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ON THE ZEROS OF A CLASS OF POLYNOMIALS AND RELATED ANALYTIC FUNCTIONS

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We have considered polynomials of type $\sum_{j=0}^{n} (a_j e^{i\phi} + b_j e^{i\psi}) z^j$ with $|\phi - \psi| < \pi$ and real a_j's, b_j's, and have obtained discs containing their zeros under various conditions on a_j's and b_j's. We have also considered associated analytic functions $\sum_{j=0}^{\infty} (a_j e^{i\phi} + b_j e^{i\psi}) z^j$ and have obtained zero free discs for them, under various conditions on a_j's and b_j's.

1. INTRODUCTION AND STATEMENT OF RESULTS

The following result is well known in the theory of the distribution of zeros of polynomials.

Theorem A (Eneström-Kakeya) — If $p(z) = \sum_{j=0}^{n} \alpha_j z^j$ is a polynomial of degree n such that

$$\alpha_n \geq \alpha_{n-1} \geq \alpha_{n-2} \geq .... \geq \alpha_1 \geq \alpha_0 > 0,$$

then all the zeros of $p(z)$ lie in $|z| \leq 1$.

With the help of Theorem A, one gets the following equivalent form of the Eneström-Kakeya theorem by considering the polynomial $z^n p(1/z)$.

Theorem A′ — If $p(z) = \sum_{j=0}^{n} \alpha_j z^j$ is a polynomial of degree n such that

$$\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq .... \geq \alpha_n > 0,$$

then $p(z)$ has no zeros in $|z| < 1$.

Karanicoloff[1] extended this result to polynomials with complex coefficients and proved the following

Theorem B — Let $p(z) = \sum_{j=0}^{n} \alpha_j z^j$ be a polynomial of degree n such that

$$\alpha_j = a_j e^{i\phi} + b_j e^{i\psi},\ j = 0,\ 1,\ 2,\ ....,\ n,$$

$$|\phi - \psi| < \pi, \quad a_0 \geq a_1 \geq ...... \geq a_n \geq 0, \quad b_0 \geq b_1 \geq ..... \geq b_n \geq 0.$$

Then $p(z)$ has no zeros in

$$|z| < r,$$

where

$$r = \frac{-(a_0 + b_0 - a_1 - b_1) + \left\{ (a_0 + b_0 - a_1 - b_1)^2 + 4\,|\alpha_0|\,(a_1 + b_1) \right\}^{1/2}}{2(a_1 + b_1)}.$$

With the help of Theorem B, one easily gets the following equivalent form of Karanicoloff's result by considering the polynomial $z^n p(1/z)$.

Theorem B′ — Let $p(z) = \sum_{j=0}^{n} \alpha_j z^j$ be a polynomial of degree n such that

$$\alpha_j = a_j e^{i\phi} + b_j e^{i\psi},\ j = 0,\ 1,\ 2,\ ....,\ n, \qquad ...\ (1.1)$$

$$|\phi - \psi| < \pi, \qquad ...\ (1.2)$$

$$a_n \geq a_{n-1} \geq .... \geq a_1 \geq a_0 > 0, \qquad ...\ (1.3)$$

$$b_n \geq b_{n-1} \geq .... \geq b_1 \geq b_0 > 0. \qquad ...\ (1.4)$$

Then all the zeros of $p(z)$ lie in

$$|z| \leq r', \qquad ...\ (1.5)$$

where

$$r' = \frac{1}{2\,|\alpha_n|} \Big(\left\{ (a_n + b_n) - (a_{n-1} + b_{n-1}) \right\} \left[\left\{ (a_n + b_n) - (a_{n-1} + b_{n-1}) \right\}^2 + 4\,|\alpha_n|\,(a_{n-1} + b_{n-1}) \right]^{1/2} \Big). \qquad ...\ (1.6)$$

We have generalised Theorem B′ and have obtained the following.

Theorem 1 — Let $p(z) = \sum_{j=0}^{n} \alpha_j z^j$ be a polynomial of degree n such that

$$\alpha_j = a_j e^{i\phi} + b_j e^{i\psi},\ j = 0, 1, 2, \ldots\ldots, n,$$

$$|\phi - \psi| < \pi.$$

If $t_1 > t_2 \geq 0$ can be found such that

$$\left.\begin{array}{l} a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2} \geq 0, \\ b_r t_1 t_2 + b_{r-1}(t_1 - t_2) - b_{r-2} \geq 0, \end{array}\right| \quad r = 0, 1, \ldots, n+1, \qquad \ldots (1.7)$$

$$a_{-1} = a_{-2} = a_{n+1} = b_{-1} = b_{-2} = b_{n+1} = 0,$$

then all the zeros of $p(z)$ lie in

$$|z| \leq \frac{1}{2|\alpha_n|}\Big[\big\{(a_n + b_n)(t_1 - t_2) - (a_{n-1} + b_{n-1})\big\}$$
$$+ \left(\{(a_n + b_n)(t_1 - t_2) - (a_{n-1} + b_{n-1})\}^2 + 4|\alpha_n| t_1 \{(a_n + b_n) t_2 + (a_{n-1} + b_{n-1})\}\right)^{1/2}\Big]. \qquad \ldots (1.8)$$

Remark : For $t_1 = 1$ and $t_2 = 0$, Theorem 1 reduces to Theorem B′.

By considering $t_2 = 0$, in Theorem 1, we obtain

Corollary 1 — Let $p(z) = \sum_{j=0}^{n} \alpha_j z^j$ be a polynomial of degree n such that

$$\alpha_j = a_j e^{i\phi} + b_j e^{i\psi},\ j = 0, 1, 2, \ldots, n,$$

$$|\phi - \psi| < \pi,$$

$$a_j > 0, b_j > 0, j = 0, 1, 2, \ldots, n.$$

Then all the zeros of $p(z)$ lie in

$$|z| \leq R', \qquad \ldots (1.9)$$

where

$$R' = \frac{1}{2|\alpha_n|}\Big[\big\{t_1(a_n + b_n) - (a_{n-1} + b_{n-1})\big\}$$
$$+ \left(\big\{t_1(a_n + b_n) - (a_{n-1} + b_{n-1})\big\}^2 + 4|\alpha_n| t_1 (a_{n-1} + b_{n-1})\right)^{1/2}\Big], \qquad \ldots (1.10)$$

where

$$t_1 = \max\left(\frac{a_0}{a_1}, \frac{a_1}{a_2}, \ldots, \frac{a_{n-1}}{a_n}, \frac{b_0}{b_1}, \frac{b_1}{b_2}, \ldots, \frac{b_{n-1}}{b_n}\right). \qquad \ldots (1.11)$$

We have further refined the bound (1.9) and have obtained the following

Theorem 2 — Under the same hypotheses as in Corollary 1, all the zeros of the polynomial $p(z)$ lie in

$$\{z : |z| \le R', |z-a| \ge R_a \text{ for every positive } a\},$$

where

$$R_a = \frac{a}{n}\left[\frac{-(c_0^a + d_0^a - c_1^a - d_1^a) + \left\{(c_0^a + d_0^a - c_1^a - d_1^a)^2 + 4\,|p(a)|\,(c_1^a + d_1^a)\right\}^{1/2}}{2(c_1^a + d_1^a)}\right], \qquad \text{... (1.12)}$$

and $c_0^a, d_0^a, c_1^a, d_1^a$ are given by

$$p(a) = c_0^a e^{i\phi} + d_0^a e^{i\psi}, \qquad \text{... (1.13)}$$

$$\left(\frac{a}{n}\right)\frac{p'(a)}{1!} = c_1^a e^{i\phi} + d_1^a e^{i\psi}. \qquad \text{... (1.14)}$$

Simultaneously, we have been able to refine Karanicoloff's bound (i.e. Theorem B′) also and have obtained

Theorem 3 — Let $p(z) = \sum_{j=0}^{n} \alpha_j z^j$ be a polynomial of degree n such that

$$\alpha_j = a_j e^{i\phi} + b_j e^{i\psi},\ j = 0, 1, 2, \ldots, n,$$

$$|\phi - \psi| < \pi,$$

$$a_n \ge a_{n-1} \ge \ldots \ge a_1 \ge a_0 > 0,$$

$$b_n \ge b_{n-1} \ge \ldots \ge b_1 \ge b_0 > 0.$$

Then all the zeros of $p(z)$ lie in $\{z : |z| \le r', |z-a| \ge R_a$ for every positive $a\ (\ne 1)$, $|z-1| \ge R_1''\,|z|\}$, where r' is as in Theorem B′, R_a and associated quantities are as in Theorem 2,

$$R_1'' = \frac{2}{n}\left[\frac{-(c_0'' + d_0'' - c_1'' - d_1'') + \left\{(c_0'' + d_0'' - c_1'' - d_1'')^2 + 4\,|q(1)|\,(c_1'' + d_1'')\right\}^{1/2}}{2(c_1'' + d_1'')}\right], \qquad \text{... (1.15)}$$

$$q(z) = z^n p(1/z), \qquad \text{... (1.16)}$$

$$q(1) = c_0'' e^{i\phi} + d_0'' e^{i\psi} \qquad \text{... (1.17)}$$

$$\left(\frac{2}{n}\right)\frac{q'(1)}{1!} = c_1'' e^{i\phi} + d_1'' e^{i\psi}. \qquad \text{.... (1.18)}$$

Finally, we have considered a class of associated analytic functions and have obtained the following results about the zeros of these analytic functions.

Theorem 4 — Let $f(z) = \sum_{j=0}^{\infty} \alpha_j z^j \ (\not\equiv 0)$ be analytic in $|z| \leq t$. If

$$\alpha_j = a_j e^{i\phi} + b_j e^{i\psi}, \ j = 0, 1, 2, \ldots., \qquad \ldots (1.19)$$

$$0 < |\phi - \psi| < \pi, \qquad \ldots (1.20)$$

and for some finite non-negative integer k,

$$0 < a_0 \leq t a_1 \leq \ldots. \leq t^k a_k \geq t^{k+1} a_{k+1} \geq \ldots., \qquad \ldots (1.21)$$

then $f(z)$ does not vanish in

$$|z| < \frac{t}{2M_k'} \{-|\alpha_0 - \alpha_1 t| + (|\alpha_0 - \alpha_1 t|^2 + 4M_k' |\alpha_0|)^{1/2}\}, \qquad \ldots (1.22)$$

where

$$M_k' = 2a_k t^k - a_1 t + |b_1| t + 2 \sum_{j=2}^{\infty} |b_j| t^j \quad (k \geq 1), \qquad \ldots (1.23)$$

$$M_0' = M_1'. \qquad \ldots (1.24)$$

Theorem 5 — Let $f(z) = \sum_{j=0}^{\infty} \alpha_j z^j \ (\not\equiv 0)$ be analytic in $|z| \leq t$. If

$$\alpha_j = a_j e^{i\phi} + b_j e^{i\psi}, \ j = 0, 1, 2\ldots$$

$$0 < |\phi - \psi| < \pi,$$

and for some finite non-negative integers k and r,

$$0 < a_0 \leq t a_1 \leq \ldots. \leq t^k a_k \geq t^{k+1} a_{k+1} \geq \ldots.. ,$$

$$0 < b_0 \leq t b_1 \leq \ldots. \leq t^r b_r \geq t^{r+1} b_{r+1} \geq \ldots.. , \qquad \ldots (1.25)$$

then $f(z)$ does not vanish in

$$|z| < \frac{t}{2m_{kr}'} \{-|\alpha_0 - \alpha_1 t| + (|\alpha_0 - \alpha_1 t|^2 + 4|\alpha_0| m_{kr}')^{1/2}\}, \qquad \ldots (1.26)$$

where

$$m_{kr}' = 2a_k t^k - a_1 t + 2b_r t^r - b_1 t \qquad (r, k \geq 1) \qquad \ldots (1.27)$$

$$m_{k0}' = m_{k1}' \quad (k \geq 1) \qquad \ldots (1.28)$$

$$m'_{0r} = m'_{1r}\ (r \geq 1) \qquad \dots (1.29)$$

$$m'_{00} = m'_{11}. \qquad \dots (1.30)$$

2. LEMMAS

For the proofs of the theorems, we need the following lemmas.

Lemma 1 — Let $p(z)$ be a polynomial of degree n. Then

$$\max_{|z|=R} |p'(z)| \leq (n/R) \max_{|z|=R} |p(z)|. \qquad \dots (2.1)$$

Lemma 1 is an immediate consequence of Bernstein's theorem on the derivative of a trigonometric polynomial[3].

Lemma 2 — If $p(z)$ is a polynomial of degree n, having no zeros in $|z| < 1$, then

$$\max_{|z|=1} |p'(z)| \leq \left(\frac{n}{2}\right) \max_{|z|=1} |p(z)|. \qquad \dots (2.2)$$

There is equality in (2.2) for $p(z) = 1 + z^n$.

Lemma 2 is due to Lax[2].

Lemma 3 — Let $p(z) = \sum_{j=0}^{n} \alpha_j z^j$ be a polynomial of degree n such that

$$\alpha_j = a_j e^{i\phi} + b_j e^{i\psi},\ j = 0,.\ 1, \dots, n,$$

$$|\phi - \psi| < \pi,$$

$$a_j \geq 0, b_j \geq 0,\ j = 0, 1, 2, \dots, n.$$

Then for every real $a > 0$, $p(z)$ does not vanish in the disc

$$|z - a| < R_a, \qquad \dots (2.3)$$

where R_a and associated quantities are, as in Theorem 2.

Proof of Lemma 3 — It is obvious that

$$\psi_1(t) = p\left(a + \frac{ta}{n}\right) = p(a) + \left(\frac{a}{n}\right)\frac{p'(a)}{1!}t + \left(\frac{a}{n}\right)^2 \frac{p''(a)}{2!}t^2 + \dots + \left(\frac{a}{n}\right)^n \frac{p^{(n)}(a)}{n!}t^n. \qquad \dots (2.4)$$

Also

$$\begin{aligned} p(z) &= \left(\sum_{j=0}^{n} a_j z^j\right) e^{i\phi} + \left(\sum_{j=0}^{n} b_j z^j\right) e^{i\psi}. \\ &= g(z)e^{i\phi} + h(z)e^{i\psi} \end{aligned} \qquad \text{... (2.5)}$$

where

$$g(z) = \sum_{j=0}^{n} a_j z^j \qquad \text{... (2.6)}$$

$$h(z) = \sum_{j=0}^{n} b_j z^j \qquad \text{... (2.7)}$$

are polynomials with non-negative coefficients.

Using (2.5), we get

$$\left.\begin{array}{l} p(a) = g(a)e^{i\phi} + h(a)e^{i\psi} = c_0^a\, e^{i\phi} + d_0^a\, e^{i\psi} \\ \left(\dfrac{a}{n}\right)\dfrac{p'(a)}{1!} = \left(\dfrac{a}{n}\right)\dfrac{g'(a)}{1!}\, e^{i\phi} + \left(\dfrac{a}{n}\right)\dfrac{h'(a)}{1!}\, e^{i\psi} = c_1^a\, e^{i\phi} + d_1^a\, e^{i\psi} \\ \qquad\vdots \\ \left(\dfrac{a}{n}\right)^k \dfrac{p^{(k)}(a)}{k!} = \left(\dfrac{a}{n}\right)^k \dfrac{g^{(k)}(a)}{k!}\, e^{i\phi} + \left(\dfrac{a}{n}\right)^k \dfrac{h^{(k)}(a)}{k!}\, e^{i\psi} = c_k^a\, e^{i\phi} + d_k^a\, e^{i\psi} \\ \qquad\vdots \\ \left(\dfrac{a}{n}\right)^n \dfrac{p^{(n)}(a)}{n!} = \left(\dfrac{a}{n}\right)^n \dfrac{g^{(n)}(a)}{n!}\, e^{i\phi} + \left(\dfrac{a}{n}\right)^n \dfrac{h^{(n)}(a)}{n!}\, e^{i\psi} = c_n^a\, e^{i\phi} + d_n^a\, e^{i\psi} \end{array}\right]$$

$$\text{... (2.8)}$$

Now $g^{(k)}(z)$, $k = 0, 1, ..., n-1$ is a polynomial of degree $(\leq n-k)$ with real and non-negative coefficients. Therefore, applying Lemma 1 to the polynomial $g^{(k)}(z)$, we get

$$\max_{|z|=a} |g^{(k+1)}(z)| \leq \frac{(n-k)}{a} \max_{|z|=a} |g^{(k)}(z)|,\ k = 0, 1, ..., n-1,$$

which implies

$$\begin{aligned} g^{(k+1)}(a) &\leq \left(\frac{n-k}{a}\right) g^{(k)}(a),\ k = 0, 1, ..., n-1, \\ &\leq \frac{n(1+k)}{a}\, g^{(k)}(a),\ k = 0, 1, ..., n-1. \end{aligned}$$

And so,

$$\left(\frac{a}{n}\right)^{k+1}\frac{g^{(k+1)}(a)}{(k+1)!}\le\left(\frac{a}{n}\right)^{k}\frac{g^{(k)}(a)}{k!},\ k = 0,\ 1,\ \ldots,\ n-1,$$

which implies that

$$c_0^a \ge c_1^a \ge c_2^a \ge \ldots \ge c_n^a \ge 0. \qquad \ldots (2.9)$$

We can similarly show that

$$d_0^a \ge d_1^a \ge d_2^a \ge \ldots \ge d_n^a \ge 0. \qquad \ldots (2.10)$$

So by (2.4), (2.8), (2.9), (2.10) and Theorem B, we can say that $\psi_1(t)$ does not vanish in the disc

$$|t| < \frac{-(c_0^a + d_0^a - c_1^a - d_1^a) + \left\{(c_0^a + d_0^a - c_1^a - d_1^a)^2 + 4\,|p(a)|\,(c_1^a + d_1^a)\right\}^{1/2}}{2(c_1^a + d_1^a)}. \qquad \ldots (2.11)$$

And as

$$\psi_1(t) = p\left(a + \frac{ta}{n}\right),$$

we can say that $p(z)$ does not vanish in the disc

$$|z-a| < \frac{a}{n}\left[\frac{-(c_0^a + d_0^a - c_1^a - d_1^a) + \left\{(c_0^a + d_0^a - c_1^a - d_1^a)^2 + 4\,|p(a)|\,(c_1^a + d_1^a)\right\}^{1/2}}{2(c_1^a + d_1^a)}\right].$$

This completes the proof of Lemma 3.

Lemma 4 — Let $p(z) = \sum_{j=0}^{n} \alpha_j z^j$ be a polynomial of degree n such that

$$\alpha_j = a_j e^{i\phi} + b_j e^{i\psi},\ j = 0,\ 1,\ 2,\ \ldots,\ n,$$

$$|\phi - \psi| < \pi,$$

$$a_0 \ge a_1 \ge a_2 \ge \ldots \ge a_n \ge 0,$$

$$b_0 \ge b_1 \ge b_2 \ge \ldots \ge b_n \ge 0.$$

Then $p(z)$ does not vanish in the disc

$$|z-1| < R_1' \qquad \ldots (2.12)$$

where

$$R_1' = \frac{2}{n}\left[\frac{-(c_0' + d_0' - c_1' - d_1') + \left\{(c_0' + d_0' - c_1' - d_1')^2 + 4\,|p(1)|\,(c_1' + d_1')\right\}^{1/2}}{2(c_1' + d_1')}\right]$$

and c_0', d_0', c_1', d_1' are given by

$$p(1) = c_0' e^{i\phi} + d_0' e^{i\psi}$$

$$\left(\frac{2}{n}\right)\frac{p'(1)}{1!} = c_1' e^{i\phi} + d_1' e^{i\psi}.$$

Proof of Lemma 4 — It is obvious that

$$\phi_1(t) = p\left(1 + \frac{2}{n}t\right) = p(1) + \left(\frac{2}{n}\right)\frac{p'(1)}{1!}t + \left(\frac{2}{n}\right)^2\frac{p''(1)}{2!}t^2 + \ldots$$

$$+ \ldots. + \left(\frac{2}{n}\right)^n \frac{p^{(n)}(1)}{n!}t^n \qquad \ldots (2.13)$$

Also, using (2.5), we get

$$\left.\begin{aligned}
p(1) &= g(1)e^{i\phi} + h(1)e^{i\psi} = c_0' e^{i\phi} + d_0' e^{i\psi} \\
\left(\frac{2}{n}\right)\frac{p'(1)}{1!} &= \left(\frac{2}{n}\right)\frac{g'(1)}{1!}e^{i\phi} + \left(\frac{2}{n}\right)\frac{h'(1)}{1!}e^{i\psi} = c_1' e^{i\phi} + d_1' e^{i\psi} \\
&\vdots \\
\left(\frac{2}{n}\right)^n\frac{p^{(n)}(1)}{n!} &= \left(\frac{2}{n}\right)^n\frac{g^{(n)}(1)}{n!}e^{i\phi} + \left(\frac{2}{n}\right)^n\frac{h^{(n)}(1)}{n!}e^{i\psi} = c_n' e^{i\phi} + d_n' e^{i\psi}.
\end{aligned}\right] \qquad \ldots (2.14)$$

Now $g^{(k)}(z)$, $k = 1, \ldots., n - 1$, is a polynomial of degree $(\leq n - k)$ with real and non-negative coefficients. Therefore, applying Lemma 1 to the polynomial $g^{(k)}(z)$, we get

$$\max_{|z|=1} |g^{(k+1)}(z)| \leq \frac{(n-k)}{1} \max_{|z|=1} |g^{(k)}(z)|, \; k = 1, 2, \ldots., n - 1,$$

which implies

$$g^{(k+1)}(1) \leq (n-k)\, g^{(k)}(1), \; k = 1, 2, \ldots., n - 1,$$

$$\leq \frac{n(1+k)}{2} g^{(k)}(1), \; k = 1, 2, \ldots., n - 1.$$

And so

$$\left(\frac{2}{n}\right)^{k+1} \frac{g^{(k+1)}(1)}{(k+1)!} \leq \left(\frac{2}{n}\right)^k \frac{g^{(k)}(1)}{k!}, \; k = 1, 2, \ldots, n - 1. \qquad \ldots (2.15)$$

Also $g(z)$ is a polynomial of degree $(\leq n)$ with real, non-negative and decreasing coefficients. Therefore, by Theorem A$'$, $g(z)$ has no zeros in $|z| < 1$ (excluding the trivial case $g(z) \equiv 0$). And so, by Lemma 2,

$$\max_{|z|=1} |g'(z)| \leq \left(\frac{n}{2}\right) \max_{|z|=1} |g(z)|,$$

which implies

$$g'(1) \leq \left(\frac{n}{2}\right) g(1),$$

i.e.

$$g(1) \geq \left(\frac{2}{n}\right) g'(1). \qquad \text{... (2.16)}$$

This inequality is also true in the trivial case ($g(z) \equiv 0$). Nowm by (2.15) and (2.16), we can say that

$$c_0' \geq c_1' \geq c_2' \geq .... \geq c_n' \geq 0. \qquad \text{... (2.17)}$$

We can similarly show that

$$d_0' \geq d_1' \geq d_2' \geq .... \geq d_n' \geq 0. \qquad \text{... (2.18)}$$

So by (2.13), (2.14), (2.17), (2.18) and Theorem B, we can say that $\phi_1(t)$ does not vanish in the disc

$$|t| < \frac{-(c_0' + d_0' - c_0' - d_1') + \left\{(c_0' + d_0' - c_1' - d_1')^2 + 4\,|p(1)|\,(c_1' + d_1')\right\}^{1/2}}{2(c_1' + d_1')}. \qquad \text{... (2.19)}$$

And, as

$$\phi_1(t) = p\left(1 + \frac{2}{n} t\right),$$

we can say that $p(z)$ does not vanish in the disc

$$|z - 1| < \frac{2}{n}\left[\frac{-(c_0' + d_0' - c_1' - d_1') + \left\{(c_0' + d_0' - c_1' - d_1')^2| + 4\,|p(1)|\,(c_1' + d_1')\right\}^{1/2}}{2(c_1' + d_1')}\right].$$

This completes the proof of Lemma 4.

Lemma 5 — Let $f(z) = \sum_{j=0}^{\infty} \alpha_j z^j \ (\not\equiv 0)$ be analytic in $|z| \leq t$. If

$$\alpha_j = a_j e^{i\phi} + b_j e^{i\psi},\ j = 0, 1, 2, ...,$$

$$0 < |\phi - \psi| < \pi,$$

then $\sum\limits_{j=0}^{\infty} |a_j| t^j$ and $\sum\limits_{j=0}^{\infty} |b_j| t^j$ converge.

PROOF : $\sum\limits_{j=0}^{\infty} |\alpha_j| t^j$ obviously converges. Now, for $j = 0, 1, 2, \ldots$

$$|\alpha_j t^j| = |(a_j e^{i\phi} + b_j e^{i\psi}) t^j|$$

$$= |a_j' \cos\phi + b_j' \cos\psi) + i(a_j' \sin\phi + b_j' \sin\psi)$$

where

$$a_j' = a_j t^j, \ j = 0, 1, 2, \ldots \qquad \ldots (2.20)$$

$$b_j' = b_j t^j, \ j = 0, 1, 2, \ldots. \qquad \ldots (2.21)$$

And so, the series

$$\sum_{j=0}^{\infty} |a_j' \cos\phi + b_j' \cos\psi|, \quad \sum_{j=0}^{\infty} |a_j' \sin\phi + b_j' \sin\psi|$$

converge. Hence the series

$$\sum_{j=0}^{\infty} |a_j' \cos\phi + b_j' \cos\psi| \, |\sin\psi|, \quad \sum_{j=0}^{\infty} |a_j' \sin\phi + b_j' \sin\psi| \, |\cos\psi|$$

will also converge. Therefore the series

$$\sum_{j=0}^{\infty} |(a_j' \sin\phi + b_j' \sin\psi)\cos\psi - (a_j' \cos\phi + b_j' \cos\psi)\sin\psi| \qquad \ldots (2.22)$$

$$= \sum_{j=0}^{\infty} |a_j' \sin(\phi - \psi)| = \sum_{j=0}^{\infty} |a_j'| \, |\sin(\phi - \psi)|$$

also converges. And as, $0 < |\phi - \psi| < \pi$, the series $\sum\limits_{j=0}^{\infty} |a_j'|$ i.e. $\sum\limits_{j=0}^{\infty} |a_j| t^j$ also converges.

By considering the series

$$\sum_{j=0}^{\infty} |(a_j' \sin\phi + b_j' \sin\psi)\cos\phi - (a_j' \cos\phi + b_j' \cos\psi)\sin\phi|,$$

instead of the series (2.22), we get the convergence of the series $\sum\limits_{j=0}^{\infty} |b_j| t^j$. This completes the proof of Lemma 5.

3. Proofs of the Theorems

Proof of Theorem 1 — Consider the polynomial

$$
\begin{aligned}
F(z) &= (t_2 + z)(t_1 - z)\,p(z) && \dots (3.1)\\
&= -\alpha_n z^{n+2} + \{\alpha_n (t_1 - t_2) - \alpha_{n-1}\} z^{n+1}\\
&\quad + \{\alpha_n t_1 t_2 + \alpha_{n-1}(t_1 - t_2) - \alpha_{n-2}\} z^n + \dots\\
&\quad + \{\alpha_2 t_1 t_2 + \alpha_1 (t_1 - t_2) - \alpha_0\} z^2 + \{\alpha_1 t_1 t_2 + \alpha_0 (t_1 - t_2)\}\, z + \alpha_0 t_1 t_2\\
&= -\alpha_n z^{n+2} + [\{a_n (t_1 - t_2) - a_{n-1}\}\, e^{i\phi} + \{b_n (t_1 - t_2) - b_{n-1}\} e^{i\Psi}] z^{n+1}\\
&\quad + [\{a_n t_1 t_2 + a_{n-1}(t_1 - t_2) - a_{n-2}\}\, z^n + \dots + \{a_2 t_1 t_2\\
&\quad + a_1 (t_1 - t_2) - a_0\} z^2 + \{a_1 t_1 t_2 + a_0 (t_1 - t_2)\}\, z + a_0 t_1 t_2] e^{i\phi}\\
&\quad + [\{b_n t_1 t_2 + b_{n-1}(t_1 - t_2) - b_{n-2}\} z^n + \dots\\
&\quad + \{b_2 t_1 t_2 + b_1 (t_1 - t_2) - b_0\}\, z^2\\
&\quad + \{b_1 t_1 t_2 + b_0 (t_1 - t_2)\}\, z + b_0 t_1 t_2]\, e^{i\Psi}\\
&= -\alpha_n z^{n+2} + [\{a_n (t_1 - t_2) - a_{n-1}\}\, e^{i\phi}\\
&\quad + \{b_n (t_1 - t_2) - b_{-n-1}\}\, e^{i\Psi}]\, z^{n+1} + R_1(z) + S_1(z), && \dots (3.2)
\end{aligned}
$$

where
$$R_1(z) = \left[\sum_{j=0}^{n} \left\{ a_j t_1 t_2 + a_{j-1}(t_1 - t_2) - a_{j-2} \right\} z^j \right] e^{i\phi} \qquad \dots (3.3)$$

$$S_1(z) = \left[\sum_{j=0}^{n} \left\{ b_j t_1 t_2 + b_{j-1}(t_1 - t_2) - b_{j-2} \right\} z^j \right] e^{i\Psi}. \qquad \dots (3.4)$$

For $|z| = Rt_1$ $(R \geq 1)$, we have

$$
\begin{aligned}
|R_1(z)| &\leq \sum_{j=0}^{n} \left| a_j t_1 t_2 + a_{j-1}(t_1 - t_2) - a_{j-2} \right| |z|^j\\
&= \sum_{j=0}^{n} \{a_j t_1 t_2 + a_{j-1}(t_1 - t_2) - a_{j-2}\} R^j t_1^j, && \text{(by (1.7))}\\
&\leq \left[\sum_{j=0}^{n} \left\{ a_j t_1 t_2 + a_{j-1}(t_1 - t_2) - a_{j-2} \right\} t_1^j \right] R^n\\
&= (a_n t_2 + a_{n-1})\, R^n t_1^{n+1}, && \dots (3.5)
\end{aligned}
$$

which implies that

$$a_n t_2 + a_{n-1} \geq 0. \qquad \dots (3.6)$$

Similarly, for $|z| = Rt_1$ $(R \geq 1)$, we can show that

$$|S_1(z)| \leq (b_n t_2 + b_{n-1}) R^n t_1^{n+1} \qquad \dots (3.7)$$

which implies that

$$b_n t_2 + b_{n-1} \geq 0. \qquad \dots (3.8)$$

Now, for $|z| = Rt_1$ $(R \geq 1)$, we have

$$\begin{aligned} |F(z)| &\geq |\alpha_n| |z|^{n+2} - \{|a_n(t_1 - t_2) - a_{n-1}| \\ &\quad + |b_n(t_1 - t_2) - b_{n-1}|\} |z|^{n+1} - |R_1(z)| - |S_1(z)| \\ &\geq |\alpha_n| R^{n+2} t_1^{n+2} - [\{ a_n(t_1 - t_2) - a_{n-1}\} \\ &\quad + \{b_n(t_1 - t_2) - b_{n-1}\}] R^{n+1} t_1^{n+1} \\ &\quad - \{(a_n + b_n) t_2 + (a_{n-1} + b_{n-1})\} R^n t_1^{n+1} \\ &\qquad\qquad \text{(by (1.7), (3.5) and (3.7))} \\ &= [|\alpha_n| R^2 t_1 - \{(a_n + b_n)(t_1 - t_2) - (a_{n-1} + b_{n-1})\} R \\ &\quad - \{(a_n + b_n) t_2 + (a_{n-1} + b_{n-1})\}] R^n t_1^{n+1}, \\ &= |\alpha_n| R^n t_1^{n+2} (R - \alpha)(R + \beta), \end{aligned} \qquad \dots (3.9)$$

where

$$\begin{aligned} \alpha = \frac{1}{2|\alpha_n| t_1} & [\{(a_n + b_n)(t_1 - t_2) - (a_{n-1} + b_{n-1})\} \\ & + (\{(a_n + b_n)(t_1 - t_2) - (a_{n-1} + b_{n-1})\}^2 \\ & + 4|\alpha_n| t_1 \{(a_n + b_n) t_2 + (a_{n-1} + b_{n-1})\})^{1/2}], \end{aligned} \qquad \dots (3.10)$$

$$\begin{aligned} -\beta = \frac{1}{2|\alpha_n| t_1} & [\{(a_n + b_n)(t_1 - t_2) - (a_{n-1} + b_{n-1}) \\ & - (\{(a_n + b_n)(t_1 - t_2) - (a_{n-1} + b_{n-1})\}^2 \\ & + 4|\alpha_n| t_1 \{(a_n + b_n) t_2 + (a_{n-1} + b_{n-1})\})^{1/2}]. \end{aligned} \qquad \dots (3.11)$$

Further (3.2) can be rewritten as

$$\begin{aligned} F(z) = -\alpha_n z^{n+2} + & [\{a_n(t_1 - t_2) - a_{n-1}\} e^{i\phi} z^{n+1} + R_1(z)] \\ & + [\{b_n(t_1 - t_2) - b_{n-1}\} e^{i\psi} z^{n+1} + S_1(z)] \end{aligned}$$

$$= -\alpha_n z^{n+2} + [R(z)] + [S(z)]. \qquad \text{... (3.12)}$$

As we have obtained (3.6) using modulus of $R_1(z)$ on $|z| = Rt_1$, we can similarly obtain

$$a_n \geq 0, \qquad \text{... (3.13)}$$

using modulus of $R(z)$ on $|z| = Rt_1$. On similar lines, we can get

$$b_n \geq 0, \qquad \text{... (3.14)}$$

using modulus of $S(z)$ on $|z| = Rt_1$.

Now we get easily that

$$-\beta \leq 0 \text{ (by (3.6) and (3.8))}, \qquad \text{... (3.15)}$$

$$\alpha \geq 1 \text{ (by (3.13) and (3.14))}. \qquad \text{... (3.16)}$$

Therefore for $|z| = Rt_1$ $(R \geq 1)$, we have

$$|F(z)| > 0,$$

if

$$R > \alpha \qquad \text{(by (3.9))}.$$

Hence $F(z)$ and therefore $p(z)$ has all its zeros in

$$|z| \leq \alpha\, t_1$$

$$= \frac{1}{2|\alpha_n|} [\{(a_n + b_n)(t_1 - t_2) - (a_{n-1} + b_{n-1})\}$$

$$+ (\{(a_n + b_n)(t_1 - t_2) - (a_{n-1} + b_{n-1})\}^2$$

$$+ 4|\alpha_n| t_1 \{(a_n + b_n) t_2 + (a_{n-1} + b_{n-1})\})^{1/2}].$$

And this completes the proof of the theorem.

Proof of Theorem 2 — It follows easily with the help of Corollary 1 and Lemma 3.

Proof of Theorem 3 — Applying Lemma 4 to the polynomial $q(z) = z^n p(1/z)$, we get the region

$$|z - 1| \geq R_1''$$

containing all the zeros of $q(z)$. And therefore all the zeros of $p(z)$ lie in

$$|z - 1| \geq R_1'' |z|. \qquad \text{... (3.17)}$$

And now, with the help of Theorem B′ and Lemma 3, the theorem follows easily.

Proof of Theorem 4 — Obviously

$$g(z) = f(tz) = \sum_{j=0}^{\infty} \alpha_j t^j z^j \qquad \ldots (3.18)$$

is analytic in $|z| \leq 1$. Now, consider the function

$$G(z) = (z-1)\,g(z) = -\alpha_0 + (\alpha_0 - \alpha_1 t)\,z + \sum_{j=2}^{\infty} (\alpha_{j-1} t^{j-1} - \alpha_j t^j)\,z^j$$

$$= -\alpha_0 + (\alpha_0 - \alpha_1 t)z + \phi_2(z). \qquad \ldots (3.19)$$

For $|z| \leq 1$ and $k \geq 2$, we have

$$|s_n(z)| = \left| \sum_{j=2}^{n} (\alpha_{j-1} t^{j-1} - \alpha_j t^j)\,z^j \right|$$

$$\leq |z|^2 \left\{ \sum_{j=2}^{n} |\alpha_{j-1} t^{j-1} - \alpha_j t^j| \right\}$$

$$\leq |z|^2 \left\{ \sum_{j=2}^{n} (|a_{j-1} t^{j-1} - a_j t^j| + |b_{j-1} t^{j-1} - b_j t^j|) \right\}$$

$$\leq |z|^2 \left\{ \sum_{j=2}^{k} (a_j t^j - a_{j-1} t^{j-1}) + \sum_{j=k+1}^{n} (a_{j-1} t^{j-1} - a_j t^j) + \sum_{j=2}^{n} (|b_{j-1}|\,t^{j-1} + |b_j|\,t^j) \right\}$$

(by (1.21))

$$= |z|^2 \left\{ 2a_k t^k - a_1 t - a_n t^n + |b_1|\,t + 2\sum_{j=2}^{n} |b_j|\,t^j - |b_n|\,t^n \right\}. \qquad \ldots (3.20)$$

Considering the limit in (3.20), as $n \to \infty$, we get, for $|z| \leq 1$ and $k \geq 2$, by Lemma 5

$$|\phi_2(z)| \leq |z|^2 \left\{ 2a_k t^k - a_1 t + |b_1|\,t + 2\sum_{j=2}^{\infty} |b_j|\,t^j \right\}. \qquad \ldots (3.21)$$

For $|z| \leq 1$ and $k = 0, 1$, one can get analogously that

$$|\phi_2(z)| \leq |z|^2 \left\{ a_1 t + |b_1|\,t + 2\sum_{j=2}^{\infty} |b_j|\,t^j \right\}. \qquad \ldots (3.22)$$

And so, from (3.21) and (3.22), we can always say that for $|z| \le 1$

$$|\phi_2(z)| \le |z|^2 M_k' \qquad \text{(by (1.23) and (1.24))} \qquad \ldots (3.23)$$

whatsoever k may be,

Now for $|z| \le 1$, we have

$$\begin{aligned} |G(z)| &\ge |\alpha_0| - |\alpha_0 - \alpha_1 t|\,|z| - |\phi_2(z)| \\ &\ge |\alpha_0| - |\alpha_0 - \alpha_1 t|\,|z| - M_k'|z|^2 \qquad \text{(by (3.23))} \\ &= -M_k'(|z| - \alpha')(|z| + \beta') \qquad \ldots (3.24) \end{aligned}$$

where

$$\alpha' = \frac{1}{2M_k'}\ \{-|\alpha_0 - \alpha_1 t| + (|\alpha_0 - \alpha_1 t|^2 + 4|\alpha_0| M_k')^{1/2}\}, \qquad \ldots (3.25)$$

$$-\beta = \frac{1}{2M_k'}\ \{-|\alpha_0 - \alpha_1 t| - (|\alpha_0 - \alpha_1 t|^2 + 4|\alpha_0| M_k')^{1/2}\}. \qquad \ldots (3.26)$$

Further it is obvious that

$$-\beta' \le 0. \qquad \ldots (3.27)$$

Also

$$\alpha' \le 1$$

iff

$$|\alpha_0| \le M_k' + |\alpha_0 - \alpha_1 t|. \qquad \ldots (3.28)$$

And (3.28) is trivially seen to be true. And so

$$\alpha' \le 1. \qquad \ldots (3.29)$$

Therefore from (3.24), we have, for $|z| \le 1$

$$|G(z)| > 0$$

if

$$|z| < \alpha'.$$

Thus $g(z)$ does not vanish in

$$|z| < \alpha'.$$

Hence by (3.18), $f(z)$ does not vanish in

$$|z| < t\alpha'.$$

And this completes the proof of the theorem.

Proof of Theorem 5 — It follows on the same lines as Theorem 4, with a change :

$$\sum_{j=2}^{n} |b_{j-1}t^{j-1} - b_j t^j| = \sum_{j=2}^{r} (b_j t^j - b_{j-1}t^{j-1}) + \sum_{j=r+1}^{n} (b_{j-1}t^{j-1} - b_j t^j), r \geq 2,$$

$$= \sum_{j=2}^{n} (b_{j-1}t^{j-1} - b_j t^j), \quad r = 0, 1$$

instead of

$$\sum_{j=2}^{n} |b_{j-1}t^{j-1} - b_j t^j| \leq \sum_{j=2}^{n} (|b_{j-1}|t^{j-1} + |b_j|t^j).$$

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