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POLYNOMIAL SOLUTIONS FOR PELL'S EQUATION REVISITED

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Ramasamy[2] looked at square-free $D \in \mathbf{N}$ such that the so-called Pell's equation $x^2 - Dy^2 = \pm 1$ has polynomial solutions. In this paper, we will show that the results of Ramasamy[2] underlie a well-known phenomenon pertaining to fundamental units in related real quadratic orders (not necessarily maximal). Thus, we will generalize the results of Ramasamy[2] and expose, for the reader, a related result of Lagrange as it pertains to the continued fraction algorithm which essentially explains all of this.

1. BACKGROUND MATERIAL

As is well-known, for any non-square $D \in \mathbf{N}$, the Diophantine equation

$$x^2 - Dy^2 = \pm 1 \qquad \ldots (1)$$

was erroneously dubbed Pell's equation by Euler, when in fact the honour goes to Lord Brouncker (see Mollin[1], footnote 1.1.5, p. 5). In any case, the solutions of eqn. (1) are derived from the powers of the fundamental unit of the underlying real quadratic order, and the related simple continued fraction expansion of $\sqrt{D}$. In order to see how this all fits together, we need to introduce some notation and background information. For what follows, all of the proofs, as well as historical background information may be found in Mollin[1].

Let $D_0 > 1$ be a square-free positive integer and set

$$\sigma_0 = \begin{cases} 2 & \text{if } D_0 \equiv 1 \pmod 4, \\ 1 & \text{otherwise.} \end{cases}$$

Define

$$w_0 = (\sigma_0 - 1 + \sqrt{D_0})/\sigma_0,$$

and

$$\Delta_0 = (w_0 - w_0')^2 = 4D_0/\sigma_0^2,$$

where w_0' is the algebraic conjugate of w_0, i.e. $w_0' = (\sigma_0 - 1 - \sqrt{D_0})/\sigma_0$. We call Δ_0 a fundamental discriminant or field discriminant with associated radicand D_0, and w_0 is called the principal fundamental surd associated with Δ_0.

Now we generalize this well-known notion of a field discriminant since we will be working in a more general setting than the field case, which we will show below is tantamount to working in the maximal order only. If we let f_Δ be any positive integer, and as above, Δ_0 is a fundamental discriminant with associated radicand D_0, then $\Delta = f_\Delta^2 \Delta_0$ is called a discriminant with conductor f_Δ.

In order to generalize the notions of radicand and principal surd, we need to develop more notation as follows. If we set $g = gcd\ (f_\Delta, \sigma_0)$, $\sigma = \sigma_0/g$, and let

$$D = (f_\Delta/g)^2 D_0,$$

then

$$\Delta = 4D/\sigma^2.$$

Thus, D is called the radicand associated with Δ. Furthermore, if we set

$$w_\Delta = (\sigma - 1 + \sqrt{D})/\sigma,$$

and let $h = (\sigma_0 - g - f_\Delta \sigma_0 + f_\Delta)/\sigma_0$, which an easy exercise shows to be an integer, then

$$w_\Delta = f_\Delta w_0 + h,$$

called the principal surd associated with the discriminant $\Delta = (w_\Delta - w_\Delta')^2$.

Now we bring ideal structure into the picture. If $[\alpha, \beta] = \alpha \mathbf{Z} + \beta \mathbf{Z}$, then $O_\Delta = [1, f_\Delta w_0] = [1, w_\Delta]$. This is an order in $K = \mathbf{Q}(\sqrt{D_0}) = \mathbf{Q}(\sqrt{\Delta})$ having conductor f_Δ and discriminant Δ. If $f_\Delta = 1$, then O_Δ is called the maximal order or ring of integers of K, since $[1, f_\Delta w_0] \subseteq [1, w_0]$, for any conductor f_Δ. In fact, $f_\Delta = | O_{\Delta_0} : O_\Delta |$ is the index of O_Δ in the maximal order O_{Δ_0}. Thus, an order is uniquely determined by its discriminant, and this provides a one-to-one correspondence between non-square integers $\Delta \equiv 0, 1 \pmod 4$ and orders in quadratic fields.

Let $I = [a, b + cw_\Delta]$, with $a > 0$. The following tells us when such a module is an ideal. In what follows, the symbol $N(x)$ when applied to an $x \in O_\Delta$ will mean the norm of x, namely xx', where x' denotes the algebraic conjugate of x (i.e. as above, if $x = a + b\sqrt{D}$, for instance, then $x' = a - b\sqrt{D}$).

Theorem 1 (*Ideal criterion*) — If $I = [a, b + cw_\Delta]$, then I is a non-zero ideal of O_Δ if and only if $c \mid a$, $c \mid b$ and $ac \mid N(b + cw_\Delta)$.

Note that I is a zero ideal if and only if $I \subseteq \mathbf{Z}$. Thus, if I is a non-zero ideal in O_Δ, then I can be written as $[a, b + cw_\Delta]$ where $a, b, c \in \mathbf{Z}$, $a > 0$, $c > 0$, $c \mid b$ and $c \mid a$. In fact, for a given non-zero ideal I in O_Δ, the integers a and c are unique. Indeed, a is the least positive rational integer in I, which we denote by $L(I)$. Also, we denote the value of $cL(I)$, called the norm of I, by $N(I)$. In other words, when the symbol N is applied to an ideal I, it will mean the norm of I as above. Thus, the context will not lead to confusion between norms of ideals and norms of elements.

We call c a rational integer factor of I. Throughout, we assume our ideals to be non-zero. If I is an O_Δ-ideal with $L(I) = N(I)$, i.e. $c = 1$, then I is called primitive which means that I has no rational integer factors other than ± 1. When I is primitive, then $N(I) = L(I) = | O_\Delta : I |$, the index of I in O_Δ. Moreover, if $I = [a, b + w_\Delta]$ is primitive, then so is its conjugate $I' = [a, b + w'_\Delta]$. Two ideals I and J of O_Δ are equivalent (denoted by $I \sim J$) if there exist non-zero $\alpha, \beta \in O_\Delta$ such that $(\alpha)I = (\beta)J$ (where (x) denotes the principal ideal generated by x).

A fundamental property of ideals which we will need is that of reduction which we now define as the last concept concerning ideals before we make the link with continued fractions, which will allow us to state the powerful continued fraction algorithm.

Definition 1 — If $\Delta > 0$ is a discriminant, then an ideal I of O_Δ is said to be reduced if it is primitive, and does not contain any non-zero element α such that both $|\alpha| < N(I)$ and $|\alpha'| < N(I)$.

In order to ensure that the uninitiated reader will be able to more readily see how to recognize reduced ideals (not always so transparent from Definition 1), we include the following facts.

Theorem 2 — Let $\Delta > 0$, be a discriminant, and let $I = [N(I), b + w_\Delta]$ be a primitive ideal in O_Δ.

1. If $\gamma = (b + w_\Delta)/N(I) > 1$, and $-1 < \gamma' < 0$, then I is reduced.
2. If $N(I) < \sqrt{\Delta}/2$, then I is reduced.
3. If I is reduced, then $N(I) < \sqrt{\Delta}$.
4. If $0 \leq b < a$, and $\sqrt{\Delta}/2 < N(I) < \sqrt{\Delta}$, then I is reduced if and only if $N(I) - w_\Delta < b < -w'_\Delta$.

PROOF : See Mollin[1], Corollaries 1.4.2-1.4.4, p. 19. □

Now we give an elucidation of the theory of continued fractions as it pertains to the above. A continued fraction is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_{i-1} + \cfrac{1}{a_i + \ddots}}}}$$

where $a_i \in \mathbf{R}$ are called the partial quotients of the continued fraction expansion. Herein, we will be concerned strictly with the following types of continued fractions. If $a_i \in \mathbf{Z}$ and $a_i > 0$ for $i > 0$ we call the expression an infinite simple continued fraction (which is equivalent to being an irrational number) and use the notational convenience $\langle a_0; a_1, a_2, ..., a_i, \ldots \rangle$ to denote it. Furthermore, we will be concerned with continued fraction expressions involving quadratics, so we need the following.

Definition 2 — A real number γ is called a quadratic irrational, associated with the radicand D, if γ can be written as

$$\gamma = (P + \sqrt{D})/Q$$

where P, Q, $D \in \mathbf{Z}$, $D > 0$, $Q \neq 0$, and $P^2 \equiv D \pmod{Q}$.

We will denote the continued fraction expansion of γ by $\gamma = \langle a_0; a_1, a_2, \ldots \rangle$. Now, if we set $P = P_0$, $Q = Q_0$, and $a_0 = \lfloor (P_0 + \sqrt{D})/Q_0 \rfloor = \lfloor \gamma \rfloor$, then we may recursively define (for $i \geq 0$) :

$$P_{i+1} = a_i Q_i - P_i, \qquad \ldots (2)$$

$$Q_{i+1} = (D - P_{i+1}^2)/Q_i, \qquad \ldots (3)$$

and

$$a_{i+1} = \lfloor (P_{i+1} + \sqrt{D})/Q_{i+1} \rfloor, \qquad \ldots (4)$$

(where $\lfloor x \rfloor$ is the greatest integer less than or equal to x, i.e. the floor of x).

Together with the following discussion, the above provides the groundwork for the introduction of the formal link between continued fractions and ideals which has come to be known as the Infrastructure (see Mollin[1], pp. 41-65), and which is embodied in the continued fraction algorithm to follow below.

Quadratic irrationals are also special in terms of periodicity, namely every quadratic irrational is periodic (Mollin[1], see p. 56). Furthermore, there is a special kind of periodicity, which will provide our link to the theory of reduced ideals. We therefore need to define this special type.

Definition 3 — An infinite simple continued fraction γ is called purely periodic if $\gamma = \langle \overline{a_0; a_1, a_2, ..., a_{l-1}} \rangle$, said to have preiod length equal to $l = l(\gamma)$. (We may also write $\gamma = \langle a_0; \overline{a_1, a_2, ..., a_l} \rangle$.)

Now we look at the quadratic irrationals which satisfy this property.

Definition 4 — A quadratic irrational γ is called reduced, provided $\gamma > 1$ and $-1 < \gamma' < 0$.

The reader is invited to compare Definition 4 with Theorem 2(1) to see the connection which is now emerging between the ideals and continued fractions.

A fundamental fact (see Mollin[1], Theorem 2.1.1, pp. 42-43) which we will use throughout the balance of the paper is the following.

Theorem 3 — The continued fraction expansion of a quadratic irrational is purely periodic if and only if it is reduced.

Now we are finally in a position to link the continued fractions with reduced ideals. If $I = [a, b + w_\Delta]$ is a primitive ideal in O_Δ, we denote the continued fraction expansion of $\gamma = (b + w_\Delta)/a$ by $\langle a_0; \overline{a_1, a_2, \ldots, a_l} \rangle$ with period length $l = l(\gamma)$ where $a = a_0$, and (for g and h as above) :

$$(P_0, Q_0) = ((\sigma_0 b + f_\Delta (\sigma_0 - 1) + h\sigma_0)/g, a\sigma_0/g), \qquad \ldots (5)$$

$$a_0 = \lfloor (P_0 + \sqrt{D})/Q_0 \rfloor. \qquad \ldots (6)$$

Now we see that eqns. (2)-(4) are satisfied for our quadratic irrational γ based upon the given O_Δ-ideal. This association allows us to present the following crucial, and fundamental algorithm which produces all of the reduced ideals equivalent to a given O_Δ-ideal.

Theorem 4 (*The continued fraction algorithm*) — Let $\Delta > 0$ be a discriminant, and let $I = I_1 = [a, b + w_\Delta]$ be a primitive ideal in the order O_Δ. Set $P = P_0$ and $Q = Q_0$, as defined in eqn. (5), and let P_i and Q_i for $i > 0$ be defined by eqns. (2)-(3) in the continued fraction expansion of $\gamma = \gamma_0 = (P + \sqrt{D})/Q$. If $I_k = [Q_{k-1}/\sigma, (P_{k-1} + \sqrt{D})/\sigma]$, then $I_1 \sim I_k$ for all $k \geq 1$. There exists a least value $m \geq 1$ such that I_m is reduced, and I_{m+i} is reduced for all $i \geq 0$.

The following consequence of Theorem 4 shows how we achieve complete solutions of eqn. (1) via continued fractions. First we need to define a well-known pair of sequences. Define two sequences of integers $\{A_k\}$ and $\{B_k\}$ inductively by

$$A_{-2} = 0,\ A_{-1} = 1,\ A_k = a_k A_{k-1} + A_{k-2} \quad (\text{for } k \geq 0),$$

$$B_{-2} = 1,\ B_{-1} = 0,\ B_k = a_k B_{k-1} + B_{k-2} \quad (\text{for } k \geq 0),$$

where $\{a_i\}$ is an infinite sequence of integers with $a_i > 0$ for $i > 0$.

Corollary 1 — Let $D > 0$ be a non-square rational integer with $\gamma = \sqrt{D} = \langle a_0; a_1, a_2, \ldots \rangle$ and $l = l(\sqrt{D})$. If l is even, then all positive solutions of $x^2 - Dy^2 = 1$ are given by $x = A_{kl-1}$ and $y = B_{kl-1}$ for $k \geq 1$, whereas there are no solutions to $x^2 - Dy^2 = -1$. If l is odd, then all positive solutions of $x^2 - Dy^2 = 1$ are given by $x = A_{2kl-1}$ and $y = B_{2kl-1}$ for $k \geq 1$, whereas all positive solutions of $x^2 - Dy^2 = -1$ are given by $x = A_{(2k-1)l-1}$, $y = B_{(2k-1)l-1}$ for $k \geq 1$.

From Theorem 4, we get all reduced ideals equivalent to a given reduced ideal $I = [N(I), b + w_\Delta]$. In other words, in the simple continued fraction expansion of $(b + w_\Delta)/a$, where $a = N(I)$, we have, for P_i and Q_i as in eqns. (2) and (3),

$$I = I_1 = [Q_0/\sigma, (P_0 + \sqrt{D})/\sigma] \sim I_2 = [Q_1/\sigma, (P_1 + \sqrt{D})/\sigma]$$

$$\sim \ldots \sim I_l = [Q_{l-1}/\sigma, (P_{l-1} + \sqrt{D})/\sigma].$$

Finally, $I_{l+1} = I_1 = I$ for a complete cycle of reduced ideals equivalent to I of length $l(I) = l$. Therefore, the $(P_i + \sqrt{D})/Q_i$ are the complete quotients of $(b + w_\Delta)/N(I) = \gamma$, and the a_i's are the partial quotients of γ. Also, the Q_i/σ's represent the norms of all reduced ideals equivalent to I. Note that $l = l(\gamma) = l(I)$ is the period length of the cycle of reduced ideals equivalent to I, which in turn, is the period length of the simple continued fraction expansion of the quadratic irrational γ.

The following is a well-known result dating back to Lagrange, which gives more information pertaining to the solutions of eqn. (1).

Theorem 5 — Let $I = [a, b + w_\Delta]$ be a reduced ideal, where ε_Δ is the fundamental unit of O_Δ. If P_i and Q_i, for $i = 1, 2, \ldots, l(I) = l$, appear in the continued fraction expansion of $(b + w_\Delta)/a$ (via eqns. (2) and (3)), then

$$\varepsilon_\Delta = \prod_{i=1}^{l} (P_i + \sqrt{D})/Q_i$$

and

$$N(\varepsilon_\Delta) = (-1)^l.$$

We now describe the interplay between the fundamental units in two special orders which has consequences for the solution of eqn. (1) as well, and will be the principal tool for presentation of results in section 2. Since an arbitrary real quadratic order O_Δ is always contained in the maximal order O_{Δ_0}, then ε_Δ is a unit in O_{Δ_0}. Therefore, $\varepsilon_\Delta = \varepsilon_{\Delta_0}^u$, where u is called the unit index of O_Δ in O_{Δ_0}.

In particular, if $O_{\Delta_0} = [1, (1 + \sqrt{D_0})/2]$ for $D_0 \equiv 1 \pmod 4$, then set $O_\Delta = [1, \sqrt{D_0}]$ where $\Delta = 4\Delta_0 = f_\Delta^2 D_0 = f_\Delta^2 \Delta_0$. In this case, either $\varepsilon_\Delta = \varepsilon_{\Delta_0}^3$ or $\varepsilon_\Delta = \varepsilon_{\Delta_0}$.[†]

[†]This follows from the general class number formula for real quadratic orders. See, for example Mollin[1], (footnote 1.5.9, pp. 25-26). If $f_\Delta > 1$ is the conductor of an order O_Δ with fundamental discriminant Δ_0 and unit index u, then the class number of O_Δ is $h_\Delta = h_{\Delta_0} \psi_{\Delta_0}(f_\Delta)/u$ where $\psi_{\Delta_0}(f_\Delta) = f_\Delta \prod (1 - (\Delta_0/p)/p)$ with the product ranging over all the distinct primes dividing f_Δ and (*/*) denotes the Kronecker symbol. It is shown that $h_\Delta \geq h_{\Delta_0}$. Indeed, $h_\Delta/h_{\Delta_0} \in \mathbb{Z}$, but it is an open question as to when $h_\Delta = h_{\Delta_0}$. In any ease, u divides ψ_{Δ_0}. Thus, in our special case we deduce that $h_\Delta = h_{\Delta_0}$ when $\Delta \equiv 1 \pmod 8$, and $h_\Delta = 3h_{\Delta_0}/u$ when $\Delta \equiv 5 \pmod 8$. In other words, either $\varepsilon_\Delta = \varepsilon_{\Delta_0}^3$ or $\varepsilon_\Delta = \varepsilon_{\Delta_0}$.

What all of the above depicts is that the minimal solution (or fundamental solution) of eqn. (1) is given by ε_Δ. In other words, if $\varepsilon_\Delta = a + b\sqrt{D}$, $a, b \in \mathbf{N}$ and $A^2 - B^2D = \pm 1$ for any $A, B \in \mathbf{N}$ is a solution of eqn. (1), then $A + B\sqrt{D} = \varepsilon_\Delta^n = (a + b\sqrt{D})^n$ for some $n \in \mathbf{N}$. This is the process which we will exploit in §2. First we make some observations which are of interest in their own right with respect to solutions of eqn. (1).

By Theorem 5, the Diophantine equation

$$x^2 - Dy^2 = -1 \qquad \ldots (7)$$

is solvable if and only if $l(\sqrt{D})$ is odd. Also, if $D = D_0 = \Delta_0 \equiv 5 \pmod 8$, then

$$x^2 - Dy^2 = \pm 4, \qquad gcd(x, y) = 1, \qquad \ldots (8)$$

is solvable if and only if ε_{Δ_0} is not in $O_\Delta = [1, \sqrt{D_0}]$ where $\Delta = 4D_0 = 4\Delta_0 = 4D$.

If $\Delta_0 = D_0 \equiv 1 \pmod 4$, then the solvability of eqn. (8) is equivalent to the ideal $I = [4, 1 + \sqrt{\Delta_0}]$ being principal in $O_\Delta = [1, \sqrt{\Delta_0}]$, where $\Delta = 4\Delta_0 = 4D_0 = 4D$. However, it is not the case that the solvability of

$$x^2 - Dy^2 = -4, \qquad gcd(x, y) = 1, \qquad \ldots (9)$$

for $\Delta_0 \equiv 5 \pmod 8$ is equivalent to ε_{Δ_0} not being in $O_\Delta = [1, \sqrt{\Delta_0}]$ for $\Delta = 4\Delta_0$. A counterexample is $\Delta_0 = 21$, since $\varepsilon_{\Delta_0} = (5 + \sqrt{21})/2$ which is not in $[1, \sqrt{21}] = O_\Delta$. Yet, eqn. (9) is not solvable.

Questions involving the solvability of eqn. (8) are attributed to Eisenstein, who looked for a criterion for the solvability of this equation. There is such a criterion for the solvability of this equation when $N(\varepsilon_\Delta) = -1$, but no criterion has been given for $N(\varepsilon_\Delta) = 1$ (see Mollin[1], pp. 59-62).

Special types of discriminants have been studied in depth, such as the following.

Definition 5 — Let $D > 0$ be a radicand which can be written in the form $D = s^2 + r$ where $r \mid 4s$. D is said to be of Extended-Richaud-Degert-type (or simply ERD-type). If $|r| \in \{1, 4\}$, then D is called narrow-Richaud-Degert type (or simply narrow RD-type).

It is these latter types which Ramasamy[2] studied, ostensibly without being aware of it. In order to generalize his results, we need some fundamental facts concerning ERD-types. The fundamental units for ERD-types are known (for instance see Ramasamy[2], Chapter 3, pp. 77-87).

Theorem 6 — Let $D = D_0 \neq 5$ be a positive fundamental radicand which is of ERD-type, with $D = s^2 + r$ where $r \mid 4s$, then

$$\varepsilon_\Delta = \begin{cases} s + \sqrt{D} & \text{if } |r| = 1, \\ (s + \sqrt{D})/2 & \text{if } |r| = 4, \\ (2s^2 + r + 2s\sqrt{D})/|r| & \text{if } |r| \notin \{1, 4\}. \end{cases}$$

Now we are in a position to state the main results.

2. RESULTS

By the results developed in section 1, we have the following.

Theorem 7 — Let $\Delta_0 > 0$ be a fundamental discriminant with radicand D_0. If $\varepsilon_{\Delta_0} \in [1, \sqrt{D_0}]$, then the fundamental solution of eqn. (2) is given by ε_{Δ_0}. If $\varepsilon_{\Delta_0} \notin [1, \sqrt{D_0}]$, then the fundamental solution is given by $\varepsilon_{\Delta_0}^3$.

For example, we have the following results from Mollin[1]. Since we will need to separate the two cases of eqn. (1), we therefore label the complement of eqn. (7)

$$x^2 - Dy^2 = 1. \qquad \ldots (10)$$

In each of the following Corollaries, $D > 0$ is assumed to be a fundamental radicand.

Corollary 2 (Ramasamy[2], Theorem 1, p. 578) — If $D = 4t^2 + 12t + 5$ is a fundamental radicand for $t \in \mathbf{N}$, then the fundamental solution of eqn. (10) is $4t^3 + 18t^2 + 24t + 9 + (2t^2 + 6t + 4)\sqrt{D}$.

PROOF : Since $D = (2t+3)^2 - 4$ for $t \geq 1$, then D is of narrow RD-type, so we may invoke Theorem 6 to get that

$$\varepsilon_{\Delta_0} = \varepsilon_D = (2t + 3 + \sqrt{D})/2 \notin [1, \sqrt{D}].$$

Therefore,

$$\varepsilon_{\Delta_0}^3 = 4t^3 + 18t^2 + 24t + 9 + (2t^2 + 6t + 4)\sqrt{D}$$

is the fundamental solution of eqn. (10). Notice here that $l(w_\Delta) = 2$ and $l(\sqrt{D}) = 6$ (for example see Mollin[1], Exercise 2.1.13 (c), p. 58).

Corollary 3 (Ramasamy[2], Theorem 2, p. 578) — If $D = m^2t^4 + 2mnt^3 + n^2t^2 + mt + n$, $(m, n, t, \in \mathbf{N})$, then the fundamental solution of eqn. (10) is $2mt^3 + 2nt^2 + 1 + 2t\sqrt{D}$.

PROOF : The radicand $D = (mt+n)^2t^2 + (mt+n)$ is of ERD-type, so by Theorem 6

$$\varepsilon_\Delta = 2(mt + n)t^2 + 1 + 2t\sqrt{D} \in [1, \sqrt{D}].$$

(Note here that $l(\sqrt{D}) = 2$.) □

Corollary 4 (Ramasamy[2], Theorem 3, pp. 578-579) — If $D = m^2t^4 + 2mnt^3 + n^2t^2 + 2mt + 2n$, $(m, n, t, \in \mathbf{N})$, then the fundamental solution of eqn. (10) is $mt^3 + nt^2 + 1 + t\sqrt{D}$.

PROOF : Again $D = (mt + n)^2 + 2(mt + n)$ is of ERD-type, so by Theorem 6

$$\varepsilon_\Delta = mt^2 + nt + 1 + t\sqrt{D} \in [1, \sqrt{D}].$$

Note that here $l(\sqrt{D}) = 2$. □

Corollary 5 (Ramasamy[2], Theorem 4, p. 579) — If $D = 4t^2 + 4t + 5$, $(t \in \mathbf{N})$, then by Theorem 6 the fundamental solution of eqn. (7) is $4t^3 + 6t^2 + 6t + 2 + (2t^2 + 2t + 1)\sqrt{D}$.

PROOF : Since $D = (2t+1)^2 + 4$ is of narrow RD-type, then by Theorem 6,

$$\varepsilon_D = (2t + 1 + \sqrt{D})/2 \notin [1, \sqrt{D}].$$

Therefore,

$$\varepsilon_D^3 = 4t^3 + 6t^2 + 6t + 2 + (2t^2 + 2t + 1)\sqrt{D}$$

is the fundamental solution of eqn. (7). Notice here that $l(\sqrt{D}) = 5$ and $l(w_D) = 1$ (see Mollin[1], Exercise 2.1.13(c), p. 58). □

Corollary 6 (Ramasamy[2], Theorem 5, p. 579) — If $D = 4t^4 + 4t + 2$, $(t \in \mathbb{N})$, then the fundamental solution of eqn. (7) is $4t^4 - 4t^3 + 2t^2 + 2t - 1 + (2t^2 - 2t + 1)\sqrt{D}$.

PROOF : This D is not necessarily of ERD-type. However, from eqns. (2)-(4) and Theorem 4 we get the simple continued fraction expansion of $\sqrt{D}$ as follows. To see how the following table is constructed we begin with $i = 0$, and calculate $P_0 = 0$ by eqn. (5), since $\sigma_0 = \sigma = 1$, given that $D \equiv 2$, (mod 4). Thus $Q_0 = 1$ by eqn. (5), and $a_0 = 2t^2 = \lfloor \sqrt{D} \rfloor$, by eqn. (6). Now, by eqn. (2), $P_1 = a_0 Q_0 - P_0 = 2t^2$, and by equation (3), $Q_1 = (D - P_1^2)/Q_0 = 4t^2 + 4t + 2 - 4t^4 = 4t + 2$. Also, by eqn. (4), $a_1 = \lfloor (P_1 + \sqrt{D})/Q_1 \rfloor = \lfloor (2t^2 + 2t^2)/(4t + 2) \rfloor = t - 1$. The balance of the table is constructed in this fashion.

i	0	1	2	3	4	5
P_i	0	$2t^2$	$2t^2 - 2t - 2$	$2t + 1$	$2t^2 - 2t - 2$	$2t^2$
Q_i	1	$4t + 2$	$2t^2 - 1$	$2t^2 - 1$	$4t + 2$	1
a_i	$2t^2$	$t - 1$	1	1	$t - 1$	$4t^2$

Since $l(\sqrt{D}) = 5$, then by Theorem 5, we may easily find ε_Δ as follows.

$$\varepsilon_\Delta = \prod_{i=1}^{5} (P_i + \sqrt{D})/Q_i$$

$$= \frac{(2t^2 + \sqrt{D})}{(4t+2)} \cdot \frac{(2t^2 - 2t - 2 + \sqrt{D})}{(2t^2 - 1)} \cdot \frac{(2t + 1 + \sqrt{D})}{(2t^2 - 1)}$$

$$\cdot \frac{(2t^2 - 2t - 2 + \sqrt{D})}{(4t+2)} \cdot (2t^2 + \sqrt{D})$$

$$= 4t^4 - 4t^3 + 2t^2 + 2t - 1 + (2t^2 - 2t + 1)\sqrt{D},$$

as required. □

By the same simple exercise as performed in the proof of Corollary 6 we achieve the following.

Corollary 7 (Ramasamy[2], Theorem 6, p. 579) — If $D = 16t^4 + 8t^3 + 9t^2 + 6t + 2$, $(t \in \mathbf{N})$, then the fundamental solution of eqn. (7) is $16t^4 + 4t^3 + 8t^2 + 3t + 1 + (4t^2 + 1)\sqrt{D}$.

We also have the following generalization.

Corollary 8 (Ramasamy[2], Corollary 1, p. 580) — If $D = k^2t^2 \pm 2k$, then the fundamental solution of eqn. (1) is $\varepsilon_\Delta = kt^2 \pm 1 \pm t\sqrt{D}$.

PROOF : Since D is of ERD-type, then we employ the above techniques to get the result immediately from Theorems 5 and 6. □

Finally, we have the following.

Corollary 9 (Ramasamy[2], Theorem 8, p. 580) — If $u^2 - 2v^k = \pm 1$, $(u, v \in \mathbf{N})$, and $\Delta_0 = D_0 = u^2 \mp 4$ is a fundamental discriminant, then $\varepsilon_{\Delta_0}^3 = \varepsilon_{4\Delta_0} = u(v^k - 1) + v^k\sqrt{D}$ is the fundamental solution of eqn. (1). In other words, $u(v^k - 1) + v\sqrt{D}$ is the fundamental solution of $x^2 - Dy^{2k} = \pm 1$.

PROOF : Since Δ_0 is of narrow RD-type, then by Theorem 6,

$$\varepsilon_{\Delta_0} = (u \mp 1)/4 \notin [1, \sqrt{D}].$$

Hence,

$$\varepsilon_\Delta^3 = u(v^k - 1) + v^k\sqrt{D}$$

is a fundamental solution of eqn. (1). Equivalently, $u(v^k - 1) + v\sqrt{D}$ is the fundamental solution of $x^2 - Dy^{2k} = \pm 1$.

The primary thrust of this paper has been to enlighten the reader in terms of the solution of Pell-like equations via the interplay between the two special orders described in section 1, involving the continued fraction algorithm. This methodology, as we have illustrated in the numerous applications above, is a powerful and beautiful tool.

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