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POINTS OF BOUNDED VARIATION

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In the paper we discuss some properties of points of bounded variation. Also introducing the notion of points of right (left) bounded variation we study some of their properties.

1. INTRODUCTION AND DEFINITIONS

Let f be a function defined and finite in $[\alpha, \beta]$. f is said to be of bounded variation on $[\alpha, \beta]$ if its total variation $V_f[\alpha, \beta]$ is finite {cf. Jeffery[2], pp. 118, 137}.

Clearly the function $f(x) = x \sin \frac{\pi}{x}, 0 < x \leq 1$

$$= 0, \; x = 0$$

is not of bounded variation on [0, 1], but it is of bounded variation on $[\delta, 1], 0 < \delta < 1$. This consideration leads to the concept of a point of bounded variation as introduced by Jeffery[2] (p. 137).

Definition 1 — Let f be defined and finite in (a, b) and $x \in (a, b)$. If there exists a closed interval containing x in its interior, contained in (a, b), on which f is of bounded variation, then the point x is called a point of bounded variation or a B_v -point of f.

If there exists no such interval, x is called a point of nonbounded variation or NB_v-point of f.

Let us now consider the following example.

Example 1 — Suppose $f(x) = x \sin \frac{\pi}{x}, \quad 0 < x < 1$

$$= x, \quad -1 < x \leq 0.$$

Then clearly $x = 0$ is an NB_v -point of f but we see that f is of bounded variation in $[-\delta, 0], 0 < \delta < 1$.

This example motivates one to introduce the following definition.

Definition 2 — Let f be a function defined and finite in (a, b). A point x, $a < x < b$, is called a point of left bounded variation or an LB_v -point of f if there exists a $\delta\,(>0)$ such that $[x-\delta, x] \subset (a, b)$ and $V_f\,[x-\delta, x] < +\infty$.

If no such $\delta\,(>0)$ exists, the point x is called a point of nonleft bounded variation or an NLB_v-point of f.

A point of right bounded variation or an RB_v -point and a point of non right bounded variation or an NRB_v-point of f can be defined likewise.

The aim of the paper is to study some properties of the sets of LB_v -points, RB_v -points B_v -points of a function and their interrelations. It is known (Jeffery[2], p. 137; Bhakta and Lahiri[1], Corollary 1) that the set of all B_v-points of a function is open; in the paper we prove that any nonvoid open set is precisely the set of B_v -points of some function. We also prove similar results for the sets of LB_v -points and RB_v -points of a function. Throughout the paper, when it is not explicitly stated, we mean that the usual topology of the real line is concerned. Now we state a number of definitions which are useful in the sequel.

Definition 3 — Let f be defined and finite in (a, b). Let $x \in (a, b)$ and $\delta\,(>0)$ be such that $[x-\delta, x] \subset (a, b)$. Suppose that

$$V_L\,(x, \delta) = \frac{1}{1 + V_f\,[x-\delta, x]} \quad \text{and} \quad v_L\,(x) = \lim_{\delta \to 0+} V_L\,(x, \delta).$$

Then $v_L(x)$ is called the left variation function of f.

In a parallel way we can define the right variation function $v_R(x)$ of f.

Definition 4 (cf. Kelley[3], p. 59) — A set G of real numbers is called open in the upper (lower) limit topology of the real line if for every $x \in G$ there exists a $\delta\,(>0)$ such that $(x-\delta, x] \subset G$ $([x, x+\delta) \subset G)$.

Clearly the upper and lower limit topology is finer than the usual topology of the real line.

Definition 5 — Let G be an open set in the upper limit topology of the real line. An interval Δ is called a right uncertain component interval of G in the upper limit topology if

(i) $\Delta \subset G$,

(ii) the left end point a, say, of Δ does not belong to G,

(iii) the right end point b, say, of Δ may or may not belong to G.

If $b \in G$ then $\Delta = (a, b]$ and if $b \notin G$ then $\Delta = (a, b)$.

In a like manner we can define the left uncertain component interval of an open set in the lower limit topology of the real line.

We denote by $\{LB_v\}$, $\{RB_v\}$, $\{B_v\}$ $\{NLB_v\}$ etc. the set of LB_v -points, RB_v -points, B_v -points, NLB_v -points etc. of a function f defined and finite (a, b). Clearly $\{B_v\} = \{LB_v\} \cap \{RB_v\}$ and $\{NB_v\} = \{NLB_v\} \cup \{NRB_v\}$.

2. Propositions and Theorems

Proposition 1 — $0 \leq v_L(x) \leq 1$ and $0 \leq v_R(x) \leq 1$.

Proposition 2 — $v_L(x) = 0$ ($v_R(x) = 0$) if and only if x is NLB_v (NRB_v)-point of f.

Proposition 3 — $v_L(x) > 0$ ($v_R(x) > 0$) if and only if x is an LB_v (RB_v)-point of f.

Proposition 4 — $v_L(x)$ ($v_R(x)$) is a lower semicontinuous function from left (right).

The above propositions can be proved in the line of Bhakta and Lahiri[1].

Proposition 5 — (i) $\{B_v\} \subset \{LB_v\} \subset D(\{B_v\}) = D(\{LB_v\})$, and (ii) $\{B_v\} \subset \{RB_v\} \subset D(\{B_v\}) = D(\{RB_v\})$, where $D(A)$ represents the derived set of A in the usual topology of the real line.

We omit the proof because it is straight forward.

Remark 1 : Though the sets $\{B_v\}$, $\{LB_v\}$, $\{RB_v\}$ are dense-in-themselves they are not always perfect. Considering Example 1 we see that $x = 0$ is an NRB_v -point of f whereas every point of (0, 1) is an RB_v -point of f. So $\{RB_v\} \neq D(\{RB_v\})$ and hence $\{RB_v\}$ is not perfect. From Proposition 5 we see that if $\{B_v\}$ is a perfect set then $\{LB_v\}$, $\{RB_v\}$ are perfect sets and $\{B_v\} = \{LB_v\} = \{RB_v\}$.

Proposition 6 — The set $\{LB_v\}$ ($\{RB_v\}$) is open in upper (lower) limit topology of the real line.

Proof : Let $x_0 \in \{LB_v\}$. Then by Proposition 3, $v_L(x_0) > 0$. Since $v_L(x)$ is lower semicontinuous from left at x_0, there exists $\delta\,(>0)$ such that $(x_0 - \delta, x_0] \subset (a, b)$ and $v_L(x) > \frac{1}{2} v_L(x_0)$ for $x \in (x_0 - \delta, x_0]$. Hence by Proposition 3 we get $(x_0 - \delta, x_0] \subset \{LB_v\}$. So the set $\{LB_v\}$ is open in the upper limit topology of the real line. Similarly we can prove that the set $\{RB_v\}$ is open in the lower limit topology of the real line. This process the proposition.

Following example shows that the set $\{LB_v\}$ is not, in general, open in the usual topology of the real line.

Example 2 — Let $f(x) = x, \quad -1 < x \leq 0$

$= 0,\ x \in (0, 1)$ and x is rational

$= 1,\ x \in (0, 1)$ and x is irrational.

Clearly $x = 0$ is an LB_v -point of f. Now we show that any $\alpha, 0 < \alpha < 1$, is an NLB_v -point of f. Let us choose any $\delta, 0 < \delta < \alpha$ and consider a subdivision of $[\alpha - \delta, \alpha]$ in the following manner

$$\alpha - \delta = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = \alpha$$

where x_{i+1} is rational (irrational) if x_i irrational (rational) for $i = 0, 1, 2, \ldots, n - 1$. Then $\sum_{i=0}^{n-1} \left| f(x_{i+1}) - f(x_i) \right| = n \to \infty$ as $n \to \infty$ so that $v_f[\alpha - \delta, \alpha] = \infty$. Hence α is an NLB_v-point of f. Therefore, $0 \in \{LB_v\}$ is not an interior point of $\{LB_v\}$ in the usual topology of the real line.

Similarly we can show that the set $\{RB_v\}$ is not, in general, open in the usual topology of the real line.

Now we prove a converse of Corollary 1 of Bhakta and Lahiri[1].

Theorem 1 — Let E be a nonvoid open set. Then there exists a function defined on the entire real line of which E is precisely the set of all B_v -points.

PROOF : If E is the entire real line, any constant function serves the purpose. So we suppose that E is a proper subset of the real line. Since E is open, we can write (Jeffery[2], p. 47), $E = \bigcup_n (a_n, b_n)$ where $(a_n, b_n) \cap (a_m, b_m) = \phi$ for $n \neq m$; $a_n, b_n \notin E$ for all n and n runs over a set of positive integers.

Now we consider the function f defined as follows :

$$
\begin{aligned}
f(x) &= \frac{x-a_n}{b_n-a_n} \sin \frac{\pi (b_n - a_n)}{x-a_n} \text{ if } a_n < x \leq b_n \\
&= 0 \text{ if } x = a_n \\
&= 0 \text{ if } x \text{ is rational and } x \notin E \cup \{a_n\} \cup \{b_n\} \\
&= 1 \text{ if } x \text{ is irrational and } x \notin E \cup \{a_n\} \cup \{b_n\}.
\end{aligned}
$$

Let $\alpha \in E$. Then $\alpha \in (a_{n_0}, b_{n_0})$ for some positive integer n_0. Also we can find a $\delta\,(>0)$ such that $[\alpha-\delta, \alpha+\delta] \subset (a_{n_0}, b_{n_0})$. Then clearly $f'(x)$ exists and is bounded in $[\alpha-\delta, \alpha+\delta]$ and so f is of bounded variation on $[\alpha-\delta, \alpha+\delta]$. Hence α is a B_v -point of f. Since α is arbitrary, it follows that every point of E is a B_v -point of f.

If $\beta \notin E$, then following three cases arise : (i) $\beta = a_{n_1}$ for some positive integer n_1, (ii) $\beta = b_{n_2}$ for some positive integer n_2, or (iii) $b_{n_3} < \beta < a_{n_4}$ for some positive integers n_3, n_4 such that $[b_{n_3}, a_{n_4}] \cap E = \phi$.

If $\beta = a_{n_1}$, then clearly β is an NB_v -point of f.

If $\beta = b_{n_2}$ and $b_{n_2} = a_{n'_2}$ for some positive integer n'_2, then β is clearly an NB_v -point of f.

If $\beta = b_{n_2}$ and $b_{n_2} \notin \{a_n\}$, we can choose a $\delta_1\,(>0)$ such that for $0 < \delta \leq \delta_1$ $[b_{n_2}, b_{n_2}+\delta] \subset CE$, the complement of E. Considering a subdivision of $[b_{n_2}, b_{n_2}+\delta]$ as in Example 2 we can show that β is an NRB_v-point of f.

Finally if $b_{n_3} < \beta < a_{n_4}$, we choose a $\delta_2\,(>0)$ such that for $0 < \delta \leq \delta_2$ $[\beta-\delta, \beta+\delta] \subset CE$. Now considering a subdivision of $[\beta-\delta, \beta+\delta]$ as in Example 2 it is easy to show that β is an NB_v-point of f.

Since $\beta \notin E$ is arbitrary, it follows that every point not belonging to E is an NB_v -point of f. This proves the theorem.

Now we prove a sort of converse of Porposition 6. To do this we require some ideas of the structure of an open set in the upper (lower) limit topology of the real line.

Lemma 1 — A nonvoid bounded open set G in the upper (lower) limit topology of the real line can be expressed as a countable union of pairwise disjoint right (left) uncertain component intervals.

PROOF : We prove the lemma for upper limit topology in the following steps.

Step I : Let $x_0 \in G$. Then the set $F = (-\infty, x_0] \cap CG$ is nonvoid and closed in the upper limit topology. Also clearly the set is bounded above. Let $a = \sup F$. If possible suppose $a \notin F$ and O be an open neighbourhood of a in the upper limit topology. Then there exists an $\varepsilon (>0)$ such that $(a-\varepsilon, a] \subset O$ and so $(a - \varepsilon\, a) \subset O \setminus \{a\}$. From the definition of a it follows that there exists $y \in F$ such that $y \in (a-\varepsilon, a) \subset O \setminus \{a\}$.

Therefore, a is a limit point of F in the upper limit topology. This is a contradiction and so $a \in F$. Since $x_0 \in G$ and $a \notin G$, it follows that $a < x_0$. If possible let $b \in (a, x_0]$ and $b \notin G$ then $b \in F$ and so $\sup F = a < b$, a contradiction. Hence $(a, x_0] \subset G$. Since x_0 is arbitrary, every element of G belongs to an interval contained in G whose left end point does not belong to G.

Step II : Let $I_\alpha \subset G$ and $I_\alpha = (a_\alpha, b_\alpha]$ or (a_α, b_α) according as $b_\alpha \in G$ or $b_\alpha \notin G$ and $a_\alpha \notin G$.

Also let $I_\alpha \cap I_\beta \neq \phi$ and $p \in I_\alpha \cap I_\beta$. If $a_\alpha < a_\beta$ then $a_\alpha < a_\beta < p \leqq b_\alpha$ and so $a_\beta \in I_\alpha \subset G$, a contradiction. Again if $a_\beta < a_\alpha$ then $a_\beta < a_\alpha < p \leqq b_\beta$ and so $a_\alpha \in I_\beta \subset G$, a contradiction. So $a_\alpha = a_\beta$.

Step III : Let $x \in G$ and $\{I_\alpha\}$ be the collection of all intervals as defined in Step II containing x. Let $a = \inf_\alpha a_\alpha$ and $b = \sup_\alpha b_\alpha$. In virtue of Step II it follows that $a = a_\alpha$ for all α and so $a \notin G$. Let $I_x = \bigcup_\alpha I_\alpha$ and $I_x^* = (a, b]$ or (a, b) according as $b \in G$ or $b \notin G$.

Let $y \notin I_x^*$. Then the following cases come up for consideration.

(i) If $y \leqq a$ then $y \leqq a_\alpha$ for all α and so $y \notin I_\alpha$ for any α. Hence $y \notin I_x$.

(ii) If $y > b$ then $y > b_\alpha$ for all α and so $y \notin I_\alpha$ for any α. Hence $y \notin I_x$.

(iii) If $y = b$ then $b \notin G$. So $y > b_\alpha$ for all $b_\alpha \in G$ and y may be equal to some $b_\alpha \notin G$. Hence $y \notin I_\alpha$ for any α and so $y \notin I_x$. Therefore $I_x \subset I_x^*$.

Next let $y \in I_x^*$. Then we consider the following cases.

(i) If $y = x$ then $y \in I_x$.

(ii) If $a < y < x$ then $a \leq a_{\alpha_0} < y < x \leq b_{\alpha_0}$ for some α_0. So $y \in I_{\alpha_0} \subset I_x$.

(iii) If $x < y < b$ then $a_{\alpha_1} < x < b_{\alpha_1} \leq b$ for some α_1. So $y \in I_{\alpha_1} \subset I_x$.

(iv) If $y = b$ then $b \in G$. So by Step I there exists $c \notin G$ such $(c, b] \subset G$. If possible let b be not attained. Then there exists I_{α_2} such that $I_{\alpha_2} \cap (c, b] \neq \phi$. So by Step II this implies $x \in (c, b]$ and hence $(c, b]$ is a member of $\{I_\alpha\}$. This implies that $b < b$, a contradiction. So $b = b_{\alpha_3}$ for some α_3 and hence $y \in I_{\alpha_3} \subset I_x$.

Therefore $I_x^* \subset I_x$ and so $I_x^* = I_x$.

Step IV : Let for some $x, y \in G, I_x^* \cap I_y^* \neq \phi$. Then by Step II the left end points of I_x^* and I_y^* are same. If I_x^* is a proper subset of I_y^* then $x \in I_y^*$ and so I_y^* is a member of $\{I_\alpha\}$, which contradicts the definition of I_x^*. Similarly I_y^* is not a proper subset of I_x^*. So $I_x^* = I_y^*$.

Step V : By virtue of Step IV we see that the collection $\{I_x^* : x \in G\}$ is a collection of pairwise disjoint right uncertain component intervals of G and so this collection is countable. Also by Step III it follows that $G = \bigcup_{x \in G} I_x^*$. This proves the lemma.

Lemma 2 — A nonvoid open set G, not necessarily bounded, in the upper (lower) limit topology of the real line can be expressed as a countable union of pairwise disjoint right (left) uncertain component intervals in the upper (lower) limit topology.

PROOF : We prove the lemma for the upper limit topology. First we note that

$$G = \bigcup_{n=0}^{\infty} [\{(-n-1, -n] \cup (n, n+1]\} \cap G].$$

Since the set $\{(-n-1, -n] \cup (n, n+1]\} \cap G$ is bounded and open in the upper limit topology, by Lemma 1 each nonvoid set $\{(-n-1, -n] \cap (n, n+1]\} \cap G$ can be expressed as a countable union of pairwise disjoint right uncertain component intervals in the upper limit topology. Since the sets $\{(-n-1, -n] \cup (n, n+1]\} \cap G$, $n = 0, 1, 2, \ldots$ are pairwise disjoint, the lemma is proved.

Now we are in a position to prove a sort of converse of Proposition 6 which is given in the following theorem.

Theorem 2 — Given a nonvoid open set G in the upper (lower) limit topology of the real line, there exists a function f defined on the entire real line such that

(i) every point of G is an LB_v (RB_v)-point of f, and

(ii) every point of CG, the complement of G, except possibly a countable subset, is an NLB_v (NRB_v)-point of f.

PROOF : We prove the theorem for upper limit topology. If G is the entire real line, any constant function serves the purpose; so we suppose that G is a proper subset of the real line. By Lemma 2 we can write $G = \bigcup_i \Delta_i$ where $\Delta_i \cap \Delta_j = \phi$ for $i \neq j$ and Δ_i is a right uncertain component interval of G (in the upper limit topology) with left end point a_i and right end point b_i, and i runs over a set of positive integers.

Let $f(x) = x$ if $a_i < x \leqq b_i$

$= 1$ if x is rational and $x \notin G \cup \{b_i\}$

$= 0$ if x is irrational and $x \notin G \cup \{b_i\}$.

Now it is easy to verify that every point of G is an LB_v-point of f and every point of $CG \setminus \{b_i\}$ is an NLB_v-point of f. This proves the theorem.

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REFERENCES

1. P. C. Bhakta and B. K. Lahiri, *Am. Math. Monthly* **70** (1963), 644-47.
2. R. L. Jeffery, *The Theory of Functions of Real Variable*, Second Edition, University of Toronto Press Toronto, 1953.
3. J. L. Kelley, *General Topology*, D. Van Nostrand Company Inc. 1968.