Indian J. pure appl. Math., 28(4) : 563-573, April 1997

SHEAR FLOW INSTABILITY OF AN INCOMPRESSIBLE VISCO-ELASTIC SECOND ORDER FLUID IN A POROUS MEDIUM

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(*Received* 22 *November* 1995; *after revision* 10 *October* 1996; *accepted* 5 *December* 1996)

The present paper deals with the shear flow instability of visco-elastic second order fluid in a porous medium within the framework of linear theory of stability. Following Jaimala and Agrawal[1] and unlike Beck[2] and Irmay[3], the inertia term is not modified when the flow and its stability is governed by Darcy's law[4]. The sufficient conditions of Miles[5] and those of Jaimala and Agrawal[1] are extended to the present case and the analysis reveals a stabilizing character of viscosity and visco-elasticity and a destabilizing character of medium permeability and the shear of velocity.

INTRODUCTION

A detailed account of the instability of fluid flows in a porous medium under varying assumptions has been well summarized by Scheidegger[6] and Yih[7,8] and further, the last three decades have witnessed an enormous amount of work in this direction under various untouched but physically important conditions. The Darcy's law completely neglects the viscous force. Brinkman[9] argued that for flows through a porous medium with high permeability, the momentum equation for porous medium flow must reduce to the viscous limit and he advocated that the classical frictional term be included in the Darcy's law. This led to the celebrated Brinkman's model. In a recent study, Jaimala and Agrawal[1] unlike Irmay[3] and Beck[2], have investigated a more physically realistic model where at high flow rates or in high permeability porous media there is a departure from Darcy's law and there is no necessity of modifying the inertia term.

The investigations of the instability problems of flows through porous media in non-Newtonian fluids are however very few in number as compared to those in Newtonian fluids though the non-Newtonian viscosity is one of the most interesting rheological phenomena on high polymeric systems. In particular, the theory of second order fluids, being recently built by Green *et al.*[10], Coleman[11] and Noll[12] and most up-to-date to characterize the behaviour of the most general and simple fluids, has attracted the attention of research workers in past few decades.

Thus the motivation of present analysis for second order fluids is provided by Brinkman[9] and Jaimala and Agrawal[1] who pointed out the situations in which the inertia and viscous terms should be included in their usual forms to account for the flows with high flow rates, high permeability and high viscous force.

MATHEMATICAL ANALYSIS

Physical Problem

Consider a uni-directional flow of an incompressible visco-elastic second order fluid saturated in a horizontal, homogeneous and isotropic porous medium, bounded by two infinite parallel plates situated at a distance d apart. In a cartesian frame, the axis of x is in the main flow direction and the axis of z is against gravity. Thus the equations governing the stability of the flows through porous media using Brinkman[9] and visco-elastic model are

$$\nabla \cdot \mathbf{u} = 0, \qquad \dots (1)$$

$$\frac{\rho}{\Phi}\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{\Phi^2}\rho\,(\mathbf{u}\cdot\nabla)\,\mathbf{u} = -\nabla p - g\rho\lambda + \left(\mu + \mu'\frac{\partial}{\partial t}\right)\nabla^2\mathbf{u} - \frac{\mu_0\,\mathbf{u}}{\kappa} \qquad \dots (2)$$

and

$$\frac{\partial \rho}{\partial t} + \frac{1}{\Phi}(\mathbf{u}\cdot\nabla)\rho = 0, \qquad \dots (3)$$

together with the boundary conditions

$$\mathbf{u} = 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad d, \qquad \dots (4)$$

where g is the magnitude of the acceleration due to gravity and $\lambda = (0, 0, 1)$. The dependent variables are the seepage velocity $\mathbf{u}$, the density ρ and the pressure $p \cdot \mu_0$ is the effective viscosity, μ' is the coefficient of visco-elasticity and Φ is the medium porosity.

Basic state solution of eqns. (1)-(4) is given by

$$\mathbf{u} = (U(z), 0, 0),\ \rho = \rho(z) \ \text{and}\ p = Px - \int_0^z g\rho(z)\,dz, \qquad (5)$$

where
$$U(z) = \frac{P\kappa}{\mu_0}\left\{\frac{(1 - e^{(-d\sqrt{\mu_0/\mu\kappa})}\,(e^{\sqrt{\mu_0/\mu\kappa})} + e^{(d\sqrt{\mu_0/\mu\kappa})}\cdot e^{(-z\sqrt{\mu_0/\mu\kappa})}}{(e^{d\sqrt{\mu_0/\mu\kappa})} - e^{(-d\sqrt{\mu_0/\mu\kappa})}} - 1\right\}$$

and P denotes the constant pressure gradient in the flow-direction.

Perturbed State Solution and the Linearized Perturbation Equations

The initial state solution (5) is slightly perturbed so that the unknown perturbed state solution (u, v, w, ρ, p) is given by

$$(u, v, w) = (U(z) + u', 0 + v', 0 + w'), \rho = \rho + \rho' \text{ and } p = p + p',$$

where u', v', w', ρ', p' are the perturbations in u, v, w, ρ and p respectively and are the functions of x, y, z and t.

It is supposed that the unknown perturbed state (u, v, w, ρ, p) also satisfies eqns. (1)-(4). This determines the linearized perturbation equations of motion which are given by

$$\nabla \cdot \mathbf{u}' = 0$$

$$\frac{\rho}{\Phi} L\mathbf{u}' + \frac{\rho}{\Phi^2}(DU)\, w'\nu = \nabla p' - g\rho'\lambda + \left(\mu + \mu' \frac{\partial}{\partial t}\right)\nabla^2 \mathbf{u}' - \frac{\mu_0}{\kappa}\mathbf{u}';$$

and $$L\rho' + \frac{1}{\Phi}(D\rho)w' = 0$$

where $$\nu = (1, 0, 0), D \equiv \frac{d}{dz} \quad \text{and} \quad L = \frac{\partial}{\partial t} + U\frac{\partial}{\partial x}.$$

The boundary conditions are

$$\mathbf{u}' = 0 \quad \text{at} \quad z = 0 \text{ and } d.$$

The Normal Mode Analysis and Eigenvalue Problem

The perturbations (u', v', w', ρ', p') are analysed into wave-like components as

$$(u', v', w', \rho', p')$$

$$= (u(z), v(z), w(z), \rho(z), p(z)) \;.\; \exp\left(k_x \cdot x + k_y \cdot y - \frac{k_x C}{\Phi} \cdot t\right) \quad \text{... (6)}$$

where $k = (k_x^2 + k_y^2)^{1/2}$ is the total wave number and C, in general, is complex. Eliminating various physical quantities from the equations in u, v, w, ρ, p and non-dimensionlysing the resulting equations, using

$$(k^*, k_x^*, D^*) = d(k, k_x, D); (U^*, w^*, C^*) = \frac{1}{U_0}(U, w, C) \text{ and } L^* = \frac{d}{U_0}L,$$

where d is the characteristic length and U_0 is the characteristic velocity, we obtain, after omitting the asterisks, the following non-dimensional equation, namely,

$$k^2\rho(U - C - ik_x^{-1}R_D^{-1})\, w = D[\rho(U - C - ik_x^{-1}R_D^{-1})\, Dw - \rho(DU)\, w]$$

$$+ \rho\,(ik_x^{-1}R_e^{-1} + CR_{ve}^{-1})\,(D^2 - k^2)^2 w + \frac{J_0 k^2 \rho}{k_x^2(U - C)} w, \quad \text{... (7)}$$

where $R_D^{-1} = \frac{\mu_0\, d\Phi^2}{\kappa\, U_0\, \rho}$, $R_e^{-1} = \frac{\mu\, \Phi^2}{U_0\, \rho d}$, $R_{ve}^{-1} = \frac{\mu'\, \Phi}{\rho d^2}$ and $J_0 = \frac{g\Phi^2\, \beta d}{U_0^2}$ with $\beta = -\frac{D\rho}{\rho}$.

The necessary boundary conditions are

$$w = 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad z = 1.$$

Following Squire[13] it is easy to show that the three-dimensional disturbance problem is equivalent to two-dimensional disturbance problem at a reduced permeability, at increased viscosity and increased gravity. We shall therefore confine ourselves to two-dimensional disturbances only for convenience without any loss of generality. Thus for $k_y = 0$ and $v = 0$, eqn. (7) reduces to

$$\begin{aligned} k^2\rho\,(U - C - ik^{-1}R_D^{-1})\,w &= D[\rho(U - C - ik^{-1}R_D^{-1})\,Dw - \rho(DU)w] \\ &\quad + \rho(ik^{-1}R_e^{-1} + CR_{ve}^{-1})\,(D^2 - k^2)^2\, w + \frac{\rho J_0}{(U - C)}\, w. \qquad \text{... (8)} \end{aligned}$$

STATICALLY UNSTABLE SYSTEM ($D\rho > 0$)

In the absence of shear, this top heavy arrangement of fluid ($D\rho > 0 \Rightarrow J_0 < 0$) is statically unstable and the presence of shear further adds to the instability of the system. It is, therefore, of great importance to establish, as has been done in this section, some sufficient conditions of instability and the bounds on the complex wave velocity of unstable modes which are likely to exist in the present case.

Multiplication of eqn. (8) by w^*, the complex conjugate of w and integration of the resulting equation over the vertical range of z by making use of the boundary conditions lead to

$$\begin{aligned} &\int \rho(U - C - iR_D^{-1}k^{-1})\,(|Dw|^2 + k^2|w|^2)\,dz - \int \rho(DU)\,w \cdot Dw^*\,dz \\ &- \int \rho(ik^{-1}R_e^{-1} + CR_{ve}^{-1})\,(|D^2w|^2 + 2k^2|Dw|^2 + k^4|w|^2)\,dz \\ &- \int \frac{J_0\,\rho}{(U - C)}\,|w|^2\,dz = 0. \qquad \text{... (9)} \end{aligned}$$

The real and imaginary parts of eqn. (9) respectively yield

$$\begin{aligned} &\int \rho(U - C_r)\,(|Dw|^2 + k^2|w|^2) + \frac{1}{2}\int D(\rho DU)\,|w|^2\,dz \\ &- C_r R_{ve}^{-1} \int \rho(|D^2w|^2 + 2k^2|Dw|^2 + k^4|w|^2)\,dz \\ &- J_0 \int \frac{\rho(U - C_r)}{(U - C)}\,|w|^2\,dz = 0 \qquad \text{... (10)} \end{aligned}$$

and

$$(C_i + R_D^{-1} k^{-1}) \int \rho(|Dw|^2 + k^2 |w|^2)\, dz - \frac{q}{2k} \int (|Dw|^2 + k^2 |w|^2)\, dz$$

$$+ \; C_i \int \rho R_{ve}^{-1} (|D^2 w|^2 + 2k^2 |Dw|^2 + k^4 |w^2|\, dz$$

$$+ \int \rho R_e^{-1} k^{-1} (|D^2 w|^2 + 2k^2 |Dw|^2 + k^4 |w|^2) dz$$

$$+ \; C_i \int \frac{\rho J_0}{|U - C|^2} |w|^2\, dz \le 0, \qquad \text{... (11)}$$

where $\quad q = \max |\rho DU|$.

That the modes whether stable, neutral or unstable, are oscillatory follows from eqn. (10) if the velocity and density profiles satisfy the conditions that $D(\rho DU) \ge 0$ holds everywhere in the flow domain.

Two Semi-circle Theorems

Let the modes be unstable under the condition $R_D^{-1} > q/2\rho_{\min}$. Then for the consistency of inequality (11), we must necessarily have

$$(C_r - U_1)^2 + C_i^2 < \frac{|J_0|}{k^2 (k^2 R_{ve}^{-1} + 1)} \qquad \text{... (12)}$$

where $U_1 = U(z_1)$ and z_1 is some point in the flow domain. Rewriting inequality (11) as

$$C_i \int \rho(|Dw|^2 + k^2 |w|^2)\, dz$$

$$- \rho_{\min} \int \left[\frac{J_0 C_i}{|U - C|^2} - k R_D^{-1} + \frac{qk}{2\rho_{\min}} - R_e^{-1} k^3 \right] |w|^2\, dz$$

$$+ \; \frac{\rho_{\min}}{k} \left(R_D^{-1} - \frac{q}{2\rho_{\min}} \right) \int |Dw|^2\, dz$$

$$+ \int \rho R_{ve}^{-1} C_i (|D^2 w|^2 + 2k^2 |Dw|^2 + k^4 |w|^2)\, dz$$

$$+ \; \frac{R_e^{-1}}{k} \int \rho\, (|D^2 w|^2 + 2k^2 |w|^2)\, dz \le 0 \qquad \text{... (13)}$$

and arguing as above, another semi-circular region is given by

$$(C_r - U_2)^2 + \left\{ C_i - \frac{|J_0|}{2k \left(R_D^{-1} - \frac{q}{2\rho_{\min}} + k^2 R_e^{-1} \right)} \right\} \le \frac{1}{4k^2} \left\{ \frac{|J_0|}{R_D^{-1} - \frac{q}{2\rho_{\min}} + k^2 R_e^{-1}} \right\}^2, \qquad \text{... (14)}$$

where $U_2 = U(z_2)$ and z_2 is some point in the flow domain.

It is to be noted that the position of the centres of both the semi-circles given by (12) and (14) is not exactly known because of the fact that the points z_1 and z_2 are not exactly known. Therefore, the intersection of the two regions obtained from (12) and (14) by taking envelope, in each case, of all the circles with centres varying from $(U_{\min}, 0)$ to $(U_{\max}, 0)$ and

$$\left(U_{\min}, \frac{|J_0|}{2k\left(R_D^{-1} - \frac{q}{2\rho_{\min}} + R_e^{-1} k^2\right)} \right) \text{ to } \left(U_{\max}, \frac{|J_0|}{2k\left(R_D^{-1} - \frac{q}{2\rho_{\min}} + R_e^{-1} k^2\right)} \right)$$

is the actual region which arrests the complex wave velocity of unstable modes.

A Sufficient Condition for Stability

Clearly, a condition providing an empty intersection of the two regions given by (12) and (14) ensures the non-existence of unstable modes. Such a condition is obtained as

$$\left\{ (U_2 - U_1)^2 - \frac{|J_0|}{k^4 (R_{ve}^{-1} + k^{-2})} \right\}^2 > \frac{|J_0|}{k^4 (R_{ve}^{-1} + k^{-2})} \left\{ \frac{|J_0|}{k\left(R_D^{-1} - \frac{q}{2\rho_{\min}} + R_e^{-1} k^2\right)} \right\}^2 . \quad \ldots (15)$$

The validity of condition (15) and hence the existence of stable modes is ensured if either $(U_2 - U_1)^2 > F$ or $(U_2 - U_1)^2 < G$,

where $$F = \frac{|J_0|}{k^4 (R_{ve}^{-1} + k^{-2})} + \frac{|J_0|}{k\left(R_D^{-1} - \frac{q}{2\rho_{\min}} + R_e^{-1} k^2\right)} \left\{ \frac{|J_0|}{k^4 (R_{ve}^{-1} + k^{-2})} \right\}^{1/2}$$

and $$G = \frac{|J_0|}{k^4 (R_{ve}^{-1} + k^{-2})} - \frac{|J_0|}{k\left(R_D^{-1} - \frac{q}{2\rho_{\min}} + R_e^{-1} k^2\right)} \left\{ \frac{|J_0|}{k^4 (R_{ve}^{-1} + k^{-2})} \right\}^{1/2} .$$

The expressions for F and G and hence the stability of the system depends upon the variations of the density, shear of velocity, Darcy resistance, viscosity, visco-elasticity, magnitude of acceleration due to gravity and the wave number k.

The problem reduces to that of Sharma and Kumar[14] in the absence of shear and to that of Jaimala and Agrawal[1] for a Newtonian fluid in the absence of viscosity. Our analysis confirms their findings, namely, the stabilizing character of viscosity as well as of visco-elasticity and the destabilizing character of permeability and the shear of velocity.

STATICALLY STABLE SYSTEM ($D\rho < 0$)

If $D\rho < 0$ everywhere in the flow domain ($J_0 > 0$), then the density decreases in the vertically upward direction and such an arrangement is known to be statically stable in the absence of shear. The following results can be established as above :

(a) Modes whether stable, neutral or unstable may be non-oscillatory for the wave numbers

$$k^2 < \frac{1}{U^2}\left[J_0 - \frac{1}{2\rho} UD(\rho DU) \right]. \qquad \text{... (16)}$$

Proof follows from equation (10).

(b) An upper bound on C_i for an unstable mode, if exists, is given by

$$C_i < \frac{1}{k}\left\{ \frac{\frac{q}{2\rho_{\min}} - R_D^{-1} - R_e^{-1} k^2}{1 + R_{ve}^{-1} k^2} \right\}. \qquad \text{... (17)}$$

Proof follows from inequality (11).

(c) The inconsistency of inequality (17) for the wave numbers given by

$$k^2 > R_e \left\{ \frac{q}{2\rho_{\min}} - R_D^{-1} \right\} \qquad \text{... (18)}$$

immediately leads to a sufficient condition for stability which provides stability for large wave numbers given by (18).

(d) If the non-oscillatory modes exist in the wave number range given by (16), then an upper bound on C_i is given by

$$C_i < \frac{2\rho_{\max} U_{\max} J_0}{[D(\rho DU)]_{\min}} - U_{\min}^2.$$

The problem investigated by us is an extension of the one investigated by Miles[5] and Jaimala and Agrawal[1] to non-Newtonian shear flow through a porous medium and though some useful results in the form of sufficient conditions for stability and semi-circular bounds have been obtained in the above analysis, the sufficient conditions of the type of Miles[5] and Jaimala and Agrawal[1] are missing which could have provided a clear-cut effect of different physical parameters. This has been successfully achieved in the next section where we have discussed an approximate case.

AN APPROXIMATION

Miles[5], while discussing the small perturbations of shear flow $U(z)$ in an inviscid and incompressible fluid of variable density, established the sufficient conditions for stability, namely $U'(z) \neq 0$ and $J(z) > 1/4$ (where the Richardson number $J(z) = g\beta/U'^2$, $\beta = -D\rho/\rho > 0$) throughout the flow domain. Considering the flow in a porous medium, Jaimala and Agrawal[1] have successfully modified this Richardson criterion showing a clear-cut effect of the porous medium. In order to achieve our above claim, we have, in this section, restricted our discussion to the case when the wave number k is large and the parameters R_e^{-1} and R_{ve}^{-1} are small. These approximations have considerably simplified the mathematical analysis and we could apply the transformation used by Howard[15] in eqn. (8), namely

$$w = W^{3/2} G,$$

where $W = U - C - ik^{-1} R_D^{-1}$.

Equation (8) is then transformed to

$$D(\rho WDG) - \rho k^2 WG - \frac{1}{2} GD(\rho DW) - \frac{1}{4} \rho W^{-1} (DW)^2 G$$

$$+ \rho(ik^{-1} R_e^{-1} + CR_{ve}^{-1}) k^4 G + \frac{\rho J_0}{W + iR_D^{-1} k^{-1}} G = 0. \quad \ldots (19)$$

Associated boundary conditions are

$$G(z) = 0 \quad \text{at} \quad z = 0 \text{ and } z = 1. \quad \ldots (20)$$

Multiply equation (19) by G^*, the complex conjugate of G, integrate over the range of z and make use of the boundary conditions (20). The real and imaginary parts of the resulting equation respectively yield

$$\int \rho(U - C_r)(|DG|^2 + k^2 |G|^2)\, dz + \frac{1}{2} \int D(\rho\, DU) |G|^2 dz$$

$$+ \frac{1}{4} \int \frac{\rho (DU)^2 (U - C_r)}{(U - C_r)^2 + (C_i + R_D^{-1} k^{-1})^2} |G|^2 dz$$

$$- \int \rho k^4 C_r R_{ve}^{-1} |G|^2 dz - \int \frac{\rho (U - C_r) J_0}{(U - C_r)^2 + C_i^2} |G|^2 dz = 0 \quad \ldots (21)$$

and

$$- (C_i + R_D^{-1} k^{-1} \int \rho (|DG|^2 + k^2 |G|^2)\, dz + \frac{1}{4} (C_i + R_D^{-1} k^{-1})$$

$$\int \frac{\rho (DU)^2}{(U - C_r)^2 + (C_i + R_D^{-1} k^{-1})^2} |G|^2 dz$$

$$- \int \rho k^4 (R_e^{-1} k^{-1} + C_i R_{ve}^{-1}) |G|^2 dz - \int \frac{\rho C_i J_0}{(U - C_r)^2 + C_i^2} |G|^2 dz = 0. \quad \ldots (22)$$

Case 1 : Statically Stable System ($D\rho < 0$)

In this section, an attempt has been made to extend the sufficient conditions of Jaimala and Agrawal[1] so as to include the effects of viscosity and visco-elasticity of the fluid.

Sufficient conditions for stability

If U' is nowhere zero in the flow domain, eqn. (22) yields

$$- C_i \int \rho \left(| DG |^2 + k^2 | G |^2\right) dz - R_D^{-1} k^{-1} \int \rho | DG |^2 dz$$

$$+ C_i \int \frac{\rho (DU)^2/4}{(U - C_r)^2 + C_i^2} | G |^2 dz$$

$$- R_D^{-1} k \int \rho | G |^2 dz + \frac{1}{R_D^{-1} k^{-1}} \int \frac{\rho (DU)^2}{4} | G |^2 dz$$

$$- \int \rho k^4 (R_D^{-1} k^{-1} + C_i R_{ve}^{-1}) | G |^2 dz - J_0 C_i \int \frac{\rho}{(U - C_r)^2 + C_i^2} | G |^2 dz > 0. \quad \text{... (23)}$$

Further, let the system be unstable under the conditions $D\rho < 0$ and $U' \neq 0$ in the flow domain, so that $C_i > 0$. Then inequality (23) yields

$$- C_i \int \rho \left(| DG |^2 + k^2 | G |^2\right) dz - R_D^{-1} k^{-1} \int \rho | DG |^2 dz$$

$$- C_i \int \rho k^4 R_{ve}^{-1} | G |^2 dz + C_i \int \frac{\rho (DU)^2 \left(\frac{1}{4} - J\right)}{(U - C_r)^2 + C_i^2} | G |^2 dz$$

$$+ \frac{1}{R_D^{-1} k^{-1}} \int \rho \left(\frac{(DU)^2}{4} - R_D^{-2} - k^2 R_D^{-1} R_e^{-1} \right) | G |^2 dz > 0, \quad \text{... (24)}$$

where $J = \frac{g^* \beta}{(DU)^2}$ and $g^* = \frac{g d \Phi^2}{U_0^2}$.

Inequality (24) becomes mathematically inconsistent if the conditions

$$\text{(a)} \quad J > \frac{1}{4} \quad \text{and} \quad \text{(b)} \quad R_D^{-2} + k^2 R_D^{-1} R_e^{-1} > \frac{U'^2}{4} \quad \text{... (25)}$$

hold everywhere in the flow domain, implying thereby, that C_i cannot be positive and hence the system will be stable under these conditions.

Therefore conditions (25 a, b) are the sufficient conditions for stability obtained in the form we claimed in the beginning of this section.

It follows from these conditions that the situation which was known to be stable for $J > 1/4$ (due to Miles[5]) and which was partly destabilized in the presence of a porous medium (Jaimala and Agrawal[1]) is expected to be stabilized due to the viscosity of the fluid. Instability might be expected if the condition (25 b) is violated though the condition (25 a) might still hold. Below, we obtain the bounds on the complex wave velocity C of unstable modes if exist as explained above. These bounds are also important in view of the fact that these bounds indicate a stabilizing role of visco-elasticity.

Bounds on C_i for unstable modes

If the unstable modes exist under the conditions

$$U' \neq 0, \quad J > \frac{1}{4} \quad \text{and} \quad R_D^{-2} + k^2 R_D^{-1} R_e^{-1} > \frac{U'^2}{4},$$

then it follows from inequality (24) that

$$C_i < \frac{1}{k R_D^{-1}} \left\{ \frac{1}{4} U'^2_{\max} - R_D^{-2} \right\}. \quad \text{... (26)}$$

Another bound on C_i for unstable modes is obtained from eqn. (22) as

$$C_i < \frac{1}{k^3 R_{ve}^{-1} R_D^{-1}} \left\{ \frac{1}{4} U'^2_{\max} - R_D^{-2} - R_e^{-1} R_D^{-1} k^2 \right\}. \quad \text{... (27)}$$

Case 2 : Statically unstable system ($D\rho > 0$)

There is a possibility of the system to become unstable when the density increases in the vertically upward direction. Then eqn. (22) yields

$$- (C_i + R_D^{-1} k^{-1}) \int \rho \, | DG |^2 dz + \frac{1}{C_i} \int \rho \left[\frac{(DU)^2 R_D k C_i}{4} - C_i k R_D^{-1} - C_i k^3 R_e^{-1} - k^4 C_i^2 R_{ve}^{-1} - C_i^2 k^2 + | J_0 | \right] | G |^2 > 0. \quad \text{... (28)}$$

The consistency of this equation requires that

$$\frac{1}{4} (DU)^2 R_D k C_i - k C_i R_D^{-1} - k^3 C_i R_e^{-1} - k^4 C_i^2 R_{ve}^{-1} + C_i^2 k^2 + | J_0 | > 0 \quad \text{... (29)}$$

somewhere in the flow domain.

The required bounds on C_i are obtained as

$$C_i < \frac{\left\{ R_D (DU)^2_{\max} - 4R_D^{-1} - 4k^2 R_e^{-1} \right\} \left[\left\{ R_D (DU)^2_{\max} - 4R_D^{-1} - 4k^2 R_e^{-1} \right\}^2 + 64 | J_0 | (1 + k^2 R_{ve}^{-1}) \right]^{1/2}}{8k (1 + k^2 R_{ve}^{-1})}.$$

As claimed earlier, the analysis in this section clearly establishes a stabilizing character of R_D^{-1}, R_e^{-1} and R_{ve}^{-1}.

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