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ON STABILITY OF STRATIFIED COMPRESSIBLE SHEAR FLOWS

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In this paper we study the linear normal mode stability of inviscid, compressible shear flows with variable temperature distribution in the presence of gravity. First, for plane, parallel flows, we prove the equivalent of Squire's theorem, namely, that for every unstable three-dimensional disturbance there corresponds a more unstable two-dimensional disturbance. Consequently, we study the instability problem only to two-dimensional disturbances and we obtain a sufficient condition for stability and an estimate for the growth rate of an unstable mode. Then, we prove the semicircle theorem. Moreover, we obtain instability regions to subsonic and a class of supersonic disturbances which depend on the Mach number, the wave number and the depth of the fluid layer. Finally, with the help of a set of transformations, we reduce the problem of non-parallel flows to the problem of parallel flows and obtain many results for the stability problem for non-parallel flows.

1. INTRODUCTION

Eckart[2] studied the linear stability of stratified compressible parallel shear flows to two-dimensional normal mode disturbances and obtained a semicircle instability region for the complex wave velocity of an unstable mode. Later, Chimonas[1] obtained a sufficient condition for stability, namely, that the flow is stable if the Richardson number is greater than or equal to one quarter throughout the flow domain. Chimonas[1] also obtained an estimate for the growth rate of an unstable mode. Recently, Subbiah and Jain[5] modified Chimonas' proof of the sufficient condition for stability, and it enabled them to improve upon the instability regions for subsonic disturbances and for a class of supersonic disturbances. Thulasi and Subbiah[6] have proved the equivalent of Squire's theorem. All these authors have taken the temperature of the basic flow to be a constant.

In this paper, we study the same problem but with a variable temperature distribution. First, for plane, parallel flows, we prove the equivalent of Squire's theorem, namely, that for every unstable three-dimensional disturbance there corresponds a more unstable two-dimensional disturbance. Consequently, we study the instability problem only to two-dimensional disturbances. First, we obtain a sufficient condition for stability and an estimate for the growth rate of an unstable mode which are generalizations of the results of Chimonas[1] for parallel flows with constant

temperature distribution. Then, we prove the semicircle theorem which gives a bound for the complex wave velocity of an unstable mode. Moreover, we obtain instability regions to subsonic and a class of supersonic disturbances, which depend on the Mach number, the wave number and the depth of the fluid layer. Finally, with the help of a set of transformations, we reduce the problem of non-parallel flows to the problem of parallel flows and obtain many results, which are generalizations of the above results for parallel stratified flows with variable temperature distribution.

2. Formulation of the Problem and Squire's Theorem

The governing equations for the motion of an inviscid, stratified compressible fluid, confined between two infinite horizontal rigid planes at $y = y_1, y_2$, are

$$\rho [\vec{u_t} + (\vec{u} \cdot \nabla)\vec{u}] = -\frac{\nabla p}{\gamma M^2} - \rho F^{-2} \hat{y} \qquad \ldots (1)$$

$$\rho_t + \nabla \cdot \rho \vec{u} = 0 \qquad \ldots (2)$$

$$p = \rho T \qquad \ldots (3)$$

and

$$T_t + \vec{u} \cdot \nabla T + (\gamma - 1) T \nabla \cdot \vec{u} = 0 \qquad \ldots (4)$$

where $\vec{u} = (u, v, w)$ is the velocity vector, ρ the density, p the pressure, T the temperature, γ the ratio of specific heats, M the Mach number of the reference state, and a suffix t denotes partial differentiation with respect to t. Non-dimensionalization is done by introducing a reference length scale L, velocity V, pressure P_r, temperature T_r and density ρ_r satisfying $P_r = \rho_r R T_r$; R is the universal gas constant, $M = V(\gamma R T_r)^{-1/2}$ and the Froude number $F = V(gL)^{-1/2}$.

Consider the basic parallel flow given by $(U(y), 0, 0)$, $\rho_0(y)$, $T_0(y)$ and $p_0(y)$ with $\rho_0 T_0 = p_0$ which satisfies (1)-(4) and the boundary conditions $v = 0$ at $y = y_1, y_2$, provided $p_0' = -\gamma M^2 F^{-2} \rho_0(y)$. Here $U(y)$, $\rho_0(y)$, $p_0(y)$ and $T_0(y)$ are twice continuously differentiable functions of y in the flow domain and $' = \frac{d}{dy}$.

If the perturbed flow variables are $(U(y) + u, v, w)$, $\rho_0(y) + \rho$, $T_0(y) + \theta$ and $p_0(y) + p$, then the linearised perturbation equations, for normal modes of the form (function of y) exp $i(kx + lz - kct)$, yield

$$\rho_0 [ik(U - c)u + U'v] = -\frac{ikp}{\gamma M^2} \qquad \ldots (5)$$

$$\rho_0 ik(U - c)v = -\frac{p'}{\gamma M^2} - \rho F^{-2} \qquad \ldots (6)$$

$$\rho_0 ik(U-c)w = -\frac{ilp}{\gamma M^2} \qquad \ldots (7)$$

$$ik(U-c)\rho + \rho_0' v + \rho_0 [iku + v' + ilw] = 0 \qquad \ldots (8)$$

$$p = \rho_0\theta + T_0\rho \qquad \ldots (9)$$

$$ik(U-c)\theta + T_0' v + (\gamma-1) T_0 [iku + v' + ilw] = 0. \qquad \ldots (10)$$

Now, we shall prove the equivalent of Squire's theorem[4].

Theorem 1 — For every unstable three-dimensional disturbance there corresponds a more unstable two-dimensional disturbance.

PROOF : Introduce the transformations

$$\tilde{k}^2 = k^2 + l^2, \quad \tilde{k}\,\tilde{u} = ku + lw, \quad \tilde{v} = v, \quad \tilde{c} = c, \quad \tilde{M}^2\,\tilde{k}^2 = M^2 k^2,$$

$$\tilde{F}^2\,\tilde{k}^2 = F^2 k^2, \quad \tilde{k}\,\tilde{\rho} = k\rho, \quad \tilde{k}\,\tilde{\theta} = k\theta, \quad \tilde{k}\,\tilde{p} = kp. \qquad \ldots (11)$$

Then, proceeding as in Thulasi and Subbiah[6], we get the result.

As a consequence, it is enough to study the instability problem only to two-dimensional disturbances.

3. EIGEN VALUE PROBLEM

Let us put $l = 0 = w$ in (5)-(10). Then, we eliminate the variables u, ρ, θ and p in terms of v to get the stability equation

$$\left[\frac{\rho_0(U-c)v' - \rho_0 U' v - \frac{\rho_0}{T_0} M^2 F^{-2}(U-c)v}{T_0 - M^2(U-c)^2}\right]'$$

$$+ \frac{M^2 F^{-2}}{T_0}\left[\frac{\rho_0(U-c)v' - \rho_0 U' v - \frac{\rho_0}{T_0} M^2 F^{-2}(U-c)v}{T_0 - M^2(U-c)^2}\right]$$

$$- \frac{\rho_0}{T_0} k^2 (U-c)v + \frac{\rho_0}{T_0}\frac{N^2 v}{(U-c)} = 0, \qquad \ldots (12)$$

where $N^2 = -\frac{F^{-2}}{T_0}\left[\frac{\rho_0'}{\rho_0} + \frac{M^2 F^{-2}}{T_0}\right]$ is the square of the Brunt Vaisala frequency.

The associated boundary conditions are

$$v = 0, \qquad y = y_1, y_2. \qquad \ldots (13)$$

If we define

$$H(y) = v(y) \exp\left[- \int^{y} \frac{M^2 F^{-2}}{T_0} dy' \right]$$

and

$$r(y) = \rho_0(y) \exp\left[\int^{y} \frac{2M^2 F^{-2}}{T_0} dy' \right],$$

then (12) and (13) become

$$\left[\frac{r(U-c)H' - rU'H}{T_0 - M^2 (U-c)^2} \right]' - \frac{rk^2 (U-c)H}{T_0} + \frac{rN^2 H}{U-c} = 0, \qquad \ldots (14)$$

and

$$H = 0, \qquad y = y_1, y_2. \qquad \ldots (15)$$

4. Stability Analysis

Theorem 2 — A sufficient condition for stability is that $N^2 \geq \frac{U'^2}{4T_0}$ throughout the flow domain.

Proof : For unstable modes ($c_i > 0$), the transformation $v = (U-c)^{1/2} G$ reduces (14) and (15) to

$$\left[\frac{r(U-c)G'}{T_0 - M^2 (U-c)^2} \right]' - \frac{1}{2}\left[\frac{rU'}{T_0 - M^2 (U-c)^2} \right]' G$$

$$- \frac{rU'^2 G}{4(U-c)[T_0 - M^2 (U-c)^2]} - \frac{rk^2 (U-c)G}{T_0} + \frac{rN^2 G}{U-c} = 0, \qquad \ldots (16)$$

and

$$G = 0, \qquad y = y_1, y_2. \qquad \ldots (17)$$

Equation (16) can be rewritten as

$$\left[\frac{r(U-c)G'}{T_0 - M^2 (U-c)^2} \right]' - \frac{1}{2}\left[\frac{rU'}{T_0 - M^2 (U-c)^2} \right]' G - \frac{rU'^2 G}{4T_0 (U-c)}$$

$$- \frac{rM^2 (U-c) U'^2 G}{4T_0 [T_0 - M^2 (U-c)^2]} - \frac{rk^2 (U-c)G}{T_0} + \frac{rN^2 G}{U-c} = 0. \qquad \ldots (18)$$

Imaginary part of the equation obtained, by multiplying (18) by G^* and then integrating it using (17), is

$$\int_{y_1}^{y_2} \frac{r}{|T_0 - M^2 (U-c)^2|^2} \{[T_0 + M^2 (U-c_r)^2]\left[|G'|^2 + \frac{M^2 U'^2}{4T_0} |G|^2 \right]$$

$$+ M^2 c_i^2 \left[|G'|^2 + \frac{M^2 U'^2}{4T_0} |G|^2 \right] - M^2 U' (U-c_r) (GG^*)'$$

$$+ \frac{k^2}{T_0} |T_0 - M^2 (U-c)^2|^2 |G|^2\} \, dy$$

$$+ \int_{y_1}^{y_2} \frac{r}{|U-c|^2} \left[N^2 - \frac{U'^2}{4T_0} \right] |G|^2 \, dy = 0. \qquad \ldots (19)$$

Since

$$M^2 U' (U-c_r) (GG^*)' \leq [T_0 + M^2 (U-c_r)^2] \left[|G'|^2 + \frac{M^2 U'^2}{4T_0} |G|^2 \right]$$

we can write

$$[T_0 + M^2 (U-c_r)^2] \left[|G'|^2 + \frac{M^2 U'^2}{4T_0} |G|^2 \right] - M^2 U' (U-c_r) (GG^*)' = q^2$$

a non-negative quantity.

Equation (19) then implies that

$$\int_{y_1}^{y_2} \frac{r}{|T_0 - M^2 (U-c)^2|^2} \{M^2 c_i^2 \left[|G'|^2 + \frac{M^2 U'^2}{4T_0} |G|^2 \right]$$

$$+ \frac{k^2}{T_0} |T_0 - M^2 (U-c)^2|^2 |G|^2\} \, dy$$

$$+ \int_{y_1}^{y_2} \frac{r}{|U-c|^2} \left[N^2 - \frac{U'^2}{4T_0} \right] |G|^2 \, dy \leq 0. \qquad \ldots (20)$$

Inequality (20) is impossible if $N^2 \geq \frac{U'^2}{4T_0}$ throughout the flow domain. This proves the theorem.

Theorem 3 — An estimate for the growth rate of an unstable mode is given by

$$k^2 c_i^2 \leq \left[\frac{U'^2}{4T_0} - N^2 \right]_{\max_y} \qquad \ldots (21)$$

PROOF : The estimate (21) follows immediately from the inequality (20).

Theorem 4 — The complex wave velocity of an unstable mode must lie within a semicircle in the upper half of the complex plane given by

$$\left(c_r - \frac{a+b}{2}\right)^2 + c_i^2 \leq \left(\frac{b-a}{2}\right)^2 \qquad \ldots (22)$$

where $a = \min_y U(y)$ and $b = \max_y U(y)$.

PROOF : For unstable modes ($c_i > 0$), the transformation $H = (U - c)F$ reduces eqns. (14) and (15) to

$$\left[\frac{r(U-c)^2 F'}{T_0 - M^2 (U-c)^2}\right]' - \frac{rk^2 (U-c)^2 F}{T_0} + rN^2 F = 0 \qquad \ldots (23)$$

and

$$F = 0, \quad y = y_1, y_2. \qquad \ldots (24)$$

Then, following the standard procedure (Howard[3]), we get

$$\left[\left(c_r - \frac{a+b}{2}\right)^2 + c_i^2 - \left(\frac{b-a}{2}\right)^2\right] \int_{y_1}^{y_2} rQ\, dy$$

$$+ \int_{y_1}^{y_2} \frac{rM^2 |U-c|^4 |F'|^2\, dy}{|T_0 - M^2 (U-c)^2|^2} + \int_{y_1}^{y_2} rN^2 |F|^2\, dy \leq 0 \qquad \ldots (25)$$

where

$$Q = \frac{T_0 |F'|^2}{|T_0 - M^2 (U-c)^2|^2} + \frac{k^2}{T_0} |F|^2.$$

The last two terms in the left hand side of the inequality (25) are non-negative terms and therefore can be dropped. Since $\int_{y_1}^{y_2} rQ\, dy > 0$, the theorem follows.

The instability region given by the semicircle of Theorem 4 does not involve the compressibility parameter, namely the Mach number M. So, it is desirable to get instability regions involving the Mach number. This we achieve in the next two theorems.

Now, the disturbances are classified as subsonic, sonic or supersonic according as $T_0 - M^2 |U-c|^2 \gtreqless 0$.

Theorem 5 — The complex wave velocity of an unstable supersonic mode satisfying $c_i^2 \geq \frac{T_{0\max}}{M^2}$ must lie within the semi ellipse region, in the upper half of the complex plane, given by

$$\left(c_r - \frac{a+b}{2}\right)^2 + c_i^2 \left[1 + \frac{4J_0\, T_{0\max}}{\left[1 + \frac{1}{T_{0\min}}\left\{T_{0\max} + \frac{5M^2(b-a)^2}{4}\right\}\left\{1 - 4J_0\, T_{0\min}\right\}^{1/2}\right]^2}\right] \leq \left(\frac{b-a}{2}\right)^2 \quad \ldots (26)$$

where

$$J_0 = \min_y \frac{N^2}{U'^2}.$$

PROOF : For supersonic unstable mode satisfying $c_i^2 \geq \frac{T_{0\max}}{M^2}$, inequality (20) implies that

$$\int_{y_1}^{y_2} \frac{r}{|T_0 - M^2(U-c)^2|^2} \left\{T_0 |G'|^2 + \frac{M^2 U'^2 |G|^2}{4T_0} + \frac{k^2}{T_0}|T_0 - M^2(U-c)^2|^2 |G|^2\right\} dy + \int_{y_1}^{y_2} \left[N^2 - \frac{U'^2}{4T_0}\right] \frac{r|G|^2}{|U-c|^2} \leq 0. \quad \ldots (27)$$

Now, let

$$E^2 = \int_{y_1}^{y_2} r|U-c|Q\, dy$$

and

$$B^2 = \int_{y_1}^{y_2} \frac{rT_0 U'^2 |F|^2\, dy}{4|U-c||T_0 - M^2(U-c)^2|^2}.$$

Substituting $G = (U-c)^{1/2} F$ in (27) and then using the Cauchy-Schwartz's inequality, we get

$$E^2 \leq \left[1 + \frac{1}{T_{0\min}}\left\{T_{0\max} + \frac{5M^2(b-a)^2}{4}\right\}\left\{1 - 4J_0\, T_{0\min}\right\}^{1/2}\right]^2 B^2. \quad \ldots (28)$$

In obtaining the above inequality, the fact that

$$|T_0 - M^2(U-c)^2| \leq T_{0\max} + \frac{5M^2(b-a)^2}{4} \text{ is used.}$$

Since $|T_0 - M^2 (U-c)^2|^2 \geq T_{0\max}^2$ for an unstable supersonic mode satisfying $c_i^2 \geq \frac{T_{0\max}}{M^2}$, we have

$$\int_{y_1}^{y_2} rN^2 |F|^2 \, dy \geq 4 T_{0\max} J_0 c_i B^2. \qquad \ldots (29)$$

Using the inequality (28) in (29), we get

$$\int_{y_1}^{y_2} rN^2 |F|^2 \, dy \geq \frac{4T_{0\max} J_0 c_i E^2}{\left[1 + \frac{1}{T_{0\min}} \left\{ T_{0\max} + \frac{5M^2 (b-a)^2}{4} \right\} \left\{ 1 - 4J_0 T_{0\min} \right\}^{1/2} \right]^2},$$

which in turn implies that

$$\int_{y_1}^{y_2} rN^2 |F|^2 \, dy \geq \frac{4T_{0\max} J_0 c_i^2 \int_{y_1}^{y_2} rQ \, dy}{\left[1 + \frac{1}{T_{0\min}} \left\{ T_{0\max} + \frac{5M^2 (b-a)^2}{4} \right\} \left\{ 1 - 4J_0 T_{0\min} \right\}^{1/2} \right]^2}. \qquad \ldots (30)$$

Using this estimate for the term $\int_{y_1}^{y_2} rN^2 |F|^2 \, dy$ in (25), we get the inequality (26). This proves the theorem.

Theorem 6 — The complex wave velocity of an unstable subsonic mode must be inside the semi ellipse type region in the upper half of the complex plane given by

$$\left(c_r - \frac{a+b}{2} \right)^2 + c_i^2 + \frac{J_0 U_{\min}^{\prime 2} T_{0\min} (\lambda^2 + k^2) \left[T_{0\min} - \frac{5M^2 (b-a)^2}{4} \right]^2 M^4 c_i^8}{U_{\max}^{\prime 4} T_{0\max}^5 \left[\frac{1}{4T_{0\min}} - J_0 \right] \left[\frac{1}{2} + 2T_{0\max} \left\{ \frac{1}{4T_{0\min}} - J_0 \right\}^{1/2} \right]^2} \leq \left(\frac{b-a}{2} \right)^2, \qquad \ldots (31)$$

where

$$\lambda^2 = \frac{r_{\min} T_{0\min} \pi^2}{4 r_{\max} T_{0\max} (y_2 - y_1)^2}.$$

PROOF : For an unstable subsonic mode, $c_i^2 \le \frac{T_0}{M^2} \le \frac{T_{0\max}}{M^2}$ and therefore, inequality (20) yields

$$\int_{y_1}^{y_2} \frac{r}{|T_0 - M^2 (U-c)^2|^2} \times \left\{ \frac{M^2 c_i^2}{T_0} \left[T_0 |G'|^2 + \frac{k^2}{T_0} |T_0 - M^2 (U-c)^2|^2 |G|^2 \right] \right\} dy \le \left[\frac{1}{4T_{0\min}} - J_0 \right] \int_{y_1}^{y_2} \frac{rU'^2 |G|^2 dy}{|U-c|^2}. \quad \ldots (32)$$

Now, let $E_1^2 = \int_{y_1}^{y_2} \frac{rM^2}{T_0} c_i^2 |U-c| Q \, dy$ and

$$B_1^2 = \int_{y_1}^{y_2} \frac{rM^2 c_i^2 U'^2 |F|^2 dy}{|U-c| |T_0 - M^2 (U-c)^2|^2}.$$

Then, proceeding as in Theorem 5, we get

$$E_1^2 \le \frac{T_{0\max}}{M^2 C_i^2} \left[\frac{1}{2} + 2 T_{0\max} \left\{ \frac{1}{4T_{0\min}} - J_0 \right\}^{1/2} \right]^2 B_1^2. \quad \ldots (33)$$

Also, we get

$$\int_{y_1}^{y_2} \frac{rM^2 c_i^2 |G'|^2 dy}{|T_0 - M^2 (U-c)^2|^2} \ge \frac{\lambda^2}{k^2} \int_{y_1}^{y_2} \frac{rM^2 c_i^2}{T_0} \frac{k^2}{T_0} |G|^2 dy, \quad \ldots (34)$$

with the help of Rayleigh-Ritz inequality.

Using (34) in (32), we get

$$\lambda^2 + k^2 \le \left[\frac{1}{4T_{0\min}} - J_0 \right] \frac{U'^2_{\max} T^2_{0\max}}{M^2 C_i^4}. \quad \ldots (35)$$

If $\nu^2 = \int_{y_1}^{y_2} \frac{rT_0 |F'|^2 dy}{|T_0 - M^2 (U-c)^2|^2} \Big/ \int_{y_1}^{y_2} \frac{r}{T_0} |F|^2 dy$, then

$$\frac{\int_{y_1}^{y_2} rQ\,dy}{\int_{y_1}^{y_2} \frac{r|F|^2\,dy}{T_0}} = \nu^2 + k^2$$

$$\leq \frac{(\nu^2+k^2)\left[\frac{1}{4T_{0\min}} - J_0\right] U'^2_{\max} T^2_{0\max}}{(\lambda^2+k^2) M^2 C_i^4}, \quad \ldots (36)$$

where (35) is used to get (36). Further, we have

$$\frac{E_1^2}{B_1^2} \geq \frac{c_i^2(\nu^2+k^2)\left[T_{0\min} - \frac{5M^2(b-a)^2}{4}\right]^2}{U'^2_{\max} T^2_{0\max}},$$

which together with (33) gives

$$\nu^2 + k^2 \leq \frac{U'^2_{\max} T^3_{0\max}\left[\frac{1}{2} + 2T^{1/2}_{0\max}\left\{\frac{1}{4T_{0\min}} - J_0\right\}^{1/2}\right]^2}{M^2 c_i^4\left[T_{0\min} - \frac{5M^2(b-a)^2}{4}\right]^2}. \quad \ldots (37)$$

Using this estimate for $\nu^2 + k^2$ in (49), we get

$$\int_{y_1}^{y_2} rQ\,dy \Big/ \int_{y_1}^{y_2} \frac{r}{T_0}|F|^2\,dy$$

$$\leq \frac{U'^4_{\max} T^5_{0\max}\left[\frac{1}{2} + 2T^{1/2}_{0\max}\left\{\frac{1}{4T_{0\min}} - J_0\right\}^{1/2}\right]^2\left[\frac{1}{4T_{0\min}} - J_0\right]}{M^4 c_i^3(\lambda^2+k^2)\left[T_{0\min} - \frac{5M^2(b-a)^2}{4}\right]^2}. \quad \ldots (38)$$

Since

$$\int_{y_1}^{y_2} rN^2|F|^2\,dy \geq J_0 U'^2_{\min} T_{0\min} \int_{y_1}^{y_2} \frac{r|F|^2\,dy}{T_0},$$

inequality (38) gives

$$\int_{y_1}^{y_2} rN^2 |F|^2 dy$$

$$\geq \frac{J_0 U'^2_{\min} T_{0\min} M^4 c_i^8 (\lambda^2 + k^2) \left[T_{0\min} - \frac{5M^2 (b-a)^2}{4} \right]^2 \int_{y_1}^{y_2} rQ\, dy}{U'^4_{\max} T_{0\max}^5 \left[\frac{1}{4T_{0\min}} - J_0 \right] \left[\frac{1}{2} + 2T_{0\max}^{1/2} \left\{ \frac{1}{4T_{0\min}} - J_0 \right\}^{1/2} \right]^2} \quad \ldots (39)$$

This proves the theorem.

5. Stability Results for Non-Parallel Flows

Consider the non-parallel flow given by $\vec{U} = (U(y), 0, W(y))$, $\rho_0 = \rho_0(y)$, $T_0(y)$ and $p_0 = p_0(y)$ with $p_0 = \rho_0 T_0$ and $\frac{dp_0}{dy} = -\gamma M^2 F^{-2} \rho_0$. Assuming the disturbances to be of the form (function of y) exp $i(kx + lz - kct)$ we get the stability equation as

$$\left[\frac{r(k(U-c) + lW) H' - r(kU' + lW')H}{(k^2 + l^2) T_0 - M^2 (k(U-c) + lW)^2} \right]' - \frac{r}{T_0} (k(U-c) + lW)H + \frac{rN^2 H}{k(U-c) + lW} = 0. \quad \ldots (40)$$

The transformations

$$\tilde{k}^2 = k^2 + l^2, \quad \tilde{k}\,\tilde{U} = kU + lW, \quad \tilde{k}\,\tilde{c} = kc$$

reduce the stability equation (40) to the equation

$$\left[\frac{r(\tilde{U} - \tilde{c}) H' - r\tilde{U}' H}{T_0 - M^2 (\tilde{U} - \tilde{c})^2} \right]' - \frac{rk^2 (\tilde{U} - \tilde{c}) H}{T_0} + \frac{rN^2 H}{\tilde{U} - \tilde{c}} = 0. \quad \ldots (41)$$

Equation (41) is similar to the stability equation (14). Now, the following results follow directly from the corresponding theorems for parallel flows proved in the previous section.

Theorem 7 — A sufficient condition for stability is that $N^2 \geq \frac{U'^2 + W'^2}{4T_0}$ throughout the flow domain.

Theorem 8 — An estimate for the growth rate of an unstable mode is given by

$$k^2 c_i^2 \leq \left[\frac{U'^2 + W'^2}{4T_0} - N^2 \right]_{\max_y} \quad \ldots (42)$$

Theorem 9 — A necessary condition for stability to an unstable mode is that

$$\left(c_r - \frac{\tilde{a}+\tilde{b}}{2}\right)^2 + c_i^2 \le \left(\frac{\tilde{b}-\tilde{a}}{2}\right)^2, \qquad \ldots (43)$$

where $\tilde{a} = \min_y \left(U + \frac{1}{k} W\right)$ and $\tilde{b} = \max_y \left(U + \frac{1}{k} W\right)$.

We may classify the disturbances as subsonic, sonic or supersonic according as

$$T_0 - M^2 \frac{|k(U-c) + lW|^2}{(k^2+l^2)} \gtreqless 0.$$

Theorem 10 — A necessary condition for instability to a class of unstable supersonic modes satisfying $c_i^2 > \frac{T_{0\max}}{M^2}$ is that

$$\left(c_r - \frac{\tilde{a}+\tilde{b}}{2}\right)^2 + c_i^2 \left[\frac{4T_{0\max} J_0}{\left[1 + \frac{1}{T_{0\min}}\left\{T_{0\max} + \frac{5M^2(\tilde{b}-\tilde{a})^2}{4}\right\}\left\{1 - 4J_0 T_{0\min}\right\}^{1/2}\right]^2}\right] \le \left(\frac{\tilde{b}-\tilde{a}}{2}\right)^2 \qquad \ldots (44)$$

where $J_0 = \min_y \frac{N^2}{U'^2 + W'^2}$.

Theorem 11 — A necessary condition for instability to an unstable subsonic mode is that

$$\left(c_r - \frac{\tilde{a}+\tilde{b}}{2}\right)^2 + c_i^2 + \frac{J_0 M^4 c_i^8 (U'^2+W'^2)_{\min} T_{0\min} (\lambda^2+k^2)\left[T_{0\min} - \frac{5M^2(\tilde{b}-\tilde{a})^2}{4}\right]^2 M^4 c_i^3}{(U'^2+W'^2)_{\max} T_{0\max}^5 \left[\frac{1}{4T_{0\min}} - J_0\right]\left[\frac{1}{2} + 2T_{0\max}^{1/2}\left\{\frac{1}{4T_{0\min}} - J_0\right\}^{1/2}\right]^2} \le \left(\frac{\tilde{b}-\tilde{a}}{2}\right)^2, \qquad \ldots (45)$$

where $\lambda^2 = \frac{r_{\min} T_{0\min} \pi^2}{4 r_{\max} T_{0\max} (y_2 - y_1)^2}$.

6. Concluding Remarks

In this paper, we have studied the linear normal mode stability of inviscid, compressible shear flows with variable temperature distribution in the presence of buoyancy force. For plane parallel flows, we have proved the equivalent of Squire's theorem. Consequently, we have studied the instability of parallel shear flows to two-dimensional disturbances. First, we have obtained a sufficient condition for stability and an estimate for the growth rate of an unstable mode. Then, we have shown that the complex wave velocity of an unstable mode must lie inside a semicircle in the upper half of the complex plane. Also, we have obtained instability regions to subsonic and a class of supersonic disturbances. These instability regions depend on the compressibility parameter, namely the Mach number, the wave number and the depth of the fluid layer. Finally, the stability problem for non-parallel flows has been reduced to the stability problem for parallel flows with the help of suitable transformations and we have generalized our stability results for parallel flows to non-parallel flows.

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