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UNIQUENESS THEOREMS FOR MEROMORPHIC FUNCTIONS II

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This paper studies the problem of uniqueness of meromorphic functions and shows that there exist two finite sets S_j $(j = 1, 2)$ such that any two nonconstant meromorphic functions f and g satisfying $\overline{E}_f(S_j) = \overline{E}_g(S_j)$ for $j = 1, 2$ must be identical. The results in this paper improve some theorems given by Nevanlinna[7], Brosch[8], Yi[6] and other authors. As a special case, these results answer an open question posed by Gross[5].

1. INTRODUCTION AND MAIN RESULTS

By a meromorphic function we shall always mean a meromorphic function in the complex plane $\mathbb{C}$. We use the usual notations of Nevanlinna theory of moromorphic functions (see, for example, Hayman[1]). We use I to denote any set of positive real numbers of infinite linear measure and use E to denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence (see Yi[2]). We denote by $S(r, f)$ any quantity satisfying $S(r,f) = o(T(r,f))$ $(r \to \infty, r \notin E)$.

Let h be a nonconstant meromorphic function and let S be a subset of distinct elements in $\hat{\mathbb{C}}$. Set

$$E_h(S) = h^{-1}(S),$$

where we take due account of multiplicities (see Gross[3]). Usually, the notation $\overline{E}_h(S)$ expresses the set which contains the same points as $E_h(S)$ but without counting multiplicities.

Let f and g be two nonconstant meromorphic functions and S be a subset of distinct elements in $\hat{\mathbb{C}}$. If $E_f(S) = E_g(S)$, we say that f and g share the set S *CM* (counting multiplicity). If $\overline{E}_f(S) = \overline{E}_g(S)$, we say f and g share the set S *IM* (ignoring multiplicity). As a special case, let $S = \{a\}$, where $a \in \hat{\mathbb{C}}$. If $E_f(\{a\}) = E_g(\{a\})$, we say f and g share the value a *CM*. If $\overline{E}_f(\{a\}) = \overline{E}_g(\{a\})$, we say f and g share the value a *IM* (see Gundensen[4]).

In 1976, Gross posed the following open question (see Gross[5], Question 6):

Question 1 — Can one find two finite sets $S_j (j=1,2)$ such that any two non-constant entire functions f and g satisfying $E_f(S_j)=E_g(S_j)$ for $j = 1, 2$ must be identical?

Throughout this paper, we shall use w and u to denote the constants $\exp(2\pi i/n)$ and $\exp(2\pi i/m)$ respectively, where n and m are positive integers. Recently, the present author proved the following results which provide positive answers to Question 1.

Theorem A[6] — Let $S_1 = \{a_1+b_1, a_1+b_1 w, \ldots, a_1+b_1 w^{n-1}\}$ and $S_2 = \{a_2+b_2, a_2+b_2 u, \ldots, a_2+b_2 u^{m-1}\}$, where $n > 6$, $m > 6$, a_1, b_1, a_2 and b_2 are constants such that $b_1 b_2 \neq 0$ and $a_1 \neq a_2$. Suppose that f and g are two nonconstant entire functions satisfying $F_f(S_j)=E_g(S_j)$ for $j = 1, 2$, then $f \equiv g$.

Theorem B[6] — Let $S_1 = \{a+b_1, a+b_1 w, \ldots, a+b_1 w^{n-1}\}$ and $S_2 = \{a_2+b_2, a_2+b_2 u, \ldots, a_2+b_2 u^{m-1}\}$, where n (> 6) and m (> 6) have no common factors, and a, b_1 and b_2 are constants such that $b_1 b_2 \neq 0$ and $b_1^{2mn} \neq b_2^{2mn}$. Suppose that f and g are two nonconstant entire functions satisfying $E_f(S_j)=E_g(S_j)$ for $j = 1$, 2, then $f \equiv g$.

Now it is natural to ask the following questions :

Question 2 — Can one find two finite sets $S_j (j = 1, 2)$ such that any two nonconstant entire functions f and g satisfying $\overline{E}_f(S_j) = \overline{E}_g(S_j)$ for $j = 1, 2$ must be identical?

Question 3 — Can one find two finite sets S_j ($j = 1, 2$) such that any two nonconstant meromorphic functions f and g satisfying $\overline{E}_f(S_j) = \overline{E}_g(S_j)$ for $j = 1, 2$ must be identical?

In this paper, we prove the following theorems, which provide positive answers to Question 2 and Question 3.

Theorem 1 — Let $S_1 = \{a+b, a+bw, \ldots, a+bw^{n-1}\}$ and $S_2 = \{c\}$, where $n > 7$, a, b and c are constants such that $b \neq 0$, $c \neq a$ and $(c-a)^{2n} \neq b^{2n}$. Suppose that f and g are nonconstant entire functions satisfying $\overline{E}_f(S_j) = \overline{E}_g(S_j)$ for $j = 1, 2$, then $f \equiv g$.

Theorem 2 — Let $S_1 = \{a_1+b_1, a_1+b_1 w, \ldots, a_1+b_1 w^{n-1}\}$ and $S_2 = \{a_2+b_2, a_2+b_2 u, \ldots, a_2+b_2 u^{m-1}\}$, where $n > 7$, $m > 7$, a_1, b_1, a_2 and b_2 are constants such that $b_1 b_2 \neq 0$ and $a_1 \neq a_2$. Suppose that f and g are two nonconstant entire functions satisfying $\overline{E}_f(S_j) = \overline{E}_g(S_j)$ for $j = 1, 2$, then $f \equiv g$.

Theorem 3 — Let $S_1 = \{a+b_1, a+b_1 w, \ldots, a+b_1 w^{n-1}\}$ and $S_2 = \{a+b_2, a+b_2 u, \ldots, a+b_2 u^{m-1}\}$, where n (> 7) and m (> 7), have no common factor a, b_1 and b_2 are constants such that $b_1 b_2 \neq 0$ and $b_1^{2mn} \neq b_2^{2mn}$. Suppose that f and g are two nonconstant entire functions satisfying $\overline{E}_f(S_j) = \overline{E}_g(S_j)$ for $j = 1, 2$, then $f \equiv g$.

Theorem 4 — Let $S_1 = \{a+b, a+bw, \ldots, a+bw^{n-1}\}$ and $S_2 = \{c_1, c_2\}$, where $n > 14$, a, b, c_1 and c_2 are constants such that $b \neq 0, c_1 \neq a, c_2 \neq a$, $(c_1-a)^n \neq (c_2-a)^n$, $(c_1-a)^{2n} \neq b^{2n}$, $(c_2-a)^{2n} \neq b^{2n}$ and $(c_1-a)^n(c_2-a)^n \neq b^{2n}$. Suppose that f and g are two nonconstant meromorphic functions satisfying $\overline{E}_f(S_j) = \overline{E}_g(S_j)$ for $j = 1, 2$, then $f \equiv g$.

Theorem 5 — Let $S_1 = \{a_1+b_1, a_1+b_1 w, \ldots, a_1+b_1 w^{n-1}\}$ and $S_2 = \{a_2+b_2, a_2+b_2 u, \ldots, a_2+b_2 u^{m-1}\}$, where $n > 14$, $m > 14$, a_1, b_1, a_2 and b_2 are constants such that $b_1 b_2 \neq 0$ and $a_1 \neq a_2$. Suppose that f and g are two nonconstant meromorphic functions satisfying $\overline{E}_f(S_j) = \overline{E}_g(S_j)$ for $j = 1, 2$, then $f \equiv g$.

Theorem 6 — Let $S_1 = \{a+b_1, a+b_1 w, \ldots, a+b_1 w^{n-1}\}$ and $S_2 = \{a+b_2, a+b_2 u, \ldots, a+b_2 u^{m-1}\}$, where n (> 14) and m (> 14) have no common factors, and a, b_1 and b_2 are constants such that $b_1 b_2 \neq 0$ and $b_1^{2mn} \neq b_2^{2mn}$. Suppose that f and g are two nonconstant meromorphic functions satisfying $\overline{E}_f(S_j) = \overline{E}_g(S_j)$ for $j = 1, 2$, then $f \equiv g$.

Nevanlinna[7] proved the following well-known theorem.

Theorem C (see Brosch[8] or Yi-Yang[9]) — Let f and g be two nonconstant meromorphic functions such that f and g share 1 *IM*. If $\overline{N}\left(r, \frac{1}{f}\right) + \overline{N}(r, f) = S(r, f)$ and $\overline{N}\left(r, \frac{1}{g}\right) + \overline{N}(r, g) = S(r, g)$, then $f \equiv g$ or $f \cdot g \equiv 1$.

In this paper we improve the above result and obtain the following :

Theorem 7 — Let f and g be two nonconstant meromorphic functions such that f and g share 1 *IM*. If

$$\limsup_{\substack{r \to \infty \\ r \in I}} \frac{\overline{N}\left(r, \frac{1}{f}\right) + \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}(r, g)}{T(r, f) + T(r, g)} < \frac{1}{7} \qquad \ldots (1.1)$$

then $f \equiv g$ or $f \cdot g \equiv 1$.

By Theorem 7 we immediately obtain the following corollary.

Corollary — Let f and g be two nonconstant meromorphic functions such that f and g share 1 *IM*. If

$$\overline{N}\left(r, \frac{1}{f}\right) + \overline{N}(r, f) \leq \lambda \cdot T(r, f) + S(r, f) \qquad \ldots (1.2)$$

and

$$\overline{N}\left(r, \frac{1}{g}\right) + \overline{N}(r, g) \leq \lambda \cdot T(r, g) + S(r, g), \qquad \ldots (1.3)$$

where $\lambda < 1/7$, then $f \equiv g$ or $f \cdot g \equiv 1$.

Recently, the present author proved the following result.

Theorem D[6] — Let $S_1 = \{a+b, a+bw, \ldots, a+bw^{n-1}\}$ and $S_2 = \{\infty\}$, where $n > 6$, a and $b (\neq 0)$ are constants. If f and g are nonconstant meromorphic functions such that $E_f(S_j) = E_g(S_j)$ $(j = 1, 2)$, then $f - a \equiv t(g-a)$, where $t^n = 1$, or $(f-a) \cdot (g-a) \equiv s$, where $s^n = b^{2n}$.

In this paper, we prove the following theorems which are improvements of Theorem D. These results will be needed in the proof of our theorems.

Theorem 8 — Let $S = \{a+b, a+bw, \ldots, a+bw^{n-1}\}$, where $n > 7$, a and $b(\neq 0)$ are constants. If f and g are nonconstant entire functions such that $\overline{E}_f(S) = \overline{E}_g(S)$, then $f - a \equiv t(g-a)$, where $t^n = 1$, or $(f-a) \cdot (g-a) \equiv s$, where $s^n = b^{2n}$.

Theorem 9 — Let $S = \{a+b, a+bw, \ldots, a+bw^{n-1}\}$, where $n > 14$, a and $b(\neq 0)$ are constants. If f and g are nonconstant meromorphic functions such that $\overline{E}_f(S) = \overline{E}_g(S)$, then $f-a \equiv t(g-a)$, where $t^n = 1$, or $(f-a) \cdot (g-a) \equiv s$, where $s^n = b^{2n}$.

2. Some Lemmas

Let f and g be two nonconstant meromorphic functions such that f and g share 1 *IM*. Let z_0 be a 1-point of f of order p, a 1-point of g of order q. We denote by $\overline{N}_L\left(r, \frac{1}{f-1}\right)$ the counting function of those 1-points of f where $p > q$; by $N_E^{1)}\left(r, \frac{1}{f-1}\right)$ the counting function of those 1-points of f where $p = q = 1$; by $\overline{N}_E^{(2}\left(r, \frac{1}{f-1}\right)$ the counting function of those 1-points of f where $p = q \geq 2$, each point in these counting functions is counted only once. In the same way, we can define $\overline{N}_L\left(r, \frac{1}{g-1}\right)$, $N_E^{1)}\left(r, \frac{1}{g-1}\right)$ and $\overline{N}_E^{(2}\left(r, \frac{1}{g-1}\right)$ (see Yi-Yang[9]). Particularly, if f and g share 1 *CM*, then

$$\overline{N}_L\left(r, \frac{1}{f-1}\right) = \overline{N}_L\left(r, \frac{1}{g-1}\right) = 0. \quad \ldots (2.1)$$

With these notations, it is easy to see that

$$N_E^{1)}\left(r, \frac{1}{f-1}\right) = N_E^{1)}\left(r, \frac{1}{g-1}\right), \quad \ldots (2.2)$$

$$\overline{N}_E^{(2}\left(r, \frac{1}{f-1}\right) = \overline{N}_E^{(2}\left(r, \frac{1}{g-1}\right) \quad \ldots (2.3)$$

and

$$\overline{N}\left(r, \frac{1}{f-1}\right) = N_E^{1)}\left(r, \frac{1}{f-1}\right) + \overline{N}_L\left(r, \frac{1}{f-1}\right) + \overline{N}_L\left(r, \frac{1}{g-1}\right) + \overline{N}_E^{(2}\left(r, \frac{1}{g-1}\right)$$
$$= \overline{N}\left(r, \frac{1}{g-1}\right). \quad \ldots (2.4)$$

Lemma 1 — Let f and g be two nonconstant meromorphic functions such that f and g share 1 *IM*. Then

$$T(r,f) \le \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{g}\right) + \overline{N}(r,g) + N_E^{1)}\left(r,\frac{1}{f-1}\right)$$
$$+ \overline{N}_L\left(r,\frac{1}{f-1}\right) - N_0\left(r,\frac{1}{f'}\right) - N_0\left(r,\frac{1}{g'}\right) + S(r,f) + S(r,g), \quad \text{... (2.5)}$$

where $N_0(r, 1/f')$ denotes the counting function corresponding to the zeros of f' that are not zeros of f and $f-1$, $N_0(r, 1/g')$ denotes the counting function corresponding to the zeros of g' that are not zeros of g and $g-1$.

PROOF : By the second fundamental theorem, we have

$$T(r,f) + T(r,g) \le \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{f-1}\right)$$
$$+ \overline{N}\left(r,\frac{1}{g}\right) + \overline{N}(r,g) + \overline{N}\left(r,\frac{1}{g-1}\right)$$
$$- N_0\left(r,\frac{1}{f'}\right) - N_0\left(r,\frac{1}{g'}\right) + S(r,f) + S(r,g). \quad \text{... (2.6)}$$

Noting that f and g share 1 *IM*, we get from (2.4)

$$\overline{N}\left(r,\frac{1}{f-1}\right) + \overline{N}\left(r,\frac{1}{g-1}\right) = N_E^{1)}\left(r,\frac{1}{f-1}\right) + \overline{N}_L\left(r,\frac{1}{f-1}\right) + \overline{N}_L\left(r,\frac{1}{g-1}\right)$$
$$+ \overline{N}_E^{(2}\left(r,\frac{1}{g-1}\right) + \overline{N}\left(r,\frac{1}{g-1}\right)$$
$$\le N_E^{1)}\left(r,\frac{1}{f-1}\right) + \overline{N}_L\left(r,\frac{1}{f-1}\right) + N\left(r,\frac{1}{g-1}\right)$$
$$\le N_E^{1)}\left(r,\frac{1}{f-1}\right) + \overline{N}_L\left(r,\frac{1}{f-1}\right) + T(r,g) + O(1).$$

Combining this and (2.6), we obtain the conclusion of Lemma 1.

Lemma 2 — Let

$$H = \left(\frac{f''}{f'} - \frac{2f'}{f-1}\right) - \left(\frac{g''}{g'} - \frac{2g'}{g-1}\right), \quad \text{... (2.7)}$$

where f and g are two nonconstant meromorphic functions. If f and g share 1 *IM* and $H \not\equiv 0$, then

$$N_E^{1)}\left(r,\frac{1}{f-1}\right) \le N(r,H) + S(r,f) + S(r,g). \quad \text{... (2.8)}$$

PROOF: Suppose that z_0 is a simple 1-point of f and g. Let

$$f(z) = 1 + a_1(z-z_0) + a_2(z-z_0)^2 + O((z-z_0)^3),$$

$$g(z) = 1 + b_1(z-z_0) + b_2(z-z_0)^2 + O((z-z_0)^3),$$

where $a_1 \neq 0$ and $b_1 \neq 0$. Then an elementary calculation gives that $H(z) = O(z - z_0)$, which proves that z_0 is a zero of H. Thus,

$$N_E^{1)}\left(r, \frac{1}{f-1}\right) \leq N\left(r, \frac{1}{H}\right) \leq T(r, H) + O(1). \quad \ldots (2.9)$$

From (2.7) we obtain $m(r, H) = S(r, f) + S(r, g)$. Combining this and (2.9), we obtain the conclusion of Lemma 2.

Lemma 3 — Let H be given by (2.7) and $H \equiv 0$. If f and g share 1 *IM*, then

$$T(r, f) \leq 2\overline{N}\left(r, \frac{1}{f}\right) + 2\overline{N}(r, f) + 2\overline{N}\left(r, \frac{1}{g}\right) + 2\overline{N}(r, g)$$

$$+ 2\overline{N}_L\left(r, \frac{1}{f-1}\right) + \overline{N}_L\left(r, \frac{1}{g-1}\right) + S(r, f) + S(r, g). \quad \ldots (2.10)$$

PROOF : Since f and g share 1 *IM*, by Lemma 1 and Lemma 2 we can obtain (2.5) and (2.8). From (2.7), we have

$$N(r, H) \leq \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}(r, g) + \overline{N}_L\left(r, \frac{1}{f-1}\right)$$

$$+ \overline{N}_L\left(r, \frac{1}{g-1}\right) + N_0\left(r, \frac{1}{f'}\right) + N_0\left(r, \frac{1}{g'}\right) \quad \ldots (2.11)$$

Combining (2.5), (2.8) and (2.11), we get the conclusion of Lemma 3.

Lemma 4 (see Yi[10]) — Suppose that H is given by (2.7) and $H \equiv 0$, then

$$T(r, g) = T(r, f) + O(1). \quad \ldots (2.12)$$

If further suppose that

$$\limsup_{\substack{r \to \infty \\ r \in I}} \frac{\overline{N}\left(r, \frac{1}{f}\right) + \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}(r, g)}{T(r, f)} < 1 \quad \ldots (2.13)$$

then $f \equiv g$ or $f \cdot g \equiv 1$.

Lemma 5 (see Yi[11]) — Let h be a nonconstant meromorphic function. Then

$$N\left(r, \frac{1}{h'}\right) \leq N\left(r, \frac{1}{h}\right) + \overline{N}(r, h) + S(r, h).$$

Lemma 6 — Let h be a nonconstant meromorphic function, and let a_j be distinct finite complex numbers such that $a_j \neq 0$ $(j = 1, 2, \ldots, q)$. Then

$$\sum_{j=1}^{q} \left(N\left(r, \frac{1}{h-a_j}\right) - \overline{N}\left(r, \frac{1}{h-a_j}\right)\right) \leq \overline{N}\left(r, \frac{1}{h}\right) + \overline{N}(r, h) + S(r, h).$$

PROOF : Obviously,

$$\sum_{j=1}^{q}\left(N\left(r,\frac{1}{h-a_j}\right)-\overline{N}\left(r,\frac{1}{h-a_j}\right)\right)+\left(N\left(r,\frac{1}{h}\right)-\overline{N}\left(r,\frac{1}{h}\right)\right)\leq N\left(r,\frac{1}{h'}\right).$$

From this and Lemma 5, we arrive at the conclusion of Lemma 6.

Remark : Suppose that f and g are two nonconstant meromorphic functions such that f and g share 1 *IM*. Using Lemma 6, we can obtain that

$$\overline{N}_L\left(r,\frac{1}{f-1}\right)\leq\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}(r,f)+S(r,f) \qquad \text{... (2.14)}$$

and

$$\overline{N}_L\left(r,\frac{1}{g-1}\right)\leq\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}(r,g)+S(r,g). \qquad \text{... (2.15)}$$

3. PROOF OF THEOREM 7

Let H be given by (2.7). If $H\equiv 0$, by Lemma 3 and Lemma 6 we can obtain (2.10), (2.14) and (2.15). Combining (2.10), (2.14) and (2.15) we have

$$T(r,f)\leq 4\overline{N}\left(r,\frac{1}{f}\right)+4\overline{N}(r,f)+3\overline{N}\left(r,\frac{1}{g}\right)$$
$$+\ 3\overline{N}(r,g)+S(r,f)+S(r,g). \qquad \text{... (3.1)}$$

Similarly, we have

$$T(r,g)\leq 3\overline{N}\left(r,\frac{1}{f}\right)+3\overline{N}(r,f)$$
$$+4\overline{N}\left(r,\frac{1}{g}\right)+4\,\overline{N}(r,g)+S(r,f)+S(r,g). \qquad \text{... (3.2)}$$

Combining (3.1) and (3.2) we get

$$T(r,f)+T(r,g)\leq 7\overline{N}\left(r,\frac{1}{f}\right)+7\overline{N}(r,f)+7\overline{N}\left(r,\frac{1}{g}\right)$$
$$+\ 7\overline{N}(r,g)+S(r,f)+S(r,g)$$

which contradicts (1.1). From this we derive $H\equiv 0$. By Lemma 4 we can obtain (2.12). From (1.1) and (2.12) we can obtain (2.13). Again by Lemma 4 we get the conclusion of Theorem 7.

4. Proofs of Theorem 8 and Theorem 9

Proof of Theorem 9

Let

$$F=\left(\frac{f-a}{b}\right)^n,\ G=\left(\frac{g-a}{b}\right)^n.$$

By $\overline{E}_f(S) = \overline{E}_g(S)$, we know that F and G share 1 *IM*. It is obvious that

$$\overline{N}\left(r,\frac{1}{F}\right)=\overline{N}\left(r,\frac{1}{f-a}\right)\le\frac{1}{n}T(r,F)+O(1).$$

Similarly, we have

$$\overline{N}(r,F)\le\frac{1}{n}T(r,F)+O(1),$$

$$\overline{N}\left(r,\frac{1}{G}\right)\le\frac{1}{n}T(r,G)+O(1),$$

$$\overline{N}(r,G)\le\frac{1}{n}T(r,G)+O(1).$$

Thus,

$$\limsup_{\substack{r\to\infty\\ r\in I}}\frac{\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}(r,F)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}(r,G)}{T(r,F)+T(r,G)}\le\frac{2}{n}<\frac{1}{7},$$

since $n > 14$. By Theorem 7, we get $F\equiv G$ or $F\cdot G\equiv 1$. From this we obtain the conclusion of Theorem 9.

Proof of Theorem 8

Noting that f and g are entire functions, we have $N(r,f)=N(r,g)=0$. Using this and proceeding as in the proof of Theorem 9, we can obtain the conlcusion of Theorem 8.

5. Proofs of Theorem 1 and Theorem 4

Proof of Theorem 1

By the assumption $\overline{E}_f(S_1) = \overline{E}_g(S_1)$, we have from Theorem 8

$$f-a\equiv t(g-a), \qquad \text{... (5.1)}$$

where $t^n=1$, or

$$(f-a)\cdot(g-a)\equiv s, \qquad \text{... (5.2)}$$

where a is a Picard value of f and g and $s^n=b^{2n}$. We discuss the following two cases.

Case 1 — Suppose that f and g satisfy (5.1).

If c is a Picard value of f, by the assumption $\overline{E}_f(S_2) = \overline{E}_g(S_2)$, we know that c is a Picard value of g. Again from (5.1), we know that $a + t(c - a)$ is a Picard value of f. Since f is an entire function, we have $c = a + t(c - a)$. Thus, $t = 1$, and hence $f \equiv g$.

If c is not a Picard value of f, then exists z_0 such that $f(z_0) = g(z_0) = c$. By (5.1), we obtain $c - a = t(c - a)$. Thus, $t = 1$, and hence $f \equiv g$.

Case 2 — Suppose that f and g satisfy (5.2).

It is easy to see that c is not a Picard value of f. Then exists z_0 such that $f(z_0) = g(z_0) = c$. By (5.2), we obtain $(c - a)^2 = s$. Thus, $(c - a)^{2n} = s^n = b^{2n}$, this contradicts the assumption.

This completes the proof of Theorem 1.

Proof of Theorem 4

By the assumption $\overline{E}_f(S_1) = \overline{E}_g(S_1)$, from Theorem 9 we have

$$f - a \equiv t(g - a), \qquad \text{... (5.3)}$$

where $t^n = 1$, or

$$(f - a) \cdot (g - a) \equiv s, \qquad \text{... (5.4)}$$

where $s^n = b^{2n}$. We discuss the following two cases.

Case 1 — Suppose f and g satisfy (5.3). We discuss the following three subcases.

Case 1.1 — Assume that c_1 is not a Picard value of f. Then there exists z_0 such that $f(z_0) = c_1$. By $\overline{E}_f(S_2) = \overline{E}_g(S_2)$, we obtain $g(z_0) = c_1$ or $g(z_0) = c_2$. If $g(z_0) = c_1$, by (5.3) we have $c_1 - a = t(c_1 - a)$. Thus $t = 1$, and $f \equiv g$. If $g(z_0) = c_1$, by (5.3) we have $c_1 - a = t(c_2 - a)$. Thus $t = 1$, and $f \equiv g$. If $g(z_0) = c_2$, by (5.3) we have $c_1 - a = t(c_2 - a)$. Thus $(c_1 - a)^n = (c_2 - a)^n$, which contradicts the assumption.

Case 1.2 — Assume that c_2 is not a Picard value of f. In like manner, we have $f \equiv g$.

Case 1.3 — Assume that c_1 and c_2 are Picard values of f. By $\overline{E}_f(S_2) = \overline{E}_g(S_2)$, we know that c_1 and c_2 are Picard values of g. Again by (5.3) we know that $a + t(c_1 - a)$ and $a + t(c_2 - a)$ are Picard values of f. Since a meromorphic function has at most two Picard values, $c_1 = a + t(c_1 - a)$ or $c_1 = a + t(c_2 - a)$. If $c_1 = a + t(c_1 - a)$, then $t = 1$ and $f \equiv g$. If $c_1 = a + t(c_2 - a)$, then $(c_1 - a)^n = (c_2 - a)^n$, which contradicts the assumption.

Case 2 — Suppose that f and g satisfy (5.4). We discuss the following three subcases.

Case 2.1 — Assume that c_1 is not a Picard value of f. Then there exists z_0 such that $f(z_0) = c_1$. By $\overline{E}_f(S_2) = \overline{E}_g(S_2)$, we obtain $g(z_0) = c_1$ or $g(z_0) = c_2$. If $g(z_0) = c_1$, from (5.4) we have $(c_1 - a)^2 = s$. Thus, $(c_1 - a)^{2n} = b^{2n}$, which contradicts the assumption. If $g(z_0) = c_2$, from (5.4) we have $(c_1 - a)(c_2 - a) = s$. Thus, $(c_1 - a)^n (c_2 - a)^n = b^{2n}$, which is again a contradiction.

Case 2.2 — Assume that c_2 is not a Picard value of f. In like mmanner, we have a contradiction.

Case 2.3— Assume that c_1 and c_2 are Picard values of f. By $\overline{E}_f(S_2) = \overline{E}_g(S_2)$, we know that c_1 and c_2 are Picard values of g. Again from (5.4), we know that $a+\frac{s}{c_1-a}$ and $a+\frac{s}{c_2-a}$ are Picard values of f. Thus $c_1=a+\frac{s}{c_1-a}$ or $c_1=a+\frac{s}{c_2-a}$. In like manner, we also have a contradiction.

This completes the proof of Theorem 4.

6. PROOF OF THEOREMS 2, 3, 5 AND 6

Using Theorem 8 and Theorem 9 and proceeding as in the proof of Theorem A (see Yi[6], the proof of Theorem 1), we can obtain the conclusion of Theorem 2 and Theorem 5 respectively. Using Theorem 8 and Theorem 9 and proceeding as in the proof of Theorem B (see Yi[6], the proof of Theorem 2), we can obtain the conclusion of Theorem 3 and Theorem 6 respectively.

7. CONCLUDING REMARKS

In the same manner as the above, we have the following theorems.

Theorem 10 — Let f and g be two nonconstant meromorphic functions such that f and g share 1 CM. If

$$\limsup_{\substack{r\to\infty \\ r\in I}} \frac{\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}(r,g)}{T(r,f)+T(r,g)}<\frac{1}{4},$$

then $f\equiv g$ or $f\cdot g\equiv 1$.

PROOF : Since f and g share 1 CM, we can obtain (2.1). Using (2.1) and proceeding as in the proof of Theorem 7, we can get the conclusion of Theorem 10.

Using Theorem 10 and proceeding as in the proof of Theorem 8 and Theorem 9, we can obtain the following theorems respectively.

Theorem 11 — Let $S = \{a+b, a+bw, \ldots, a+bw^{n-1}\}$, where $n > 4$, a and $b(\neq 0)$ are constants. If f and g are nonconstant entire functions such that $E_f(S)=E_g(S)$, then $f-a\equiv t(g-a)$, where $t^n=1$, or $(f-a)\cdot(g-a)\equiv s$, where $s^n=b^{2n}$.

Theorem 12 — Let $S = \{a+b, a+bw, \ldots, a+bw^{n-1}\}$, where $n > 8$, a and $b\,(\neq 0)$ are constants. If f and g are nonconstant meromorphic functions such that $E_f(S)=E_g(S)$, then $f-a\equiv t(g-a)$, where $t^n=1$, or $(f-a)\cdot(g-a)\equiv s$, where $s^n=b^{2n}$.

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