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THE GLOBAL POINT-SET DOMINATION NUMBER OF A GRAPH

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A set D of vertices in a graph $G = (V, E)$ is a dominating set of G if every vertex in $V - D$ is adjacent to some vertex in D. D is a global dominating set if it is a dominating set of both G and its complement $\overline{G}$. D is a point-set dominating set (psd-set) if for every set $S \subset V - D$, there exists a vertex v in D such that $\langle S \cup \{v\} \rangle$ is connected. Further, D is a global psd-set if it is a psd-set of both G and $\overline{G}$. The point-set domination number γ_p of G is the minimum cardinality of a psd-set of G. The global point-set domination number γ_{pg} of G is defined similarly. The following results are obtained for a co-connected graph G of order n :

(1) If G has order at least 5, then $3 \leq \gamma_{pg} \leq n - 2$.

(2) If G has cut vertices, then $\gamma_p \leq \gamma_{pg} \leq \gamma_p + 1$.

(3) If G has diameter at least 4, then $\gamma_p = \gamma_{pg}$.

A set D of vertices in a graph $G = (V, E)$ is a dominating set if every vertex in $V - D$ is adjacent to some vertex in D. Further, D is a global dominating set if it is a dominating set of both G and its complement $\overline{G}$. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. The global domination number γ_g of G is defined similarly (see Sampathkumar[1]).

For a connected graph $G = (V, E)$, a set D of vertices is a point-set dominating set (psd-set) of G if for each set $S \subset V - D$, there exists a vertex v in D such that the subgraph $\langle S \cup \{v\} \rangle$ induced by $S \cup \{v\}$ is connected. The point-set domination number $\gamma_p(G)$ of the G is defined as the minimum cardinality of a psd-set of G (see Sampathkumar[2]). Note that every psd-set is a dominating set.

We say that a graph G is co-connected if both G and $\overline{G}$ are connected. For a co-connected graph $G = (V, E)$, we define a set $D \subset V$ to be a global psd-set if it is a psd-set of both G and $\overline{G}$. The global point-set domination number γ_{pg} of G is defined as the minimum cardinality of a global psd-set.

We now initiate a study of γ_{pg}.

Henceforth, we consider only co-connected graphs G. We begin with some elementary bounds for γ_{pg}.

Every global psd-set is a psd-set, as well as a global dominating set of G. Hence we have,

(i) $\gamma_g \leq \gamma_{pg}$,

(ii) $\gamma_p \leq \gamma_{pg}$.

The neighbourhood of u is $N(u) = \{v \in V \mid uv \in E\}$ and $d(u)$ denotes the degree of u.

Acharya and Purnima Gupta (unpublished) have characterized psd-sets of a graph G as follows.

Proposition 1 — Let $G = (V, E)$ be a graph. A set $D \subset V$ is a psd-set of G if and only if for every independent set W in $V - D$, there exists a vertex u in D such that $W \subset N(u) \cap (V - D)$.

The following characterization of global psd-sets is an immediate consequence of Proposition 1.

Proposition 2 — For a graph G, a set $D \subset V(G)$ is a global psd-set if and only if the following conditions are satisfied:

(i) For every independent set W in $V - D$, there exists u in D such that $W \subset N(u) \cap (V - D)$ in G.

(ii) For every set $S \subset V - D$ such that $\langle S \rangle$ is complete in G, there exists v in D such that $S \cap N(v) = \phi$ in G.

We now obtain some bounds for γ_{pg}.

Proposition 3 — For a graph G of order $n \geq 5$,

$$3 \leq \gamma_{pg} \leq n - 2, \text{ and both bounds are sharp.}$$

PROOF : We have $2 \leq \gamma_g \leq \gamma_{pg}$ (see Sampathkumar[1]).

Suppose $\gamma_{pg}(G) = 2$. Let $D = \{u, v\}$ be a minimum global psd-set of G. Since D is a dominating set of G, every vertex in $V - D$ is adjacent to either u or v in G. Also, no vertex is adjacent to both u and v in G or u and v would not dominate $\overline{G}$. The sets $N(u)$ and $N(v)$ must both be nonempty or one of G and $\overline{G}$ would be disconnected. As D is a psd-set of G, every $x \in N(u)$ is adjacent to every $y \in N(v)$. Then, in $\overline{G}$, for any such x and y there does not exist w in D such that $\langle \{x, y, w\} \rangle$ is connected in $\overline{G}$, contradicting that D is a psd-set of $\overline{G}$ also. Hence we conclude that $\gamma_{pg} \geq 3$.

Now, if there exist vertices u and v at distance 2 in G such that $d(u) \leq n - 3$ and $d(v) \leq n - 3$ in G, then $D = V - \{u, v\}$ is a global psd-set of G and $\gamma_{pg} \leq n - 2$.

If there do not exist two such vertices in G, then for every pair of vertices at distance 2 in G, one vertex has degree $n - 2$ and the other has degree less than $n - 2$ (if both have degree $n - 2$, then $\overline{G}$ will be disconnected).

Since G is not complete, there exist vertices at distance 2 in G. Hence there exists a vertex u of degree $n-2$ in G. As $\overline{G}$ is also connected, $\langle N(u)\rangle$ is not complete in G and hence there exist nonadjacent vertices x, y in $N(u)$. Then, one of these, say x, has degree $n-2$ and $d(y) < n-2$. Also, $d(v) < n-2$ and x is adjacent to v in G, where v is the vertex nonadjacent to u.

Lemma 3a — y and v are the only vertices in G of degree less than $n-2$.

PROOF OF LEMMA : Suppose there exists $z \in V$ different from y and v such that $d(z) < n-2$. Then z is adjacent to both x and u. Since $z, y \in N(u)$ and have degree less than $n-2$, they must be adjacent in G. Similarly, $z, v \in N(x)$ and hence are adjacent in G. Now, both y and v are adjacent to z in G and have degree less than $n-2$ and hence are adjacent in G. Thus, any two vertices of degree less than $n-2$ in G are adjacent. Also, any two vertices of degree $n-2$ are adjacent in G. If $S = \{s \in V(G) \mid d(s) < n-2\}$ and $T = \{t \in V(G) \mid d(t) = n-2\}$, then $\langle S\rangle$ and $\langle T\rangle$ are complete in G. Now in $\overline{G}$, no two vertices of S are adjacent, no two vertices of T are adjacent and every vertex in T is adjacent to exactly one vertex of S. Since S has at least two vertices, $\overline{G}$ is disconnected, a contradiction. This proves the lemma.

Now, all vertices in G except y and v have degree $n-2$. Since every vertex of degree $n-2$ is adjacent to every other vertex of degree $n-2$, each such vertex is adjacent to exactly one of y and v. Since $\overline{G}$ is connected, y and v must be nonadjacent in G. Now, since $n \geq 5$, at least one of y and v, say y, has degree at least 2 in G. Then the set $D = V - \{y, u\}$ is a psd-set of both G and $\overline{G}$. Hence, we have $\gamma_{pg} \leq n-2$.

The bounds are sharp since the lower bound is attained for the trees of diameter 3 with exactly one vertex of degree 2 and the upper bound is attained for all cycles C_n $(n \geq 5)$.

In Sampathkumar[1] it has been proved that for a graph with cut vertices, γ and γ_g differ by at most one. A similar result holds for γ_p and γ_{pg} also.

We use the phrase γ_p -set to represent a minimum psd-set.

Proposition 4 — If G is a graph with cut vertices, then

$$\gamma_p \leq \gamma_{pg} \leq \gamma_p + 1.$$

PROOF : We have only to prove the second inequality.

In Sampathkumar[2] it is shown that for a graph G with cut vertices, $\gamma_p = \min [n-\Delta, n-k]$ where, as usual, Δ is the maximum degree of a vertex of G, and $k = \max [|V(B)| = -\gamma_p(B)]$, the maximum being taken over all blocks B of G.

Let $\gamma_p(G) = n-\Delta$ and let u be a vertex of maximum degree in G. Then, $V - N(u)$ is a γ_p-set of G. If u is a cut vertex of G, there exists a vertex v not adjacent to u in a block B of G. Let w be a vertex in a block B_1 of G different from B and let $D = (V - N(u)) \cup \{w\}$. Now, D is a dominating set of $\overline{G}$ since all vertices of $V - D$ are adjacent either to v or to w in $\overline{G}$.

We now prove that D is a psd-set of $\overline{G}$. Let S be a subset of $V - D$. If all vertices of S are in a single block of G, then all of them are adjacent to v or all are adjacent to w in $\overline{G}$. If S contains vertices from at least two blocks of G, then the subgraph $\langle S \rangle$ is connected in $\overline{G}$. In either case, at least one of $\langle S \cup \{v\} \rangle$ or $\langle S \cup \{w\} \rangle$ is connected in $\overline{G}$. Thus, D is a psd-set of $\overline{G}$.

On the other hand, if u is not a cut vertex of G, all vertices in $N(u)$ lie in a single block B of G. Let v be a cut vertex of G in B. Then $D = (V - N(u)) \cup \{v\}$ is a psd-set of $\overline{G}$ since any vertex in a block other than B is adjacent in $\overline{G}$ to all vertices of $N(u)$ except possibly v.

Now suppose $\gamma_p(G) = n - k$. Then three exists a block B in G with $k = |V(B)| - \gamma_p(B)$. Let S be a minimum psd-set of B, and let $D = V(G) - (V(B) - S)$. Then D is a γ_p-set of G such that all vertices of $V - D$ lie in the block B.

Now, if v is a cut vertex of G in B and w is a vertex in a block B' at v different from B, then all vertices of $V - D - \{v\}$ are adjacent to w in $\overline{G}$. Hence $D' = D \cup (v)$ is a psd-set of $\overline{G}$.

Thus, in all cases, $\gamma_{pg} \leq \gamma_p + 1$.

Corollary 4.1 — If G is a tree, a block graph or a cactus,

$$n - \Delta \leq \gamma_{pg} \leq n - \Delta + 1.$$

PROOF : Each of these graphs have cut vertices, and also $\gamma_p = n - \Delta$ (see Sampathkumar[2]).

Proposition 2 enables us to prove that $\gamma_{pg} = \gamma_p$ when G has diameter at least 4.

Let diam(G) denote the diameter of G, and let $d(u, v)$ denote the distance between u and v.

Proposition 5 — If G is a graph of diameter at least 4, then

$$\gamma_{pg} = \gamma_p.$$

PROOF : Let D be a γ_p-set of G. If D is not a psd-set of $\overline{G}$, then, by Proposition 2, there exists a set $S \subset V - D$, with $\langle S \rangle$ complete in G such that for every $d \in D, N(d) \cap S \neq \phi$.

Let $u, v \in V(G)$. We consider three cases.

Case 1 — $u, v \in D$. Then u is adjacent to a vertex in S and v is adjacent to a vertex in S and $\langle S \rangle$ is complete. Hence $d(u, v) \leq 3$ in G.

Case 2 — $u, v \in V - D$. In this case, $d(u, v) \leq 2$ in G (see Sampathkumar[2]).

Case 3 — $u \in D$ and $v \in V - D$. Then u is adjacent to a vertex u' in S.

Thus we have $d(u, v) \leq d(u, u') + d(u', v) \leq 3$.

Thus $d(u, v) \leq 3$ for all pairs of vertices u, v in G, and hence diam(G) ≤ 3, contradicting that the diameter of G is at least 4. Hence D is a psd-set of $\overline{G}$ also, $\gamma_{pg} = \gamma_p$.

Exact value of γ_{pg} can be determined for all trees, with the help of Proposition 5.

Corollary 5.1 — For a tree T order $n \geq 4$,

$\gamma_{pg} = n - \Delta + 1$ if diam(T) = 3, and

$\gamma_{pg} = n - \Delta$ if diam(T) ≥ 4.

PROOF : Let diam(T) = 3 and let u and v be the vertices in T of degree at least 2. If $d(u) > d(v)$, then $D = V - N(u)$ is the only γ_p-set in T and it is not a psd-set of $\overline{G}$ since the vertex v is not dominated by D in $\overline{G}$. Hence $\gamma_{pg} \geq |V - N(u)| = n - \Delta$. Now, for $x \in N(u)$, $D' = (V - N((u)) \cup \{x\}$ is a psd-set of $\overline{G}$, and hence is a minimum global psd-set of G. Thus, $\gamma_{pg} = n - \Delta + 1$. A similar argument holds if $d(u) = d(v)$.

If diam $(T) \geq 4$, then by Proposition 5, $\gamma_{pg} = \gamma_p = n - \Delta$.

For graphs with diameter 2, the difference between γ_p and γ_{pg} can be made as large as we please. For example, γ_p $(\overline{C}_n) = 2$, whereas

γ_{pg} $(\overline{C}_n) = n - 2$ for $n \geq 5$.

Open problem : Is it true that for graphs with diameter 3, a statement similar to the one above holds ?

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