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GENERALIZED VARIATIONAL-TYPE INEQUALITY IN HAUSDORFF TOPOLOGICAL VECTOR SPACES

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Browder and Isac have studied the variational inequality problem in reflexive real Banach spaces and in Hausdorff topological vector spaces respectively. The authors have already obtained a generalization of the variational inequality problem studied by Browder. In this note the authors have presented a certain generalization of the variational inequality problem proved by Isac.

1. INTRODUCTION

Let X be a reflexive real Banach space and let X^* be its dual endowed with weak* topology. Let the value of $f \in X^*$ at $x \in X$ be denoted by (f, x). Let K be a closed convex set in X, with $0 \in K$.

The following result is proved by Browder[3] (also see Chipot[5] and Mosco[10]).

Theorem 1.1 — Let T be a monotone and hemicontinuous map of a closed convex set K in X, with $0 \in K$, into X^*, and if K is not bounded, let T be coercive on K. Then there exists an $x_0 \in K$ such that

$$(Tx_0, y - x_0) \geq 0 \qquad \ldots (1)$$

for all $y \in K$.

The inequalities of the form (1) are called 'variational inequalities' and applicable to many cases of general interest.

The authors[2] have generalized the above theorem in the following way.

Theorem 1.2 — Let K be a closed convex set in a reflexive real Banach space X, with $0 \in K$, and X^* be the dual of X. Let $T : K \to X^*$ and $\theta : K \times K \to X$ be two continuous maps such that

(i) $(Ty, \theta(y, y)) = 0$ for all $y \in K$,

(ii) for each fixed $y \in K$, the function

$$(Ty, \theta(-, y)) : K \to \mathbb{R}$$

is convex.

Then there exists $x_0 \in K$ such that

$$(Tx_0, \theta(y, x_0)) \geq 0 \qquad \ldots (2)$$

for all $y \in K$, under each of the following conditions :

(a) For at least one $r > 0$, there exists $u \in D_r^0$ such that

$$(Ty, \theta(u, y)) \leq 0$$

for all $y \in S_r$, where

$$D_r = \{x \in K : \| x \| \leq r\}$$

$$D_r^0 = \{x \in K : \| x \| < r\}$$

$$S_r = \{x \in K : \| x \| = r\}.$$

(b) There exist a nonempty, compact and convex subset C of K and $u \in C$ such that

$$(Ty, \theta(u, y)) < 0$$

for every $y \in K - C$.

Remark 1.3 : If $\theta(y, x_0) = y - x_0$, then (2) reduces to (1).

The following result, which is parallel to Theorem 1.1, is proved by Isac[8] (Theorem 4.3.2, p. 116).

Theorem 1.4 — Let K be a nonempty convex subset of a Hausdorff topological vector space X and let X^* be the dual of X. Let $T : K \to X^*$ be a mapping such that

(i) $x \mapsto (Tx, y - x)$ is upper semicontinuous on K for every $y \in K$,

(ii) there exist a nonempty, compact and convex subset $C \subset K$ and $u \in C$ such that

$$(Ty, u - y) < 0$$

for all $y \in K - L$.

Then there exists $x_0 \in C$ such that

$$(Tx_0, y - x_0) \geq 0 \qquad \ldots (3)$$

for all $y \in K$.

The following theorem on a special variational inequality problem is studied by Isac[8].

Theorem 1.5 (Isac[8], Proposition 6.2.2, p. 170) — Let K be a nonempty compact convex set in a Hausdorff topological vector space and $T : K \to X^*$ and $f : K \to K$ be continuous maps such that

$$(Tx, x - f(x)) \geq 0$$

for all $x \in K$. Then there exists an $x_0 \in K$ such that

$$(Tx_0, y - f(x_0)) \geq 0 \qquad \ldots (4)$$

for all $y \in K$.

The following theorem on variational inequality problem is also studied by Browder[4] (also see Mosco[10]).

Theorem 1.6 — Let X be a reflexive real Banach space with dual X^*. Let $T: X \to X^*$ be a monotone and hemicontinuous operator and $f: X \to \mathbb{R}$ be a convex and lower semicontinuous map with $f(0) = 0$ such that

$$\frac{(Tx, x) + f(x)}{\| x \|} \to \infty \text{ as } \| x \| \to \infty.$$

Then there exists an $x_0 \in X$ such that

$$(Tx_0, y - x_0) + f(y) - f(x_0) \geq 0 \qquad \ldots (5)$$

for all $y \in X$.

In this paper, our aim is to obtain a certain generalization (see Theorem 2.2, below) of Theorem 1.4. Also we obtain the inequalities (1)-(5) as particular cases of Theorem 2.2.

In the proof of Theorem 1.4, Isac has used the concept of a KKM-map and applied a result of Fan[7] (also see Isac[8], Theorem 4.3.1, p. 116). However we do not use the concept of a KKM-map while proving Theorem 2.2. We use the following result of Fan[7] (also see Isac[8], Theorem 6.2.1, p. 170) for obtaining a generalization of Theorem 1.4.

Theorem 1.7 — Let K be a nonempty compact convex set in a Hausdorff topological vector space X. Let L be a subset of $K \times K$ having the following properties :

(i) For each $x \in K$, $(x, x) \in L$.

(ii) For each fixed $y \in K$, the set

$$L(y) = \{x \in K : (x, y) \in L\}$$

is closed in K.

(iii) For each $x \in K$, the set

$$M(x) = \{y \in K : (x, y) \notin L\}$$

is convex.

Then there exists an $x_0 \in K$ such that

$$\{x_0\} \times K \subset L.$$

2. Main Results

The following theorems are the main results of this paper.

Theorem 2.1 — Let K be a nonempty compact convex subset of a Hausdorff topological vector space X. Let $T: K \to X^*$, $\theta: K \times K \to X$ and $\eta: K \times K \to \mathbb{R}$ be three maps such that

(a) $(Tx, \theta(x, x)) + \eta(x, x) \geq 0$ for each $x \in K$,

(b) the map

$$y \mapsto (Tx, \theta(y, x)) + \eta(x, y)$$

of K into $\mathbb{R}$ is convex for each $x \in K$,

(c) the map

$$x \mapsto (Tx, \theta(y, x)) + \eta(x, y)$$

of K into $\mathbb{R}$ is upper semicontinuous for each $y \in K$.

Then there exists $x_0 \in K$ such that

$$(Tx_0, \theta(y, x_0)) + \eta(x_0, y) \geq 0 \qquad \ldots (6)$$

for all $y \in K$.

Proof : Let

$$L = \{(x, y) \in K \times K : (Tx, \theta(y, x)) + \eta(x, y) \geq 0\}.$$

L is nonempty since by (a), $(x, x) \in L$. Since the map

$$x \mapsto (Tx, \theta(y, x)) + \eta(x, y)$$

of K into $\mathbb{R}$ is upper semicontinuous the set

$$L(y) = \{x \in K : (x, y) \in L\}$$

$$= \{x \in K : (Tx, \theta(y, x)) + \eta(x, y) \geq 0\}$$

is closed for each $y \in K$.

We assert that for each $x \in K$, the set

$$M(x) = \{y \in K : (x, y) \notin L\}$$

$$= \{y \in K : (Tx, \theta(y, x)) + \eta(x, y) < 0\}$$

is convex : Let $y_1, y_2 \in M(x)$, $0 < t < 1$ and

$$z = ty_1 + (1 - t)y_2.$$

By (b) we have

$$(Tx, \theta(z, x)) + \eta(x, z)$$

$$\leq t((Tx, \theta(y_1, x)) + \eta(x, y_1)) + (1 - t)((Tx, \theta(y_2, x)) + \eta(x, y_2))$$

$$< 0$$

since $(x, y_1) \notin L$ and $(x, y_2) \notin L$. Thus $(x, z) \notin L$ and hence $z \in M(x)$.

Now all the conditions of Theorem 1.7 are satisfied and hence there exists $x_0 \in K$ such that

$$\{x_0\} \times K \subset L,$$

i.e., x_0 satisfies

$$(Tx_0, \theta(y, x_0)) + \eta(x_0, y) \geq 0$$

for each $y \in K$. This completes the proof of Theorem 2.1.

The following theorem generalizes Theorem 2.1 to an arbitrary nonempty convex subset of a Hausdorff topological vector space.

Theorem 2.2 — Let K be a nonempty convex subset of a Hausdorff topological vector space X. Let $T : K \to X^*$, $\theta : K \times K \to X$ and $\eta : K \times K \to \mathbb{R}$ be three maps such that

(A) $(Tx, \theta(x, x)) + \eta(x, x) \geq 0$ for each $x \in K$,

(B) the map

$$y \mapsto (Tx, \theta(y, x)) + \eta(x, y)$$

of K into $\mathbb{R}$ is convex for each $x \in K$,

(C) the map

$$x \mapsto (Tx, \theta(y, x)) + \eta(x, y)$$

of K into $\mathbb{R}$ is upper semicontinuous for each $y \in K$,

(D) there exist a nonempty, compact and convex subset C of K and $u \in C$ such that

$$(Tx, \theta(u, x)) + \eta(x, u) < 0$$

for every $x \in K - C$.

Then there exists an $x_0 \in C$ such that

$$(Tx_0, \theta(y, x_0)) + \eta(x_0, y) \geq 0 \qquad \ldots (7)$$

for all $y \in K$.

PROOF : For each $y \in K$, let

$$E(y) = \{x \in C : (Tx, \theta(y, x)) + \eta(x, y) \geq 0\}.$$

By (A), $E(y)$ is nonempty. By the upper semicontinuity of the map

$$x \mapsto (Tx, \theta(y, x)) + \eta(x, y)$$

of K into $\mathbb{R}$, it follows that the set

$$F(y) = \{x \in K : (Tx, \theta(y, x)) + \eta(x, y) \geq 0\}$$

is closed for each $y \in K$. Since C is compact in the Hausdorff space X, it is closed in X and therefore $E(y) = C \cap F(y)$ is a closed subset of C; hence $E(y)$ is compact for each $y \in K$. It is clear that (7) has a solution if

$$\bigcap_{y \in K} E(y) \neq \phi.$$

For this it is sufficient to prove that the family $\{E(y) : y \in K\}$ has the finite intersection property.

Let $y_1, y_2, ..., y_n$ be arbitrary elements of K and let C_h be the convex hull of

$$C \cup \{y_1, y_2, ..., y_n\}.$$

It is clear that C_h is a compact convex subset of K. By Theorem 2.1 there exists $\tilde{x}_0 \in C_h$ such that

$$(T\tilde{x}_0, \theta(y, \tilde{x}_0)) + \eta(\tilde{x}_0, y) \geq 0 \qquad \ldots (8)$$

for all $y \in C_h$. In fact $\tilde{x}_0 \in C$. If $\tilde{x}_0 \notin C$ (i.e., $\tilde{x}_0 \in K - C$), then by (D) there exists a $u \in C$ such that

$$(T\tilde{x}_0, \theta(u, \tilde{x}_0)) + \eta(\tilde{x}_0, u) < 0,$$

which is a contradiction to (8) when $y = u$.

Thus $\tilde{x}_0 \in C$ and in particular $\tilde{x}_0 \in E(y_i)$ for $i = 1, 2, ..., n$, i.e.,

$$\tilde{x}_0 \in \bigcap_{i=1}^{n} E(y_i).$$

Hence

$$\bigcap_{i=1}^{n} E(y_i) \neq \phi,$$

proving that the family $\{E(y) : y \in K\}$ has finite intersection property. Hence there exists an $x_0 \in K$ such that

$$(Tx_0, \theta(y, x_0)) + \eta(x_0, y) \geq 0$$

for all $y \in K$. Since $x_0 \in E(y)$ for each $y \in K$ and $E(y) \subset C$, we see that $x_0 \in C$. This completes the proof of Theorem 2.2.

It is clear that the inequalities (1)-(5) are particular cases of Theorem 2.2 (inequality (7)) with suitable choices of the functions $\theta(-, -)$ and $\eta(-, -)$.

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