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PSEUDOLINEAR VECTOR OPTIMIZATION PROBLEMS CONTAINING *n*-SET FUNCTIONS

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Characterization of pseudolinearity for n-set functions is presented and is utilized to establish sufficient efficiency conditions and duality for vector optimization problems containing n-set functions.

1. INTRODUCTION AND PRELIMINARIES

In this paper, we define pseudolinear n-set functions and present a characterization of these functions, similar to that given by Chew and Choo[6] for point functions. Sufficient efficiency and duality results are established for a multiobjective fractional programming problem, containing n-set functions, under the assumptions of pseudolinearity on the functions involved. The results developed in the present paper are extensions of those obtained previously by Bector *et al.*[3] and Kaul *et al.*[8], who studied duality for pseudolinear multiobjective programs and pseudolinear multiobjective fractional programs, respectively, for point functions.

Throughout the paper, we assume that (X, A, μ) is a finite atomless measure space with $L_1(X, A, \mu)$ separable, A^n is the n-fold product of a σ-algebra A of subsets of a given set X, d is a pseudometric on A^n, defined by $d(R, S) = \left[\sum_{i=1}^{n} \mu^2(R_i \Delta S_i)\right]^{1/2}$, $R = (R_1, ..., R_n)$, $S = (S_1, ..., S_n) \in A^n$, and Δ denotes symmetric difference. Each $I_{S_i} \in A$ can be identified with its characteristic function $I_{S_i} \in L_\infty(X, A, \mu) \subset L_1(X, A, \mu)$. Essentially, $R, S \in A^n$ will be regarded as equivalent if $d(R, S) = 0$. We admit $F(R) = F(S)$ if $d(R, S) = 0$. For $f \in L_1(X, A, \mu)$ and $S_i \in A$, the integral $\int_{S_i} f d\mu$ will be denoted by $\langle f, I_{S_i} \rangle$. It was established by Morris[11], that for any $(S_i, R_i, \lambda) \in A \times A \times [0, 1]$, there exist sequences $\{S_i^k\}$ and $\{R_i^k\}$ in A such that

$$I_{S_i^k} \xrightarrow{w^*} \lambda I_{R_i \setminus S_i} \quad \text{and} \quad I_{R_i^k} \xrightarrow{w^*} (1-\lambda) I_{S_i \setminus R_i} \qquad \text{... (1.1)}$$

imply
$$I_{S_i^k \cup R_i^k \cup (S_i \cap R_i)} \xrightarrow{w^*} \lambda I_{R_i} + (1-\lambda) I_{S_i} \qquad \text{... (1.2)}$$

where w^* stands for weak $*$ convergence. The sequence

$$\{V_i^k(\lambda)\} = \{S_i^k \cup R_i^k \cup (S_i \cap R_i)\}$$

satisfying (1.1) and (1.2) is called the Morris sequence associated with (S_i, R_i, λ).

Definition 1.1 (Lin[9]) — A subfamily α of A^n is convex if given $S, R \in \alpha$ and $\lambda \in [0, 1]$, there exists a Morris sequence $\{V_i^k(\lambda)\}$ in A, associated with (S_i, R_i, λ), for each $i = 1, ..., n$, such that $(V_1^k(\lambda), ..., V_n^k(\lambda)) \in \alpha$, for all $k \in N$, where N is the set of natural numbers.

We use the definitions of differentiability and partial differentiability as given by Corley[7]. We present the following definition from Zalmai[12] for ready reference.

Definition 1.2 — A differentiable n-set function $F : A^n \to \mathbb{R}$ is said to be pseudoconvex on A^n if for all $R, S \in A^n$ $(R \neq S)$

$$\sum_{i=1}^{n} \langle D_i F_S, I_{R_i} - I_{S_i} \rangle \geqq 0 \;\Rightarrow\; F(R) \geqq F(S)$$

or equivalently,

$$F(R) < F(S) \Rightarrow \sum_{i=1}^{n} \langle D_i F_S, I_{R_i} - I_{S_i} \rangle < 0.$$

F is said to be pseudoconcave on A^n if and only if $-F$ is pseudoconvex on A^n.

1. Pseudolinear n-Set Functions

Definition 2.1 — A differentiable n-set function $F : A^n \to \mathbb{R}$ is said to be pseudolinear on A^n if F is both pseudoconvex and pseudoconcave on A^n.

Theorem 2.1 — Let F be a differentiable n-set function defined on a convex subfamily α of A^n. Then the following three statements are equivalent :

(a) F is pseudolinear on α.

(b) For any $R, S \in \alpha$, $\sum_{i=1}^{n} \langle D_i F_S, I_{R_i} - I_{S_i} \rangle = 0 \;\Leftrightarrow\; F(R) = F(S)$.

(c) There exists a $2n$-set function $p : \alpha \times \alpha \to \mathbb{R}$ such that

(i) $p(R, S) > 0$ for all $R, S \in \alpha$,

(ii) $F(R) = F(S) + p(R, S) \sum_{i=1}^{n} \langle D_i F_S, I_{R_i} - I_{S_i} \rangle$ for all $R, S \in \alpha$.

PROOF : (a) $\Rightarrow$ (b) : Suppose F is pseudolinear on α and let $\sum_{i=1}^{n} \langle D_i F_S, I_{R_i} - I_{S_i} \rangle = 0$. Then pseudoconvexity of F implies that

$$F(R) \geqq F(S), \quad \text{... (2.1)}$$

similarly pseudoconcavity of F implies that

$$-F(R) \geqq -F(S). \quad \text{... (2.2)}$$

It follows from (2.1) and (2.2) that $F(R) = F(S)$.

Conversely, if $F(R) = F(S)$, we wish to show that

$$\sum_{i=1}^{n} \langle D_i F_S, I_{R_i} - I_{S_i} \rangle = 0.$$

We first show that

$$\lim_{k \to \infty} F(V^k(\lambda)) = F(R) \quad \text{... (2.3)}$$

where $V^k(\lambda) = (V_1^k(\lambda), ..., V_n^k(\lambda))$, each $V_i^k(\lambda)$, $i = 1, ..., n$ is a Morris sequence associated with (S_i, R_i, λ).

If (2.3) is not true, then there are two cases to be discussed :

(i) $\lim_{k \to \infty} F(V^k(\lambda)) > F(R)$,

(ii) $\lim_{k \to \infty} F(V^k(\lambda)) < F(R)$.

In case (i), it follows from the definition of the limit of the function that there exists an integer $N_1 > 0$ such that

$$F(R) < F(V^k(\lambda)) \quad \text{for all } k > N_1 \quad \text{... (2.4)}$$

which, in view of the pseudoconvexity of F, implies that

$$\sum_{i=1}^{n} \langle D_i F_{V^k(\lambda)}, I_{R_i} - I_{V_i^k(\lambda)} \rangle < \text{for all } k > N_1. \quad (2.5)$$

Now proceeding along similar lines, as in the proof of Theorem 3.5 of Lin[10], we see that

$$d(R, V^k(\lambda)) = \left[\sum_{i=1}^{n} \| I_{R_i} - I_{V_i^k(\lambda)} \|_{L_1}^2 \right]^{1/2}$$

$$\rightarrow \left[\sum_{i=1}^{n} \| I_{R_i} - \lambda I_{R_i} - (1-\lambda) I_{S_i} \|_{L_1}^2 \right]^{1/2}$$

$$= (1-\lambda) \left[\sum_{i=1}^{n} \| I_{R_i} - I_{S_i} \|_{L_1}^2 \right]^{1/2} = (1-\lambda)\, d(R, S). \qquad \text{... (2.6)}$$

Hence $\displaystyle \lim_{k \to \infty} \sum_{i=1}^{n} \langle D_i F_{V^k(\lambda)}, I_{R_i} - I_{V_i^k(\lambda)} \rangle$

$$= (1-\lambda) \lim_{k \to \infty} \sum_{i=1}^{n} \langle D_i F_{V^k(\lambda)}, I_{R_i} - I_{S_i} \rangle. \qquad \text{... (2.7)}$$

Similarly, $\displaystyle d(V^k(\lambda), S) = \left[\sum_{i=1}^{n} \| I_{V_i^k(\lambda)} - I_{S_i} \|_{L_1}^2 \right]^{1/2}$

$$\rightarrow \left[\sum_{i=1}^{n} \| \lambda I_{R_i} + (1-\lambda) I_{S_i} - I_{S_i} \|_{L_1}^2 \right]^{1/2}$$

$$= \lambda \left[\sum_{i=1}^{n} \| I_{R_i} - I_{S_i} \|_{L_1}^2 \right]^{1/2} = \lambda\, d(R, S). \qquad \text{... (2.8)}$$

Therefore, $\displaystyle \lim_{k \to \infty} \sum_{i=1}^{n} \langle D_i F_{V^k(\lambda)}, I_{S_i} - I_{V_i^k(\lambda)} \rangle$

$$= -\lambda \lim_{k \to \infty} \sum_{i=1}^{n} \langle D_i F_{V^k(\lambda)}, I_{R_i} - I_{S_i} \rangle. \qquad \text{... (2.9)}$$

It follows from (2.7) and (2.9) that

$$\lim_{k \to \infty} \sum_{i=1}^{n} \langle D_i F_{V^k(\lambda)}, I_{S_i} - I_{V_i^k(\lambda)} \rangle$$

$$= -\frac{\lambda}{1-\lambda} \lim_{k \to \infty} \sum_{i=1}^{n} \langle D_i F_{V^k(\lambda)}, I_{R_i} - I_{V_i^k(\lambda)} \rangle$$

$\geqq 0$ (by virtue of (2.5) and $0 < \lambda < 1$), that is, $$\lim_{k\to\infty} \sum_{i=1}^{n} \langle D_i F_{V^k(\lambda)}, I_{S_i} - I_{V_i^k(\lambda)} \rangle \geqq 0.$$

It again follows from the definition of the limit of the function that there exists an integer $N_2 > 0$ such that

$$\sum_{i=1}^{n} \langle D_i F_{V^k(\lambda)}, I_{S_i} - I_{V_i^k(\lambda)} \rangle \geqq 0 \quad \text{for all } k > N_2.$$

Pseudoconvexity of F now implies that

$$F(S) \geqq F(V^k(\lambda)) \quad \text{for all } k > N_2$$

$$\Rightarrow \quad F(R) \geqq F(V^k(\lambda)) \quad \text{for all } k > N_2, \text{ (as } F(R) = F(S)). \qquad \text{... (2.10)}$$

Now choose $N_3 = \max(N_1, N_2)$. For $K > N_3$, the relations (2.4) and (2.10) are not compatible. Hence a contradiction.

Case (ii) again leads us to a contradiction. Proof for this case follows along the lines similar to those adopted in case (i), by using pseudoconcavity instead of pseudoconvexity of the function F. Hence (2.3) is true.

Since F is differentiable, it follows that (Lin[9])

$$F(V^k(\lambda)) = F(S) + \sum_{i=1}^{n} \langle D_i F_S, I_{V_i^k(\lambda)} - I_{S_i} \rangle + E(V^k(\lambda), S) \qquad \text{... (2.11)}$$

for all $k \in N$,

where $\overline{\lim}_{k\to\infty} E(V^k(\lambda), S)$ is $O(\lambda)$, that is, $\lim_{\lambda\to 0} \lim_{k\to\infty} \frac{E(V^k(\lambda), S)}{\lambda} = 0.$

Hence (2.11) along with (2.8) implies that

$$\lim_{k\to\infty} F(V^k(\lambda)) = F(S) + \sum_{i=1}^{n} \langle D_i F_S, I_{R_i} - I_{S_i} \rangle + O(\lambda). \qquad \text{... (2.12)}$$

Using (2.3) we have

$$\lambda \sum_{i=1}^{n} \langle D_i F_S, I_{R_i} - I_{S_i} \rangle + O(\lambda) = 0.$$

Now dividing by λ and taking limits as $\lambda \to 0$ yields

$$\sum_{i=1}^{n} \langle D_i F_S, I_{R_i} - I_{S_i} \rangle = 0.$$

Hence the proof.

(b) $\Rightarrow$ (c) : We now need to define, under conditions (b), a $2n$-set function $p : \alpha \times \alpha \to \mathbb{R}$, that has the required properties stated in (c). Let us therefore define

$$p(R, S) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} \langle D_i F_S, I_{R_i} - I_{S_i} \rangle = 0, \\ \dfrac{F(R) - F(S)}{\sum_{i=1}^{n} \langle D_i, F_S, I_{R_i} - I_{S_i} \rangle}, & \text{otherwise.} \end{cases}$$

It suffices to show that $p(R, S) > 0$ if $\sum_{i=1}^{n} \langle D_i F_S, I_{R_i} - I_{S_i} \rangle \neq 0$. Clearly, $F(R) \neq F(S)$ if $\sum_{i=1}^{n} \langle D_i F_S, I_{R_i} - I_{S_i} \rangle \neq 0$.

Assume that $F(R) > F(S)$. Since $S, R \in \alpha$, there exists a Morris sequence $\{V_i^k(\lambda)\}$ associated with (S_i, R_i, λ) for each $i = 1, \ldots, n$, $\lambda \in (0, 1)$, such that

$$(V_1^k(\lambda), \ldots, V_n^k(\lambda)) \in \alpha \text{ for all } k \in N.$$

We claim that $\lim_{k \to \infty} F(V^k(\lambda)) > F(S)$ for all $\lambda \in (0, 1)$.

Suppose that $\lim_{k \to \infty} F(V^k(\lambda)) \leqq F(S)$, for some $\lambda \in (0, 1)$. Then by the definition of the limit of the function there exists an integer N_1 such that $F(V^k(\lambda)) \leq FS$, for some $\lambda \in (0, 1)$ for all $k > N_1$. Let

$$Q = [\delta : F(V^k(\xi)) \leqq F(S) \text{ for all } \xi \in [\lambda, \delta], \text{ for all } k > N_1.$$

This set Q is non-empty, because $\lambda \in Q$. Also Q is bounded and hence Q possesses a least upper bound. Let μ be the least upper bound of Q. Then it follows from the continuity of the function F (with respect to a pseudometric d) that (Apostol[1])

$$F(V^k(\mu)) = F(S),\ \mu > \lambda \text{ for all } k > N_1 ,$$

$$\Rightarrow \quad \lim_{k \to \infty} F(V^k(\mu)) = F(S),\ \mu > \lambda.$$

Now it follows from (2.9) that

$$\sum_{i=1}^{n} \langle D_i F_S, I_{R_i} - I_{S_i} \rangle = \frac{1}{\mu} \lim_{k \to \infty} \sum_{i=1}^{n} \langle D_i F_S, I_{V_i^k(\mu)} - I_{S_i} \rangle$$

$$= 0 \quad \text{(by hypothesis (b))},$$

i.e., $\sum_{i=1}^{n} \langle D_i F_S, I_{R_i} - I_{S_i} \rangle = 0$ which is a contradiction. Hence

$$\lim_{k \to \infty} F(V^k(\lambda)) > F(S) \text{ for all } \lambda \in (0, 1). \qquad \text{... (2.13)}$$

Again it follows from the definition of differentiability (Lin[9])

$$\sum_{i=1}^{n} \langle D_i F_S, I_{V_i^k(\lambda)} - I_{S_i} \rangle = F(V^k(\lambda) - F(S) + E(V^k(\lambda), S), \qquad \text{... (2.14)}$$

where $\overline{\lim}_{k \to \infty} E(V^k(\lambda), S) = O(\lambda)$. It follows from (2.8) and (2.14) that

$$\lambda \sum_{i=1}^{n} \langle D_i F_S, I_{R_i} - I_{S_i} \rangle = \lim_{k \to \infty} F(V^k(\lambda) - F(S) + O(\lambda). \qquad \text{... (2.15)}$$

Using (2.13) we have

$$\lambda \sum_{i=1}^{n} \langle D_i F_S, I_{R_i} - I_{S_i} \rangle > O(\lambda).$$

Dividing by λ and taking limits as $\lambda \to 0$ yields

$$\sum_{i=1}^{n} \langle D_i F_S, I_{R_i} - I_{S_i} \rangle \geqq 0.$$

Since $\sum_{i=1}^{n} \langle D_i F_S, I_{R_i} - I_{S_i} \rangle \neq 0$, it follows that

$$\sum_{i=1}^{n} \langle D_i F_S, I_{R_i} - I_{S_i} \rangle > 0.$$

Similarly, we can show that $\sum_{i=1}^{n} \langle D_i F_S, I_{R_i} - I_{S_i} \rangle < 0$ if $F(R) < F(S)$.

Hence we conclude that $p(R, S) > 0$, for all $R, S \in \alpha$, which completes the proof.

(c) $\Rightarrow$ (a) : The proof is trivial.

We shall call p defined above a proportional function of F, or we say that F is pseudolinear with respect to the proportional function p.

Remark 2.1 : If the n-set function $F : \alpha \to \mathbb{R}$ is pseudolinear with respect to the proportional function p then λF, $\lambda \geqq 0$, $\lambda \in \mathbb{R}$, is also pseudolinear with respect to the same proportional function p.

Remark 2.2 : If the n-set functions $F_i : \alpha \to \mathbb{R}$, $i = 1, \ldots, k$, are pseudolinear with respect to the same proportional function p, then

$$\lambda' F = \sum_{i=1}^{k} \lambda_i F_i,\ \lambda_i \geqq 0,\ i = 1, \ldots, k,$$

is also pseudolinear with r^spect to the same proportional function p.

Example 2.1 — In view of Proposition 3.1 of Lin[10], it may easily be seen that the n-set function $F : \alpha \to \mathbb{R}$, defined as

$$F(S) = \frac{\sum_{i=1}^{n} a_i \langle F_i, I_{S_i} \rangle + r}{\sum_{i=1}^{n} b_i \langle F_i, I_{S_i} \rangle + t}, \qquad \sum_{i=1}^{n} b_i \langle F_i, I_{S_i} \rangle + t > 0$$

where $a_i, b_i \in \mathbb{R}$, $i = 1, \ldots, n$; $r, t \in \mathbb{R}$, $S \in \alpha$, $F_1, \ldots, F_n \in L_1(X, A, \mu)$, is a pseudolinear n-set function.

Chew and Choo[6] can be referred for the corresponding example of pseudolinear point functions.

3. Optimization Problem Containing n-set Pseudolinear Functions

In this section we consider a multiobjective fractional n-set programming problem. We propose a Weir type dual problem and establish under appropriate pseudolinearity assumptions, some sufficient efficiency conditions and various duality results.

Authors were motivated to carry out the study of above mentioned results due to the existence of pseudolinear functions of the form $F(S) = \varphi(S)/\tau(S)$, where $\varphi(S)$ and $\tau(S)$ are not restricted by any kind of convexity/generalized convexity assumptions, can be seen from the following example :

Example 3.1 — Let $\varphi(S) = 2\left(\int_S f\,d\mu\right)^3 - 3\left(\int_S f\,d\mu\right)^2 - 5\left(\int_S f\,d\mu\right) + 6$ and $\tau(S) = 3\left(\int_S f\,d\mu\right)^3 - 10\left(\int_S f\,d\mu\right)^2 + 9\left(\int_S f\,d\mu\right) - 2$, where $f \in L_1(X, A, \mu)$ and $S \in A$.

Here $\varphi(S)$ and $\tau(S)$ are neither pseudoconvex nor pseudoconcave but the function $F(S) = \varphi(S)/\tau(S) = \left[2\int_S f\,d\mu + 3\right] / \left[3\int_S f\,d\mu - 1\right]$ is a pseudolinear set function on the set $\{S \in A : \int_S f\,d\mu > 1/3\}$.

Consider the following multiobjective fractional programming problem :

(P) Minimize $[\varphi_1(S)/\tau_1(S), ..., \varphi_k(S)/\tau_k(S)]$

$$\text{subject to } H_j(S) \leqq 0, \ j \in M, \quad \text{... (3.1)}$$

$$S = (S_1, ..., S_n) \in \alpha,$$

where $\varphi_i, \tau_i, i \in K$, and $H_j, j \in M$, are real-valued differentiable functions defined on α, $\tau_i(S) > 0$, $i \in K$, and $K = \{1, ..., k\}$, $M = \{1, ..., m\}$ are finite index sets. Here it may not be necessary to assume that $\varphi_i(S) \geqq 0$, $i \in K$.

The following definitions due to Bector *et al.*[2] will be needed in our discussion of efficiency and duality for (P).

Definition 3.1 — A feasible solution $S^* \in A^n$ for (P) is said to be efficient for (P) if and only if there is no other feasible solution $S^* \in A^n$ for (P) such that

$$\varphi_i(S)/\tau_i(S) < \varphi_i(S^*)/\tau_i(S^*) \text{ for some } i \in K,$$

$$\varphi_j(S)/\tau_j(S) \leqq \varphi_j(S^*)/\tau_j(S^*) \text{ for all } j \in K.$$

Definition 3.2 — An efficient solution S^* for (P) is said to be properly efficient if and only if there exists a scalar $M > 0$ such that, for all $i \in K$,

$$[\varphi_i(S^*)/\tau_i(S^*) - \varphi_i(S)/\tau_i(S)] \leqq M[\varphi_j(S)/\tau_j(S) - \varphi_j(S^*)/\tau_j(S^*)]$$

for some $j \in K$ such that $\varphi_j(S)/\tau_j(S) > \varphi_j(S^*)/\tau_j(S^*)$, whenever $S \in A^n$ is feasible for (P) and $\varphi_i(S)/\tau_i(S) < \varphi_i(S^*)/\tau_i(S^*)$.

In the sequel we shall need the following single-objective programming problem :

(SP) Minimize $F(S)$

$$\text{subject to } H_j(S) \leqq 0, \ j \in M,$$

$$S \in \alpha,$$

where F and H_j, $j \in M$, are real-valued differentiable functions defined on A^n.

Definition 3.3 — A feasible point $S^* \in \alpha$ is said to be a regular feasible solution for (P) or (SP), if there exists $S \in \alpha$ such that

$$H_j(S^*) + \sum_{i=1}^{n} \langle D_i H_{jS^*}, \ I_{S_i} - I_{S_i^*} \rangle < 0, \ j \in M.$$

Theorem 3.1 (Zalmai[12]) (*Necessary conditions*) — Let S^* be a regular optimal solution of (SP). Then there exists $u^* \in \mathbb{R}_+^m$ (non negative orthant of $\mathbb{R}^m$) such that

$$\langle D_i F_{S^*} + \sum_{j=1}^{m} u_j^* D_i H_{jS^*},\ I_{S_i} - I_{S_i^*} \rangle \geqq 0, \text{ for all } S_i \in A,\ i \in \{1, \ldots, n\},$$

$$u_j^* H_j(S^*) = 0,\ j \in M.$$

Now, as in Bector *et al.*[2] writing $F_i(S)$ for $\varphi_i(S)/\tau_i(S)$, $i \in K$, the problem (P) may be rewritten as

(P) Minimize $[F_1(S), \ldots, F_k(S)]$

subject to $H_j(S) \leqq 0,\ j \in M,$

$S \in \alpha.$

Now from (P) we write the following sequence of programs (P_r^*), one for each $r \in K$, each with a single objective function :

(P_r^*) Minimize $F_r(S)$

Subject to $F_i(S) \leqq F_i(S^*)$, for all $i \in K$, $i \neq r$,

$H_j(S) \leqq 0$, for all $j \in M$,

$S \in \alpha.$

The following lemma can be proved on the lines of Chankong and Haimes[5].

Lemma 3.1 — Let $S^* \in \alpha$ be a regular feasible solution of (P) and regular for at least one (P_r^*), $r \in K$, then S^* is an efficient solution of (P) if and only if it is an optimal solution of (P_r^*) for each $r \in K$.

Theorem 3.2 (*Sufficient efficiency conditions*) — Let S^* be feasible for (P) and let there exist $\mu \in \mathbb{R}_+^m$ and $0 < \lambda \in \mathbb{R}_+^k$ such that

$$\langle D_i (\lambda^t F)_{S^*} + D_i (\mu^t H)_{S^*},\ I_{S_i} - I_{S_i^*} \rangle \geqq 0 \qquad \ldots (3.2)$$

for all $S_i \in A$, $i \in N$,

$$\mu_k H_k(S^*) = 0,\ k \in M. \qquad \ldots (3.3)$$

Further, assume that each F_j is pseudolinear with respect to the proportional function p_j, $j \in K$, and each H_k is pseudolinear with respect to the proportional function q_k, $k \in M$. Then S^* is an efficient solution of (P).

PROOF : Suppose S^* is not efficient for (P). Then for some feasible S for (P), we have

$F_j(S) < F_j(S^*)$ for some $j \in k$,

$$F_k(S) \leqq F_k(S^*) \text{ for all } k \in k,\ k \neq j.$$

Now, pseudolinearity of F_j (w.r.t. the proportional function p_j), $j \in K$, along with the above inequalities imply

$$p_j(S, S^*) \sum_{i=1}^{n} \langle D_i F_{jS^*}, I_{S_i} - I_{S_i^*} \rangle < 0 \text{ for some } j \in K,$$

$$p_k(S, S^*) \sum_{i=1}^{n} \langle D_i F_{kS^*}, I_{S_i} - I_{S_i^*} \rangle \leqq 0 \text{ for all } k \in K,\ k \neq j,$$

and positivity of $p_j, j \in K$, implies that

$$\sum_{i=1}^{n} \langle D_i F_{jS^*}, I_{S_i} - I_{S_i^*} \rangle < 0 \text{ for some } j \in K,$$

$$\sum_{i=1}^{n} \langle D_i F_{kS^*}, I_{S_i} - I_{S_i^*} \rangle \leqq 0 \text{ for all } k \in K,\ k \neq j.$$

Since $\lambda_i > 0$, $i \in K$, multiplying each of the above inequalities by λ_i and adding we get

$$\sum_{i=1}^{n} \langle D_i (\lambda^t F)_{S^*},\ I_{S_i} - I_{S_i^*} \rangle < 0. \qquad \text{... (3.4)}$$

Now it follows from (3.1), (3.3) and the fact that $\mu_k > 0$, $k \in M$,

$$\mu_k H_k(S) \leqq \mu_k H_k(S^*) \quad \text{for all } k \in M.$$

Pseudolinearity of H_k, and hence of $\mu_k H_k$ (w.r.t. the proportional function q_k), $k \in M$, implies that

$$q_k(S, S^*) \sum_{i=1}^{n} \langle D_i \mu_k H_{kS^*},\ I_{S_i} - I_{S_i^*} \rangle \leqq 0 \text{ for all } k \in M.$$

Positivity of q_k, $k \in M$, yields that

$$\sum_{i=1}^{n} \langle D_i \mu_k H_{kS^*},\ I_{S_i} - I_{S_i^*} \rangle \leqq 0 \text{ for all } k \in M.$$

Adding the above inequalities we get

$$\sum_{i=1}^{n} \langle D_i (\mu^t H)_{S^*},\ I_{S_i} - I_{S_i^*} \rangle \leqq 0. \qquad \text{... (3.5)}$$

Adding (3.4) and (3.5), we get a contradiction to (3.2). Hence the proof.

The above necessary and sufficient theorems provide for us the motivation for introducing the following vector maximization problems as a dual problem for (P) :

(D) Maximize $[F_1(T), ..., F_k(T)]$

$$\text{subject to } \sum_{i=1}^{n} \langle D_i (\lambda^t F)_T + D_i (\mu^t H)_T,\ I_{S_i} - I_{T_i} \rangle \geqq 0, \qquad \text{... (3.6)}$$

$$\mu_j H_j (T) \geqq 0, \quad j \in M, \qquad \text{... (3.7)}$$

$$\mu_j \geqq 0,\ j \in M,\ \lambda_k > 0,\ k \in K, \qquad \text{... (3.8)}$$

$$S = (S_1, ..., S_n),\ T = (T_1, ..., T_n) \in \alpha. \qquad \text{... (3.9)}$$

Theorem 3.3 (*Weak duality*) — Let each F_i be pseudolinear with respect to the proportional function p_i, $i \in K$ and each H_k be pseudolinear with respect to the proportional function q_k, $k \in M$. If S is feasible for (P) and (T, λ, μ) is feasible for (D), then the following cannot hold :

$$F_j(S) < F_j(T), \text{ for some } j \in K,$$

$$F_k(S) \leqq F_k(T), \text{ for all } k \in K.$$

The proof of this theorem is similar to that of Theorem 3.2 and hence omitted.

Remark 3.1 : It may be remarked here that Theorem 3.2 and 3.3 also hold good if instead of assuming each F_i, $i \in K$, and H_j, $j \in M$, to be pseudolinear, we assume that $\lambda^t F$ and $\lambda^t H$ are pseudolinear with respect to the proportional functions p and q, respectively.

Theorem 3.4 — Assume that $\bar{S}$ is feasible for (P) and $(\bar{T}, \bar{\lambda}, \bar{\mu})$ is feasible for (D) with $F_j(\bar{S}) = F_j(\bar{T})$, for all $j \in K$.

(i) If each F_j is pseudolinear with respect to the proportional function p_j, $j \in K$, and each H_k is pseudolinear with respect to the proportional function q_k, $k \in M$, then $\bar{S}$ is an efficient solution for (P) and $(\bar{T}, \bar{\lambda}, \bar{\mu})$ is an efficient solution for (D).

(ii) If $\lambda^t F$ is pseudolinear with respect to the proportional function p and each H_k is pseudolinear with respect to the proportional function q_k, $k \in M$, then $\bar{S}$ and $(\bar{T}, \bar{\lambda}, \bar{\mu})$ are properly efficient solutions of (P) and (D), respectively.

PROOF : (i) Suppose $\bar{S}$ is not an efficient solution of (P). Then there exists some feasible $S \in \alpha$ for (P) such that

$$F_j(S) < F_j(\bar{S}), \text{ for some } j \in K,$$

$$F_k(S) \leqq F_k(\bar{S}), \text{ for all } k \in K.$$

Since $F_j(\bar{S}) = F_j(\bar{T})$, $j \in K$ and $(\bar{T}, \bar{\lambda}, \bar{\mu})$ is feasible for (D), we arrive at a contradiction to Theorem 3.3. Hence $\bar{S}$ is an efficient solution of (P). Similarly, it can be proved that $(\bar{T}, \bar{\lambda}, \bar{\mu})$ is an efficient solution of (D).

(ii) Now suppose that $\bar{S}$ is not a properly efficient solution of (P). Then for every scalar $G > 0$, there exists $S^0 \in \alpha$ and an index $i \in K$ such that such that $F_i(\bar{S}) - F_i(S^0) > G[F_j(S^0) - F_j(\bar{S})]$, for all j satisfying $F_j(S^0) > F_j(\bar{S})$ whenever $F_i(S^0) < F_i(\bar{S})$. This means $F_i(\bar{S}) - F_i(S^0)$ can be made arbitrarily large and hence for $\bar{\lambda} > 0$, we can obtain the following inequality

$$\sum_{i=1}^{k} \bar{\lambda}_i [F_i(\bar{S}) - F_i(S^0)] > 0. \qquad \text{... (3.10)}$$

Now S^0 is feasible for (P) and $(\bar{T}, \bar{\lambda}, \bar{\mu})$ is feasible for (D), and therefore, we have

$$H_k(S^0) \leqq 0, \qquad \text{... (3.11)}$$

$$\sum_{i=1}^{n} \langle D_i(\bar{\lambda}' F)_{\bar{T}} + D_i(\bar{\mu}' H)_{\bar{T}}, I_{S_i} - I_{\bar{T}_i} \rangle \geqq 0, \text{ for all } S \in \alpha, \qquad \text{... (3.12)}$$

$$\bar{\mu}_k H_k(\bar{T}) = 0, \ k \in M, \qquad \text{... (3.13)}$$

$$\bar{\mu}_k \geqq 0, \quad k \in M, \qquad \text{... (3.14)}$$

$$\bar{\lambda}_i > 0, \ i \in K. \qquad \text{... (3.15)}$$

It follows from (3.11), (3.13) and (3.14) that

$$\bar{\mu}_k H_k(S^0) \leqq \bar{\mu}_k H_k(\bar{T}) \text{ for all } k \in M.$$

Pseudolinearity of H_k (and hence of $\bar{\mu}_k H_k$), $k \in M$, yields

$$q_k(S^0, \bar{T}) \sum_{i=1}^{n} \langle D_i \bar{\mu}_k H_{k\bar{T}}, I_{S_i^0} - I_{T_i} \rangle \leqq 0, \text{ for all } k \in M.$$

Positivity of q_k implies that

$$\sum_{i=1}^{n} \langle D_i \bar{\mu}_k H_{k\bar{T}}, I_{S_i^0} - I_{\bar{T}_i} \rangle \leqq 0 \text{ for all } k \in M. \quad \text{... (3.16)}$$

Adding all the inequalities (3.16), we have

$$\sum_{i=1}^{n} \langle D_i \bar{\mu}^t H_{\bar{T}}, I_{S_i^0} - I_{\bar{T}_i} \rangle \leqq 0. \quad \text{... (3.17)}$$

It follows from (3.12) and (3.17) that

$$\sum_{i=1}^{n} \langle D_i (\bar{\lambda}^t F)_{\bar{T}}, I_{S_i^0} - I_{\bar{T}_i} \rangle \geqq 0.$$

On using pseudolinearity of $\bar{\lambda}^t F$ with respect to p, we get

$$\bar{\lambda}^t F(S^0) \geqq \bar{\lambda}^t F(\bar{T}). \quad \text{... (3.18)}$$

On using hypothesis $F(\bar{S}) = F(\bar{T})$ in (3.18), we get

$$\sum_{i=1}^{k} \bar{\lambda}_i [F_i(\bar{S}) - F_i(S^0)] \leqq 0,$$

which is a contradiction to (3.10). Hence $\bar{S}$ is a properly efficient solution of (P). By giving similar arguments we can prove that $(\bar{T}, \bar{\lambda}, \bar{\mu})$ is a properly efficient solution of (D).

Theorem 3.5 (*Strong duality*) — Let S^* be a regular efficient solution for (P) and regular solution for (P_r^*), $r = 1, ..., k$. Then there exist $\lambda^* \in R^k$ and $\mu^* \in \mathbb{R}^m$, such that (S^*, λ^*, μ^*) is feasible for (D). Further, if conditions (i) of Theorem 3.4 hold, then (S^*, λ^*, μ^*) is an efficient solution of (D), and if conditions (ii) of Theorem 3.4 hold, then (S^*, λ^*, μ^*) is a properly efficient solution of (D).

PROOF : Since S^* is a regular efficient solution of (P), it follows from Lemma 3.1 that S^* is an optimal solution for (P_r^*), $r = 1, ..., k$. Therefore, using Theorem 3.1, we have, that there exist $\lambda_{rp} \geqq 0$, $p = 1, ..., k$, $p \neq r$ and $\mu_{rj} \geqq 0$, $j \in M$, such that

$$\sum_{i=1}^{n} \langle D_i F_{rS^*} + \sum_{\substack{p=1 \\ p \neq r}}^{k} \langle D_i \lambda_{rp} F_{pS^*} + \sum_{j=1}^{m} D_i \mu_{rj} H_{jS^*}, I_{S_i} - I_{S_i^*} \rangle \geqq 0, \quad \text{... (3.19)}$$

for all $S \in \alpha$,

$$\mu_{rj} H_j(S^*) = 0, \quad j \in M, \quad \text{for each } r = 1, ..., k. \quad \text{... (3.20)}$$

Summing over r in (3.19) and (3.20) and setting $\lambda_r^* = 1 + \sum_{\substack{p=1 \\ p \neq r}}^{k} \lambda_{rp} > 0$, $r \in K$ and

$\mu_j^* = \sum_{r=1}^{k} \mu_{rj} \geqq 0,\ j \in M,$ we obtain

$$\sum_{i=1}^{n} \langle D_i(\lambda^{*t} F)_{S^*} + D_i(\mu^{*t} H)_{S^*}, I_{S_i} - I_{S_i^*} \rangle \geqq 0,\ \ S \in \alpha,$$

$$\mu_j^* H_j(S^*) = 0, \qquad j \in M,$$

$$\mu_j^* \geqq 0, \qquad j \in M,$$

$$\lambda_i^* > 0, \qquad i \in K.$$

The above inequalities imply that (S^*, λ^*, μ^*) is feasible for (D). The result now follows from Theorem 3.4.

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REFERENCES

1. T. M. Apostol, *Mathematical Analysis*, Addison Wesley, 1957.
2. C. R. Bector, D. Bhatia and S. Pandey. *J. Math. Anal. Appl.* **186** (3) (1994), 747-68.
3. C. R. Bector, S. Chandra and M. V. Durga Prasad, *Asia-Pacific J. Oper. Res.* **5** (1988), 150-59.
4. D. Bhatia and S. Tiwari, *J. Inf. Optim. Sci.* **14** (1993), 321-34.
5. V. Chankong and Y. Y. Haimes, *Multiobjective Decision Making : Theory and Methodology*, North Holland, New York, 1983.
6. K. L. Chew and E. U. Choo, *Math. Prog.* **28** (1984), 226-39.
7. H. W. Corley, *J. Math. Anal. Appl.* **127** (1987), 193-205.
8. R. N. Kaul, S. K. Suneja and C. S. Lalitha, *Indian J. pure Appl. Math.* **24** (1993), 279-90.
9. L. J. Lin, *J. Math. Anal. Appl.* **149** (1990), 255-70.
10. L. J. Lin, *J. Math. Anal. Appl.* **168** (1992), 351-66.
11. R. J. T. Morris, *J. Math. Anal. Appl.* **70** (1979), 540-62.
12. G. J. Zalmai, *J. Math. Anal. Appl.* **149** (1990), 322-38.