector-valued functions, which was studied by Chen[3], Chen and Yang[7] and Lee *et al.*[12];

(VVI) Find $x_0 \in D$ such that for any $x \in D$,

$$\langle T(x_0), x - x_0 \rangle \notin -\operatorname{Int} C(x_0).$$

Now we give the definition of a KKM multifunction.

Definition 1.1 — Let D be a subset of a topological vector space X. Then a multifunction $G: D \to 2^X$ is called KKM if for each nonempty finite subset N of D, $co\, N \subset G(N)$, where co denotes the convex hull and $G(N) = \bigcup \{Gx : x \in N\}$.

A convex space X is a nonempty convex set (in a vector space) with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. Thus, a convex subset D of a topological vector space X with the relative topology is automatically a convex space. For details of the convex space, see Lassonde[10].

We need the following particular form of the generalized KKM theorems due to Park[16-18], which will be used in the proof of our theorems.

Theorem 1.1 — Let D be a nonempty subset of a convex space X, and $G: D \to 2^X$ a KKM multifunction. Suppose that

(1) for each $x \in X$, $G(x)$ is closed; and

(2) there exists a nonempty compact subset K of X such that for each nonempty finite subset N of D, there exists a compact convex subset L_N of X such that $N \subset L_N$ and $L_N \bigcap\bigcap \{G(x) : x \in L_N \bigcap D\} \subset K$.

Then we have

$$K \bigcap\bigcap \{G(x) : x \in D\} \neq \phi.$$

In particular, if $X = K$, that is, X is a compact convex space, then the coercivity condition (2) is obviously held, and hence in this case, without the condition (2), it is true that $\bigcap \{G(x) : x \in D\} \neq \phi$.

2. Existence Theorems

First, we give the following definitions for the existence theorems for (GVVI).

Definition 2.1 — Let F be a multifunction from a topological space X into a topological space Y.

(1) F is said to be [respectively, weakly] closed at $x \in X$ if for each nets $\{x_\alpha\}$ [resp., weakly] converging to x and $\{y_\alpha\}$[resp., weakly] converging to y such that $y_\alpha \in F(x_\alpha)$ for all α, we have $y \in F(x)$. F is said to be [respectively, weakly] closed if it is [resp., weakly] closed at every point $x \in X$.

(2) F is said to be upper semi-continuous at $x \in X$ if for every open set V in Y containing $F(x)$, there exists a neighbourhood $N(x)$ of x such that $F(z) \subset V$ for all

$z \in N(x)$. F is said to be upper semi-continuous if it is upper semi-continuous at every $x \in X$.

(3) F is said to be compact if $F(X)$ is contained in a compact subset of Y.

Definition 2.2 — Let X and Y be two Hausdorff topological vector spaces, $T: X \to L(X, Y)$ an operator, $\{C(x): x \in D\}$ a family of convex cones in Y such that Int $C(x) \neq \phi$ and $P := \bigcap_{x \in D} C(x)$ is nonempty. Let $\eta: D \times D \to X$ be a function.

(1) T is said to be $P-\eta$-monotone over D if for any $x, y \in D, \langle T(x) - T(y), \eta(x, y)\rangle \in P$.

(2) T is said to be η-pseudomonotone over D if for any $x, y \in D, \langle T(x), \eta(y, x)\rangle \notin -$ Int $C(x)$ implies that $\langle T(y), \eta(y, x.)\rangle \notin -$ Int $C(x)$.

(3) T is said to be V-hemicontinuous if for any $x, y, z \in X$, the mapping $\alpha \mapsto \langle T(x + \alpha y), z\rangle$ is continuous at 0^+.

Definition 2.3 — Let X and Y be two Hausdorff topological vector spaces, $T: X \to 2^{L(X, Y)}$ a multifunction, $\{C(x): x \in D\}$ a family of convex cones in Y and $P := \bigcap_{x \in D} C(x)$ nonempty. Let $\eta: D \times D \to X$ be a function.

(1) T is said to be $P-\eta$-monotone over D if for any $x, y \in D, s \in T(x)$ and $t \in T(y), \langle s - t, \eta(x, y)\rangle \in P$.

(2) T is said to be η-pseudomonotone over D if for any $x, y \in X, \langle s, \eta(y, x)\rangle \notin -$ Int $C(x)$ for some $s \in T(x)$ implies that $\langle t, \eta(y, x)\rangle \notin -$ Int $C(x)$ for some $t \in T(y)$.

(3) T is said to be V-hemicontinuous if for any $x, y \in X, \alpha > 0$ and $t_\alpha \in T(x + \alpha y)$, there exists a $t_0 \in T(x)$ such that for any $z \in X, \langle t_\alpha, z\rangle \mapsto \langle t_0, z\rangle$ as $\alpha \to 0^+$.

Now we give the following assumptions needed in proving our main theorems; Let X and Y be locally convex Hausdorff topological vector spaces, D a nonempty closed and convex subset of X and $\eta: D \times D \to X$ a function.

Assumption (A)

(1) η is affine in the first variable;

(2) η is continuous in the second variable;

(3) for each $y \in D, \eta(\{y\} \times D)$ is a bounded subset of X;

(4) for any $x \in D, \eta(x, x) = 0$.

Assumption $(A)_w$

(1) η is affine in the first variable;

(2) η is weakly continuous in the second variable;

(3) for each $y \in D, \eta(\{y\} \times D)$ is a bounded subset of X;

(4) for any $x \in D, \eta(x, x) = 0$.

Now we prove the following existence theorems for (GVVLI).

Theorem 2.1—Let X and Y be locally convex Hausdorff topological vector spaces, D a nonempty closed and convex subset of X, and $C : D \to 2^Y$ a multifunction such that for each $x \in D$, $C(x)$ is a convex cone in Y with Int $C(x) \neq \phi$ and $C(x) \neq Y$. Let $W : D \to 2^Y$ be a closed multifunction defined by $W(x) = Y \setminus (-\text{Int } C(x))$ for any $x \in D$, and $L(X, Y)$ be equipped with the topology of bounded convergence.

Suppose that $\eta : D \times D \to X$ satisfies the assumption (A) and $T : X \to 2^{L(X,Y)}$ is a compact and upper semi-continuous multifunction with compact values, and T satisfies the following coercivity condition (C_1);

(C_1) there exists a nonempty compact subset K of X such that for each nonempty finite subset N of D, there exists a compact convex subset L_N of X containing N such that for each $x \in L_N \setminus K$, there exists $y \in L_N \cap D$ satisfying

$$\langle s, \eta(y, x)\rangle \in -\text{Int } C(x) \text{ for all } s \in T(x).$$

Then (GVVLI) is solvable.

PROOF : Define a multifunction $F_1 : D \to 2^X$ by

$$F_1(y) = \{x \in D : \langle s, \eta(y, x)\rangle \notin -\text{Int } C(x) \text{ for some } s \in T(x)\}$$

for $y \in D$. Then F_1 is a KKM multifunction on D.

In fact, suppose that $N = \{x_1, \ldots, x_n\} \subset D$, $\Sigma_{i=1}^n \alpha_i = 1$, $\alpha_i \geq 0$, $i = 1, \ldots, n$ and $x = \Sigma_{i=1}^n \alpha_i x_i \notin F_1(N)$. Then for any $s \in T(x)$, we have $\langle s, \eta(x_i, x)\rangle \in -\text{Int } C(x)$, $i = 1, \ldots, n$. Thus we have

$$\langle s, \eta(x, x)\rangle = \langle s, \eta\left(\sum_{i=1}^n \alpha_i x_i, x\right)\rangle = \sum_{i=1}^n \alpha_i \langle s, \eta(x_i, x)\rangle \in -\text{Int } C(x).$$

Hence $0 \in \text{Int } C(x)$, which contradicts the assumption $C(x) \neq Y$. Therefore, F_1 is a KKM multifunction on D. We claim that F_1 is closed-valued. In fact, let $\{x_\alpha\}$ be a net in $F_1(y)$ converging to $x_* \in D$ for any fixed $y \in D$. Since $x_\alpha \in F_1(y)$ for all α, there exists an $s_\alpha \in T(x_\alpha)$ such that

$$\langle s_\alpha, \eta(y, x_\alpha)\rangle \in W(x_\alpha) \text{ for all } \alpha. \quad \ldots (2.1)$$

On the other hand, since T is a compact multifunction, the closure $\overline{T(X)}$ of $T(X)$ is a compact subset of $L(X, Y)$.

Hence we can assume that there exists an $s_* \in L(X, Y)$ such that $\{s_\alpha\}$ converges to s_*. Since T is upper semi-continuous and compact-valued, it is closed(Berge[2]), and so $s_* \in T(x_*)$. Now we consider the following equality;

$$\langle s_\alpha, \eta(y, x_\alpha)\rangle - \langle s_*, \eta(y, x_*)\rangle = \langle s_\alpha - s_*, \eta(y, x_\alpha)\rangle + \langle s_*, \eta(y, x_\alpha) - \eta(y, x_*)\rangle.$$

By the assumption (A), the net $\{\eta(y, x_\alpha)\}$ is bounded. Since the $\{s_\alpha\}$ converges to

s_* and $L(X,Y)$ is equipped with the topology of bounded convergence, the net $\{s_\alpha\}$ converges to s_* uniformly on $\{\eta(y, x_\alpha)\}$, and hence the net

$$\{\langle s_\alpha - s_*, \eta(y, x_\alpha)\rangle\} \text{ converges to } 0.$$

On the other hand, since s_* is continuous, by the assumption (A), the net

$$\{\langle s_*, \eta(y, x_\alpha) - \eta(y, x_*)\rangle\} \text{ converges to } 0.$$

Hence the net

$$\{\langle s_\alpha, \eta(y, x_\alpha)\rangle\} \text{ converges to } \langle s_*, \eta(y, x_*)\rangle.$$

By the closedness of the multifunction W, from(2.1) we have $\langle s_*, \eta(y, x_*)\rangle \in W(x_*)$, hence $x_* \in F_1(y)$. Thus $F_1(y)$ is closed for any $y \in D$.

Further, note that the coercivity condition (C_1) implies that for each $x \in L_N \setminus K$, there exists $y \in L_N \cap D$ such that $x \notin F_1(y)$. Hence

$$L_N \cap \bigcap \{F_1(y) : y \in L_N \cap D\} \subset K.$$

Thus by Theorem 1.1, we obtain an $x_0 \in \bigcap \{F_1(y) : y \in D\}$. Hence there exists an $x_0 \in D$ such that for each $x \in D$, there exists an $s_0 \in T(x_0)$ such that

$$\langle s_0, \eta(x, x_0)\rangle \notin -\operatorname{Int} C(x_0).$$

Corollary 2.1 — In Theorem 2.1, the coercivity condition (C_1) can be replaced by the following condition $(C_1)'$ without affecting its conclusion :

$(C_1)'$ there exist a nonempty compact subset K of X and a $y \in K \cap D$ such that $\langle s, \eta(y, x)\rangle \in -\operatorname{Int} C(x)$ for any $x \in D \setminus K$ and $s \in T(x)$.

PROOF : It sufficies to show that $(C_1)'$ implies (C_1). In fact, for any nonempty finite subset N of D, we let $L_N = co(N \cup (K \cap D)) \subset D$. By $(C_1)'$, for any $x \in L_N \setminus K \subset D \setminus K$, there exists a $y \in K \cap D \subset L_N \cap D$ such that

$$\langle s, \eta(y, x)\rangle \in -\operatorname{Int} C(x) \quad \text{for all } s \in T(x).$$

Hence (C_1) holds.

In fact, it is sufficient to show the compactness of $\overline{T(D)}$ instead of $\overline{T(X)}$ discussed in the proof of Theorem 2.1. Hence for $D = K$ in Theorem 2.1, we can obtain the following result;

Theorem 2.2 — Let X and Y be two locally convex Hausdorff topological vector spaces, D a nonempty compact convex subset of X. Let $C : D \to 2^Y$ be a multifunction such that for each $x \in D$, $C(x)$ is a convex cone in Y with $\operatorname{Int} C(x) \neq \phi$ and $C(x) \neq Y$. Let $W : D \to 2^Y$ be a closed multifunction defined by $W(x) = Y \setminus (-\operatorname{Int} C(x))$ for any $x \in D$. If $\eta : D \times D \to X$ satisfies the assumption (A) and $T : X \to 2^{L(X,Y)}$ is upper semi-continuous and compact-valued, then (GVVLI) is solvable.

Theorem 2.3 — Let X be a semireflexive locally convex Hausdorff topological vector space, Y a locally convex Hausdorff topological space, and D a nonempty bounded, closed and convex subset of X. Let $C : D \to 2^Y$ be a multifunction such that for each $x \in D$, $C(x)$ is a convex cone in Y with Int $C(x) \neq \phi$ and $C(x) \neq Y$, and $P := \bigcap_{x \in D} C(x)$ a nonempty convex cone in Y with Int $P \neq \phi$. Let $W : D \to 2^Y$ be a weakly closed multifunction defined by $W(x) = Y \setminus (-\text{ Int } C(x))$ for any $x \in D$, and $T : X \to 2^{L(X, Y)}$ a multifunction, where $L(X, Y)$ is equipped with the topology of bounded convergence.

Suppose that $\eta : D \times D \to X$ satisfies the assumption $(A)_w$, and T is η-pseudomonotone over D, compact-valued and V-hemicontinuous.

Then (GVVLI) is solvable.

PROOF : Define a multifunction $F_1 : D \to 2^X$ by

$$F_1(y) = \{x \in D : \langle s, \eta(y, x) \rangle \notin -\text{Int } C(x) \text{ for some } s \in T(x)\} \text{ for } y \in D.$$

Then by the same argument as the proof in Theorem 2.1, F_2 is a KKM multifunction on D. Define a multifunction $F_2 : D \to 2^D$ by

$$F_2(y) = \{x \in D : \langle t, \eta(y, x) \rangle \notin -\text{Int } C(x) \text{ for some } t \in T(y)\} \text{ for } y \in D.$$

Then F_2 is also a KKM multifunction on D. In fact, for any $x \in F_1(y)$, there exists an $s \in T(x)$ such that $\langle s, \eta(y, x) \rangle \notin -$ Int $C(x)$. By the η-pseudomonotonicity of T, there exists $t \in T(y)$ such that

$$\langle t, \eta(y, x) \rangle \notin -\text{ Int } C(x).$$

Thus $x \in F_2(y)$. Hence $F_1(y) \subset F_2(y)$ for any $y \in D$. Therefore F_2 is also a KKM multifunction on D.

On the other hand, since X is semireflexive, and D is a bounded, closed and convex subset of X, D is weakly compact[20]. Now we calim that F_2 is weakly closed-valued. Indeed, for any fixed $y \in D$, let $\{x_\alpha\}$ be a net in $F_2(y)$ which converges weakly to $x_* \in D$. Since $x_\alpha \in F_2(y)$ for each α, there exists a $t_\alpha \in T(y)$ such that

$$\langle t_\alpha, \eta(y, x_\alpha) \rangle \in W(x_\alpha) \quad \text{for all } \alpha. \qquad \ldots (2.2)$$

Since $T(y)$ is compact, we may assume that $\{t_\alpha\}$ converges to some $t_* \in T(y)$. Note that

$$\langle t_\alpha, \eta(y, x_\alpha) \rangle - \langle t_*, \eta(y, x_*) \rangle = \langle t_\alpha - t_*, \eta(y, x_\alpha) \rangle - \langle t_*, \eta(y, x_\alpha) - \eta(y, x_*) \rangle.$$

Let $B = \{\eta(y, x_\alpha)\}$. By the assumption $(A)_w$, B is bounded. Since $L(X, Y)$ is equipped with the topology of bounded convergence,

$$\text{for any } y^* \in Y^*, \quad \sup_{x \in B} |y^*(\langle t_\alpha - t_*, x \rangle)| \to 0,$$

and hence

$$| y^*(\langle t_\alpha - t_*, \eta(y, x_\alpha)\rangle) | \to 0.$$

Since t_* is weakly continuous,

$$\text{for any } y^* \in Y^*, \quad | y^*(\langle t_*, \eta(y, x_\alpha) - \eta(y, x_*)\rangle) | \to 0.$$

Hence $\{\langle t_\alpha, \eta(y, x_\alpha)\rangle\}$ converges to $\langle t_*, \eta(y, x_*)\rangle$ weakly. Since W is weakly closed, by (2.2) we have $\langle t_*, \eta(y, x_*)\rangle \in W(x_*)$. Hence $\langle t_*, \eta(y, x_*)\rangle \notin -\text{Int } C(x_*)$, and thus we have $x_* \in F_2(y)$.

Thus, by Theorem 1.1, there exists an $x \in \bigcap \{F_2(y) : y \in D\}$. Then for any $y \in D$, there exists a $t_y \in T(y)$ such that $\langle t_y, \eta(y, x)\rangle \notin -\text{Int } C(x)$. By the convexity of D, for any $\alpha \in (0, 1)$, there exists a $t_\alpha \in T(\alpha y + (1-\alpha)x)$ such that $\langle t_\alpha, \eta(\alpha y + (1-\alpha)x, x\rangle \notin -\text{Int } C(x)$. Since $\eta(\cdot, x)$ is affine and $\eta(x, x) = 0$, we have

$$\langle t_\alpha, \eta(\alpha y + (1-\alpha)x, x)\rangle = \alpha\langle t_\alpha, \eta(y, x)\rangle + (1-\alpha)\langle t_\alpha, \eta(x, x)\rangle = \alpha\langle t_\alpha, \eta(y, x)\rangle.$$

Therefore, $\alpha\langle t_\alpha, \eta(y, x)\rangle \notin -\text{Int } C(x)$. Dividing by α, we have $\langle t_\alpha, \eta(y, x)\rangle \notin -\text{Int } C(x)$. By the V-hemicontinuity of T, there exists a $t_0 \in T(x)$ such that $\langle t_0, \eta(y, x)\rangle \notin -\text{Int } C(x)$. Hence $x \in \bigcap \{F_1(y) : y \in D\}$. Thus $\bigcap \{F_1(y) : y \in D\} \neq \phi$. Consequently, there exists an $x_0 \in D$ such that for each $x \in D$, there exists an $s_0 \in T(x_0)$ such that $\langle s_0, \eta(x, x_0)\rangle \notin -\text{Int } C(x_0)$.

When X and Y are Banach spaces, we can obtain the following corollary under some assumptions which are different from those of Theorem 2.1 in Ansari[1].

Corollary 2.2 — Let X be a reflexive Banach space, Y a Banach space, and D a nonempty bounded, closed and convex subset of X. Let $C : D \to 2^Y$ be a multifunction such that for each $x \in D$, $C(x)$ is a convex cone in Y with $\text{Int } C(x) \neq \phi$ and $C(x) \neq Y$, and $P := \bigcap_{x \in D} C(x)$ a nonempty convex cone in Y with $\text{Int } P \neq \phi$. Let $W : D \to 2^Y$ be a weakly closed multifunction defined by $W(x) = Y \setminus (-\text{Int } C(x))$ for any $x \in X$, and $T : X \to 2^{L(X, Y)}$ a multifunction. Suppose that $\eta : D \times D \to X$ satisfies the assumption $(A)_w$ except (3), and $T : X \to 2^{L(X, Y)}$ is η-pseudomonotone over D, compact-valued and V-hemicontinuous, then (GVVLI) is solvable.

Remark : Ansari[1] obtained the conclusion of Corollary 2.2 under the concavity and upper semi-continuity of the multifunction W, and the continuity and affinity of η.

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