MINIMAL QUASI-IDEALS IN TERNARY SEMIGROUP

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In this paper we have studied the structure of quasi-ideals in ternary semigroup without 0. We have proved that a minimal quasi-ideal $Q$ of ternary semigroup $T$, is written in the form $[eTeTe]$ where $e$ is the identity of $Q$ and if a ternary semigroup $T$ (without 0) has at least one minimal quasi-ideal then the Union of minimal quasi-ideals of $T$ is the Kernel of $T$.

1. INTRODUCTION

Definition 1.1 (Lehmer*) — A non-empty set $T$ is called a ternary semigroup if a ternary operation $[\ ]$ on $T$ is defined and satisfies the associative law

$$[[x_1 x_2 x_3] x_4 x_5] = [x_1 [x_2 x_3 x_4] x_5] = [x_1 x_2 [x_3 x_4 x_5]] = [x_1 x_2 x_3 x_4 x_5]$$

for all $x_i \in T$, $1 \leq i \leq 5$.

Sioson$^2$ gives the following definitions of ideals.

Definition 1.2 — A left (right, lateral) ideal of ternary semigroup $T$ is non-empty subset $L (R, M)$ of $T$ such that

$$[TTL] \subseteq L ([RTT] \subseteq R [TMT] \subseteq M).$$

Definition 1.3 — For each elements $t$ in $T$, the left, right and lateral ideal generated by $t$ are respectively given by:

$$(t)_L = \{t\} \cup [TTt]$$

$$(t)_R = \{t\} \cup [tTT]$$

$$(t)_M = \{t\} \cup [TtT] \cup [TTtT].$$

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Definition 1.4 — A non-empty subset $Q$ of a ternary semigroup $T$ is said to be a quasi-ideal of $T$ if

$$[TTQ] \cap ([TQT] \cup [TTQTT]) \cap [QTT] \subseteq Q.$$  

Due to the associative law in $T$, one may write Sioson\(^3\)

$$[x_1, x_2, \ldots, x_n] = [x_1 \ldots x_m x_{m-1} \ldots x_n] \ (m \leq n)$$

$$= [x_1 \ldots [x_{m-2} x_{m-1} x_m x_{m-3} x_{m-4}] \ldots x_n] \ (m \leq n).$$

Definition 1.5 (Lehmer\(^1\)) — A ternary semi-group $(G, \{\_\})$ is said to be a ternary group if it has the additional property that for all $a, b, c$ in $G$ there exist unique $x, y, z$ in $G$ such that


Definition 1.6 — $e$ is said to be the identity of ternary group $G$ if for all $a$ in $G$, there exists unique $e$ in $G$ such that


Definition 1.7 — If for $a$ in $G$, there exists unique $x$ in $G$ such that

$$[xaa] = e, [axa] = e, [aax] = e$$

then $x$ is called the inverse of $a$ in $G$.

2. Minimal Quasi-Ideals of Ternary Semigroup (Without 0)

Definition 2.1 — A ternary semigroup $T$ is said to be without 0 if it does not contain the ‘0’ element which has the following property:

$$\forall \ a, b \in T$$

$$[0 \ a \ b] = 0 = [a0b] = [ab0].$$

In this paper we are concerning the ternary semigroup without 0 and it has at least one idempotent element. Henceforth, every ternary semigroup without 0 containing the idempotent element, would be denoted by $T$.

Definition 2.2 — An ideal (left, right, lateral, quasi- or bi-) of $T$ is said to be minimal if it does not properly contain an ideal (left, right, lateral, quasi-or bi-) of $T$.

Remark 2.3 : If $e$ is an idempotent element of $T$, then

(a) $[eTT]$ is a minimal right ideal of $T$ containing $e$,

(b) $[TeT]$ is a minimal lateral ideal of $T$ containing $e$,

(c) $[TTe]$ is a minimal left ideal of $T$ containing $e$.

Remark 2.4 : If $e$ an idempotent element of $T$, then
\[ eTeTe = [eTT] \cap [TeT] \cap [TTe]. \]

**Proposition 2.5** — A quasi-ideal \( Q \) of \( T \) is minimal if and only if it is generated by any of its elements.

The proof is trivial.

**Proposition 2.6** — A quasi-ideal \( Q \) of \( T \) is minimal if and only if \( Q \) is a ternary subgroup of \( T \).

The proof is trivial.

**Proposition 2.7** — Every minimal quasi-ideal \( Q \) of \( T \) can be written in the form \( Q = [eTeTe] \) where \( e \) is the identity of \( Q \).

**Proof:** A quasi-ideal \( Q \) of \( T \) is minimal if and only if it is the intersection of minimal right, lateral and left ideal of \( T \) (Sioson\(^3\)). Remarks 2.3 and 2.4 proves the required result.

**Example 2.8** — The following example shows that a minimal quasi-ideal \( Q \) of \( T \) has an identity and \( Q \neq T \).

**Proof:** Let \( T = \{2x | x \in N\} \) be the ternary semigroup under the given operation \( \forall a, b, c \in T \).

\[ [abc] = \text{H.C.F. of } a, b \text{ and } c. \]

Then \( Q = \{2\} \) is the minimal quasi-ideal of \( T \).

Moreover it is a ternary subgroup of \( T \) and 2 is the identity of \( Q \). Also \( Q \neq T \).

**Proposition 2.9** — Let \( e \) be an idempotent element contained in a minimal left ideal \( L \) (minimal lateral ideal \( M \) and minimal right ideal \( R \)) of a ternary semigroup \( T \) then \([eeL]\) \([eeMee]\) \([Ree]\) is a ternary subgroup, moreover it is a minimal quasi-ideal of \( T \).

**Proof:** From Sioson\(^3\) we see that \([eeL]\), \([eeMee]\) and \([Ree]\) are quasi-ideals of \( T \). By hypothesis, \([eeL]\) is a ternary sub-semigroup of \( T \). But \([Leeheeh]\) is the left ideal of \( T \) for some \( h \) in \( L \).

\[ [Leeheeh] \subseteq L. \]

Since \( L \) is minimal, therefore \([Leeheeh]\) = \( L \). Now

\[ [ee(Leeheeh)] = [eeL]. \]

So there exists \( k \) in \( L \) such that

\[ [(eek) [eeh] [eeh]] = [eee] = e. \]

Hence \([eeL]\) is a ternary subgroup and therefore is a minimal quasi-ideal of \( T \). Similarly \([Ree]\) and \([eeMee]\) are the ternary subgroups of \( T \).

**Proposition 2.10** — Let \( Q \) be a minimal quasi-ideal of the ternary semigroup \( T \) then \([eQs]\) and \([sQe]\) are minimal quasi-ideals of \( T \) where \( s \) is any element of \( T \) and \( e \) is the identity element of \( Q \).
PROOF : (2.7) gives us that
\[ [eQs] = [e[TeTe]s] = [eeTeTes] \]

\([TeTes]\) is the left ideal of \(T\). Sioson\(^3\) proved that \([eeTeTes]\) is the quasi-ideal of \(T\). \(Q\) being a minimal quasi-ideal of \(T\) generated by any of its elements implies that \([eQs]\) is generated by any of its elements.

Thus by Proposition 2.5 this is a minimal quasi-ideal of \(T\). Moreover it is a ternary subgroup of \(T\). In the similar manner \([sQe]\) is a minimal quasi-ideal of \(T\).

**Proposition 2.11** — Let \(Q_1, Q_2\) and \(Q_3\) be minimal quasi-ideals of \(T\), with their identities \(e_1, e_2\) and \(e_3\) respectively. Then \([e_1Q_1Q_2Q_3e_3]\) is a minimal quasi-ideal of \(T\) such that \([e_1Q_1Q_2Q_3e_3] = [e_1TT] \cap [Te_2T] \cap [TTe_3]\).

**PROOF :** Using Proposition 2.7 we get
\[
[e_1Q_1Q_2Q_3e_3] = [e_1[e_1Te_1Te_2] [e_2Te_2Te_3] [e_3Te_3Te_3] e_3]
\[
\subseteq [e_1TT] \cap [Te_2T] \cap [TTe_3].
\]

Clearly \([e_1TT], [Te_2T]\) and \([TTe_3]\) are the minimal right, lateral and left ideal of \(T\).

So by Sioson\(^3\) \([e_1TT] \cap [Te_2T] \cap [TTe_3]\) is the minimal quasi-ideal of \(T\). Also \([Te_1Te_2Te_2Te_2Te_3Te_3] \subseteq [e_1Q_1Q_2Q_3e_3]\) is the left ideal of \(T\). Again Sioson\(^3\) proved that
\[
[e_1e_1 [Te_1e_1e_2e_2e_3e_3] = [e_1Q_1Q_2Q_3e_3]
\]
is the quasi-ideal of \(T\). Therefore, minimality implies that
\[
[e_1Q_1Q_2Q_3e_3] = [e_1TT] \cap [Te_2T] \cap [TTe_3].
\]

**Proposition 2.12** — Let \(Q_1\) and \(Q_2\) be two minimal quasi-ideals of \(T\) with the identities \(e_1\) and \(e_2\) respectively. Then

(i) \([e_1Te_1Te_2Te_2Te_1e_1]\) = \(Q_1\)

(ii) \([e_2Te_2Te_1Te_1e_1e_2e_2]\) = \(Q_2\).

**PROOF :** Since \([e_1Te_1Te_2Te_2Te_2Te_3]\) is the right ideal of \(T\). Therefore by Sioson\(^3\) \([e_1Te_1Te_2Te_2Te_1e_1]\) is the quasi-ideal of \(T\) where \(e_1\) and \(e_2\) are the idempotents of \(T\). But \([e_1Te_1[Te_2T] [e_2Te_2] [e_2Te_1] e_1]\) \(\subseteq [e_1Te_1Te_1] = Q\) (by Proposition 2.7). Minimality implies that
\[
[e_1Te_1Te_2Te_2Te_2Te_1e_1] = Q_1.
\]

Similarly, \([e_2Te_2Te_1Te_1e_1Te_2e_2] = Q_2\).

Now we give some simple but important results which we will use later.

(A) Since \(e_1\) is the identity of \(Q_1\), therefore there exists \(s_1, s_2, t_1, t_2\) and \(t_3\) in \(T\) such that
\[
[e_1e_1[e_1s_1 e_1] e_1 [e_1s_2 e_2] e_1 [e_1s_3 e_3] e_1] = e_1
\]
i.e., \([e_1e_1s_1e_1s_2e_2e_2e_1e_2e_2e_1] = e_1\)
where, \(s_1 = [e_1s_1e_1], s_2 = [e_1s_2e_2], t_1 = [t_1e_2e_1e_2e_1], t_2 = [e_2t_2s_1e_2e_2e_2] \in T.\)

(B) Similarly, there exists \(t_1, t_2, s_1\) and \(s_2\) in \(T\) such that
\([e_2e_2e_1e_2e_1e_2s_1e_1s_2e_2e_2] = e_2.\)

(C) Let \(e\) be the identity of \(T.\) Then for all \(t\) in \(T,\) we have
\([ett] = t = [tet] = [tte].\)

This implies that
\([ete] = [eet] = [tte] = t.\)

**Proposition 2.13** — All the minimal quasi-ideals of ternary semigroup \(T\) are isomorphic.

**Proof:** Let \(Q_1\) and \(Q_2\) be two minimal quasi-ideals of \(T.\) Define a map \(\psi: Q_1 \to Q_2\) as
\[\psi ([e_1x_1e_1x_2e_1]) = [e_2t_1e_2t_2e_1x_1e_1x_2e_1e_1s_1e_1s_2e_2e_2], \ \forall \ [e_1x_1e_1x_2e_1] \in Q_1\]
where \(x_1, x_2, t_1, t_2, s_1, s_2 \in T, e_1\) and \(e_2\) are the identities of \(Q_1\) and \(Q_2\) respectively.

\(\psi\) is well-defined.

By Proposition 2.12, \([e_2t_1e_2t_2e_1x_1e_1x_2e_1e_1s_1e_1s_2e_2e_2] \in Q_2.\)

By result (A) there exists \(t_1, t_2, s_1, s_2 \in T\) such that
\([e_1e_1s_1e_1s_2e_2e_2t_1e_2e_1] = e_1.\)

Now \(\psi ([e_1x_1e_1x_2e_1]) = \psi ([e_1y_1e_1y_2e_1])\)
\[\Rightarrow [e_2t_1e_2t_2e_1x_1e_1x_2e_1e_1s_1e_1s_2e_2e_2] = [e_2t_1e_2t_2e_1y_1e_1y_2e_1e_1s_1e_1s_2e_2e_2].\]

Applying \([e_1e_1s_1],\) \([e_1s_2e_2]\) on the left-hand side and \([t_1e_2t_2],\) \(e_1\) on the right-hand side under the ternary operation on both sides of the above equation we get
\[[[e_1e_1s_1] [e_1s_2e_2] [e_2t_1e_2t_2e_1x_1e_1x_2e_1e_1s_1e_1s_2e_2e_2] [t_1e_2t_2] e_1] = [[e_1e_1s_1] [e_1s_2e_2] [e_2t_1e_2t_2e_1y_1e_1y_2e_1e_1s_1e_1s_2e_2e_2] [t_1e_2t_2] e_1] \]
\[\Rightarrow [e_1x_1e_1x_2e_1] = [e_1y_1e_1y_2e_1].\]

Consider
\[\psi ([[e_1x_1e_1x_2e_1] [e_1y_1e_1y_2e_1] [e_1z_1e_1z_2e_1]])\), using \(e_1\) and \(e_2\) as the identity of \(Q_1\) and \(Q_2\) respectively.

By Proposition 2.12 and result (A) we get
\[\psi ([[e_1x_1e_1x_2e_1] [e_1y_1e_1y_2e_1] [e_1z_1e_1z_2e_1]]) = \psi ([[e_2t_1e_2t_2e_1x_1e_1x_2e_1e_1s_1e_1s_2e_2e_2] [e_2t_1e_2t_2e_1y_1e_1y_2e_1e_1s_1e_1s_2e_2e_2] [e_2t_1e_2t_2e_1z_1e_1z_2e_1e_1s_1e_1s_2e_2e_2]]) = [\psi ([[e_1x_1e_1x_2e_1]]) \psi ([[e_1y_1e_1y_2e_1]]) \psi ([[e_1z_1e_1z_2e_1]])].\]
Thus $\psi$ is a homomorphism.

Now, we define a map $\phi: Q_2 \to Q_1$ as follows:

$$\phi([e_2x_1e_2x_2e_2]) = [e_1s_1s_2e_2x_2e_2t_1e_2t_2e_1].$$

Then

$$\psi \circ \phi = I_{Q_1},$$

$$\phi \circ \psi = I_{Q_1}.$$

Hence the required result is proved.

**Proposition 2.14** — In a ternary semigroup $T$, the following are equivalent:

(i) $T$ has at least one minimal quasi-ideal,

(ii) $T$ has at least one minimal right, one minimal lateral and one minimal left ideal,

(iii) $T$ has at least one minimal right ideal and every right ideal of $T$ contains an idempotent element,

(iv) $T$ has at least one minimal lateral ideal and every lateral ideal of $T$ contains an idempotent element,

(v) $T$ has at least one minimal left ideal and every left ideal of $T$ contains an idempotent element.

**Proof:** (i) $\Rightarrow$ (II), (II) $\Rightarrow$ (III), (II) $\Rightarrow$ (IV) and (ii) $\Rightarrow$ (v) follows from (Sioson\textsuperscript{3}) Proposition 2.7. Further (iii) of (iv) of (v) implies (i) by Proposition 2.9.

**Example 2.15** — $T = \{e, a, b\}$ is a ternary semigroup under the following table.

<table>
<thead>
<tr>
<th></th>
<th>e</th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>e</td>
<td>e</td>
<td>e</td>
<td>e</td>
</tr>
<tr>
<td>a</td>
<td>e</td>
<td>e</td>
<td>b</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>b</td>
<td>b</td>
</tr>
</tbody>
</table>

For all $a, b, c$ in $T$ $[abc] = (a(bc)) = ((ab)c)$.

Then $\{e, b\}$ is a left ideal but $\{e\}$ and $\{b\}$ are two minimal disjoint quasi-ideals of $T$. Thus $\{e, b\} = \{e\} \cup \{b\}$.

**Definition 2.16** — A ternary semigroup $T$ is simple if there exists no proper three-sided ideal of $T$.

**Example 2.17** — Let $T = \{\pm 1, \pm i\}$ be a ternary semigroup under complex multiplication. Then $T$ is a simple ternary semigroup.

**Proposition 2.18** — Let $t_1, t_2$ be any two elements of a ternary semigroup $T$ and let $Q$ be a minimal quasi-ideal of $T$. Then $[t_1t_2Q], [t_1Qt_2]$ and $[Qt_1t_2]$ are minimal quasi-ideals of $T$. 
Proof : Let \( e \) be the identity of \( Q \), then \([t_1t_2Q] = [t_1t_2eeTeTee]\) which by Sioson\(^3\) is a quasi-ideal of \( T \). But \([t_1t_2eeTeTee]\) is a ternary subgroup of \( T \). Thus \([t_1t_2eeTeTee]\) is a minimal quasi-ideal of \( T \). Similarly, \([Qt_1t_2]\) is a minimal quasi-ideal of \( T \).

Also, \([t_1Qt_2]\) = \([t_1eeQeet_2]\) = \([t_1e[eQ[eeet_2]\)]\).

But by Proposition 2.9 \([eQeet_2]\) = \( Q_2 \) is minimal quasi-ideal of \( T \). So

\[
[t_1Qt_2] = [t_1eQ_2]
= [t_1e[e_2Q_2e_2]]
\]

where \( e_2 \) is the identity of \( Q_2 \). Hence \([t_1Qt_2]\) is a minimal quasi-ideal of \( T \) by Proposition 2.9.

Proposition 2.19 — Let \( A \) be a right as well as a left ideal of \( T \). Then \( A \) is a lateral ideal of \( T \). The proof is trivial.

The lateral ideal may not be a right as well as a left ideal of \( T \). This is shown by the following example.

Example 2.20 — \( T = \{e, a, b, c\} \) is a ternary semigroup under the following table

\[
\begin{array}{c|cccc}
(\ ) & e & a & b & c \\
\hline 
e & e & b & b & c \\
a & b & c & c & c \\
b & b & c & c & c \\
c & c & c & c & c \\
\end{array}
\]

For all \( a, b, c \) in \( T \), \([abc] = (a(bc)) = ((ab)c)\).

Then \( \{a, c\} \) is a lateral ideal but this is not a right as well as a left ideal of \( T \).

Proposition 2.21 — A minimal lateral ideal is a minimal quasi-ideal of \( T \).

Proof : By Sioson\(^3\) a minimal lateral ideal is an ideal of \( T \) implies that \([MMM] = M \). Hence by Proposition 2.6, \( M \) is a minimal quasi-ideal of \( T \).

Definition 2.22 — \( K \) is said to be a kernel of a ternary semigroup \( T \) if \( K \) is the intersection of all three sided ideals of \( T \).

If the intersection of all three-sided ideals of \( T \) is empty then \( T \) does not have a kernel.

Proposition 2.23 — If the ternary semigroup \( T \) has at least one minimal quasi-ideal, then the union of all minimal quasi-ideals of \( T \) is the kernel of \( T \); moreover it is a simple subsemigroup of \( T \).
PROOF: Let $K = \bigcup_{Y \in \Gamma} \{Q_Y \mid \forall Y \in \Gamma, Q_Y \text{ is a minimal quasi-ideal of } T\}$. By (2.18) for any $t_1, t_2$ in $T$ and for any minimal quasi-ideal $Q_Y, Y \in \Gamma$ we have $[t_1 t_2 Q_Y], [t_1 Q_Y t_2], [Q_Y t_1 t_2], Y \in \Gamma$ are the minimal quasi-ideals of $T$ and therefore,

$$[TTK], [TKT], [KTT] \subseteq K.$$ 

Thus $K$ is an ideal of $T$. Clearly $K$ is minimal. Hence $K$ is the kernel of $T$. Moreover $[KAKA]K$ is an ideal of $T$ where $A$ is an ideal of $K$.

Thus minimality of $K$ implies that $[KA KA K] = K$.

Now, $K = [KAKA] \subseteq [KAA] \subseteq A$

i.e., $K = A$.

Thus $K$ is a simple sub-semigroup of $T$.

Definition 2.24 — A non-zero idempotent element $e$ of a ternary semigroup $T$ is called primitive if for any non-zero idempotent $f$ of $T$, the relation

$$[eff] = [fe] = [ffe] = f \text{ implies } e = f.$$ 

Proposition 2.25 — The identity element $e$ of a minimal quasi-ideal $Q$ of a ternary semigroup $T$ is a primitive idempotent of $T$.

The proof is trivial.

Definition 2.26 — A simple ternary semigroup is called a completely simple ternary semigroup if it contains at least one primitive idempotent.

Let $T$ has at least one minimal quasi-ideal, so by Proposition 2.23 kernel $K$ of $T$ is simple ternary sub-semigroup of $T$. Hence we get the following result.

Proposition 2.27 — If the ternary semigroup $T$ has a minimal quasi-ideal then its kernel is completely simple.

Remark 2.28: Since by Proposition 3.15 of Dixit and Sarita\textsuperscript{8} a subset $Q$ of a regular ternary semigroup is a minimal quasi-ideal if and only if it is a minimal bi-ideal of it where a bi-ideal $B$ is minimal if $B = [BTB]T$.

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