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OSCILLATION CRITERIA FOR SECOND ORDER SUPERLINEAR DIFFERENTIAL EQUATIONS

WANTONG LI[1] AND JURANG YAN[2]

[1]*Department of Mathematics, Zhangye Teacher's College, Zhangye 734000, Gansu, People's Republic of China*

[2]*Department of Mathematics, Shanxi University, Taiyuan, Shanxi, 030006, People's Republic of China*

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A new oscillation criterion is given for general superlinear ordinary differential equations of second order of the form $x''(t) + a(t)f(x(t)) = 0$, where $a(t) \in C[t_0, \infty)$, $f(x) \in C(R)$ and $xf(x) > 0$, $f'(x) \geq 0$ for $x \neq 0$. Furthermore, $f(x)$ also satisfies superlinear condition, which covers the prototype nonlinear function $f(x) = |x|^\gamma \operatorname{sgn} x$ with $\gamma > 1$ known as the Emden-Fowler case. The coefficient $a(t)$ is not assumed to be eventually nonnegative. The oscillation criterion involving integral averages of $a(t)$ give a positive answer to the question given by Wong (Oscillation criterion for second order nonlinear differential equations involving integral averages, *Canad. J. Math.* **45** (1993), 1094-1103).

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

Consider the second order nonlinear ordinary differential equation

$$x''(t) + a(t)f(x(t)) = 0, \qquad \ldots (1)$$

where $a(t)$ is a continuous real-valued function on an interval $[t_0, \infty)$ without any restriction on its sign, and f is a continuous real-valued function on the real line R which is continuously differentiable on $R - \{0\}$ and satisfies $xf(x) > 0$ and $f'(x) \geq 0$ for every $x \neq 0$. The prototype of eqn. (1) is the so-called Emden-Fowler equation

$$x''(t) + a(t)|x(t)|^\gamma \operatorname{sgn} x(t) = 0, \quad \gamma > 0. \qquad \ldots (2)$$

Here we are interested in the oscillation of solutions of (1) when $f(x)$ satisfies, in addition, the following superlinear condition

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$$0 < \int_{\varepsilon}^{\infty} \frac{dy}{f(y)}, \int_{-\varepsilon}^{-\infty} \frac{dy}{f(y)} < \infty \quad \text{for all } \varepsilon > 0. \qquad \ldots (3)$$

Throughout the paper, we shall restrict our attention only to the solutions of the differential equation (1) which exist on some ray $[T, \infty)$, where $T \geq t_0$ may depend on the particular solution. Note that under quite general conditions there will always exist solutions of (1) which are continuable to an interval $[T, \infty)$, $T \geq t_0$, even though there will also exist noncontinuable solutions[1]. Such a solution is said to be oscillatory if it has arbitrary large zeros, and otherwise it is said to be nonoscillatory. Equation (1) is called oscillatory if all its solutions are oscillatory.

In the linear case, i.e. eqn. (1) when $\gamma = 1$, Kamenev[2] showed that the condition

$$\limsup_{t \to \infty} \frac{1}{t^{\alpha}} \int_{t_0}^{t} (t-s)^{\alpha} a(s)\, ds = +\infty \qquad \ldots (4)$$

for some $\alpha > 1$ alone is sufficient for oscillation of eqn. (2). Wong[3] extended this result to eqn. (2) in the sublinear case ($0 < \gamma < 1$).

Very recently, Wong[4] conjectured that whether (4) plus the following compatible condition

$$\liminf_{t \to \infty} \frac{1}{t^{\beta}} \int_{t_0}^{t} (t-s)^{\beta} a(s)\, ds > -\infty \qquad \ldots (5)$$

for some $\beta \geq 1$ is sufficient for oscillation of eqn. (2) in the superlinear case ($\gamma > 1$). The special case when $\beta = 0$ in (5) has been answered in the affirmative in an earlier paper (Wong[5]). In addition, Wong[6] proved the conjecture in the case when conditions (4) and (5) hold for $\alpha > 1$, $\beta \geq 1$ where α, β are positive integers. But it remains open problem when $\alpha > 1$, $\beta \geq 1$ are positive real numbers.

The purpose of this paper is to prove above conjecture and to extend it to the general superlinear differential equation (1), subject to nonlinear condition[3]

$$f'(x)\, G(x) \geq d > 1, \text{ for all } x, \qquad \ldots (6)$$

where $G(x) = \int_{x}^{\infty} \frac{dy}{f(y)}$. Our main result is the following theorem.

Theorem 1 — *Assume that (3) and (6) hold. Then condition (4) and (5) imply that eqn. (1) is oscillatory.*

For eqn. (2) with $\gamma > 1$, condition (6) becomes

$$f'(x)\, G(x) \geq \gamma(\gamma - 1)^{-1} = d > 1.$$

Hence, our Theorem leads to the following oscillation result.

Corollary — Equation (2) with $\gamma > 1$ is oscillatory if (4) and (5) are satisfied. This oscillation criterion is Wong's conjecture.

2. Proof of the Theorem

Assume that the differential equation (1) admits a nonoscillatory solution $x(t)$ on an interval $[T, \infty)$, $T \geq \max\{t_0, 1\}$. Without loss of generality, this solution can be supposed to be such that $x(t) \neq 0$ for $t \geq T$. Furthermore, we observe that the substitution $u = -x$ transform (1) into the equation $u''(t) + a(t)\bar{f}(u(t)) = 0$, where $\bar{f}(y) = -f(-y)$, $y \in R$. The function $\bar{f}$ is subject to the same conditions as f. So there is no loss of generality to restrict our discussion to the case, where the solution $x(t)$ is positive on the interval $[T, \infty)$. Define $w(t) = G(x(t))$. Then we obtain

$$w''(t) = a(t) + (w'(t))^2 f'(x(t)), \quad \text{for} \quad t \geq T. \qquad \ldots (7)$$

Since $f'(x) \geq 0$, we note that the integral of $f'(x(t))\, w'^2(t)$ over $[T, \infty)$ exists, finite or infinite. Hence we consider two case where $\int_T^\infty (w'(t))^2 f'(x(t))\, dt$ is finite or infinite.

First of all we consider the case $\int_T^\infty (w'(t))^2 f'(x(t))\, dt < \infty$. Here the proof given in Wong[6] is also valid for positive reals α, β, and hence for brevity is not reproduced.

Second we consider the case $\int_T^\infty (w'(s))^2 f'(x(s))\, ds = \infty$. In view of (6), we can choose a real number $k \geq \max\{\beta, 3\}$ such that

$$\frac{k-1}{k-2} < d. \qquad \ldots (8)$$

As $k \geq \beta$, it is easy to verify that condition (5) implies that

$$\liminf_{t \to \infty} \frac{1}{t^{k-1}} \int_T^t (t-s)^{k-1} a(s)\, ds > -\infty. \qquad \ldots (9)$$

Now, for each $t \geq T$, from (7) we obtain

$$\int_T^t (t-s)^{k-1} a(s)\, ds + \int_T^t (t-s)^{k-1} (w'(s))^2 f'(x(s))\, ds$$

$$= \int_T^t (t-s)^{k-1} w''(s)\, ds$$

$$= -(t-T)^{k-1} w'(T) + (k-1) \int_T^t (t-s)^{k-2} w'(s)\, ds.$$

So, we have

$$\frac{1}{t^{k-1}} \int_T^t (t-s)^{k-1} a(s)\, ds$$

$$= -\left(1-\frac{T}{t}\right)^{k-1} w'(T) + \frac{k-1}{t^{k-1}} \int_T^t (t-s)^{k-2} w'(s)\, ds$$

$$- \frac{1}{t^{k-1}} \int_T^t (t-s)^{k-1} (w'(s))^2 f'(x(s))^2\, ds, \quad \text{for} \quad t \geq T. \qquad \ldots (10)$$

Next, by taking into account (8), we choose a constant d_1 with

$$\frac{k-1}{d(k-2)} < d_1 < 1. \qquad \ldots (11)$$

We claim that, for every $t^* \geq T$, there exists a $t \geq t^*$ such that

$$\int_T^t (t-s)^{k-1} (w'(s))^2 f'(x(s))\, ds > \frac{k-1}{d_1} \int_T^t (t-s)^{k-2} w'(s)\, ds.$$

Otherwise, there is a $t^* \geq T$ such that

$$\int_T^t (t-s)^{k-1} (w'(s))^2 f'(x(s))\, ds$$

$$\leq \frac{k-1}{d_1} \int_T^t (t-s)^{k-2} w'(s)\, ds, \text{ for all } t \geq t^*. \qquad \ldots (12)$$

By using the Schwarz inequality and (12) and taking into account the definition of w, and d, for $t \geq t^*$ we obtain

$$0 \leq \int_T^t (t-s)^{k-2} w'(s)\, ds$$

$$\leq \left\{ \int_T^t (t-s)^{k-1} (w'(s))^2 f'(x(s))\, ds \right\}^{1/2} \left\{ \int_T^t \frac{(t-s)^{k-3}}{f'(x(s))}\, ds \right\}^{1/2}$$

$$\leq \left\{ \frac{k-1}{d_1} \int_T^t (t-s)^{k-2} w'(s)\, ds \right\}^{1/2} \left\{ \frac{1}{d} \int_T^t (t-s)^{k-3} w(s)\, ds \right\}^{1/2}$$

and so, we obtain

$$\int_T^t (t-s)^{k-2} w'(s)\, ds \leq \frac{k-1}{d_1 d} \int_T^t (t-s)^{k-3} w(s)\, ds, \; t \geq t^*. \qquad \ldots (13)$$

But, for any $t \geq t^*$ we have

$$\int_T^t (t-s)^{k-3} w(s)\, ds = \frac{1}{k-2}\left\{ (t-T)^{k-2} w(T) + \int_T^t (t-s)^{k-2} w'(s)\, ds \right\}.$$

So, (13) gives

$$\left(d_1 d \frac{k-2}{k-1} - 1 \right) \int_T^t (t-s)^{k-2} w'(s)\, ds \leq (t-T)^{k-2} w(T)$$

for every $t \geq t^*$. Therefore, by (12), we find

$$\left(d_1 d \frac{k-2}{k-1} - 1 \right) \frac{1}{t^{k-2}} \int_T^t (t-s)^{k-1} (w'(s))^2 f'(x(s))\, ds$$

$$\leq \frac{k-1}{d_1} \left(1 - \frac{T}{t} \right)^{k-2} w(T), \quad \text{for } t \geq t^*. \qquad \ldots (14)$$

But, one has

$$\int_T^\infty (w'(s))^2 f'(x(s))\, ds = \infty$$

$$\Rightarrow \lim_{t \to \infty} \frac{1}{t^{k-2}} \int_T^t (t-s)^{k-1} (w'(s))^2 f'(x(s))\, ds = \infty.$$

On the other hand, (11) implies that $d_1 d(k-2)/(k-1) - 1 > 0$. Thus, (14) leads to a contradiction. Hence our claim is proved and so we can consider a sequence $\{t_n\}$ $n = 1, 2, \ldots$ of points in the interval $[T, \infty)$ with $\lim_{n \to \infty} t_n = \infty$ and such that

$$\int_T^{t_n} (t_n - s)^{k-1} (w'(s))^2 f'(x(s))\, ds$$

$$> \frac{k-1}{d_1} \int_T^{t_n} (t_n - s)^{k-2} w'(s)\, ds, \quad n = 1, 2, \ldots$$

Then from (10) it follows that

$$\frac{1}{t_n^{k-1}} \int_T^{t_n} (t_n - s)^{k-1} a(s)\, ds < -\left(1 - \frac{T}{t_n} \right)^{k-1} w'(T)$$

$$+ (d_1 - 1) \frac{1}{t_n^{k-1}} \int_T^{t_n} (t_n - s)^{k-1} (w'(s))^2 f'(x(s))\, ds, \quad n = 1, 2, \ldots .$$

$$\ldots (15)$$

Since

$$\lim_{t\to\infty} \frac{1}{t^{k-1}} \int_T^t (t-s)^{k-1} (w'(s))^2 f'(x(s))\, ds$$

$$= \int_T^\infty (w'(s))^2 f'(x(s))\, ds = \infty$$

and, by (11), the constant $d_1 < 1$, inequality (15) ensures that

$$\lim_{n\to\infty} \frac{1}{t_n^{k-1}} \int_T^{t_n} (t_n - s)^{k-1} a(s)\, ds = -\infty.$$

This show that

$$\liminf_{t\to\infty} \frac{1}{t^{k-1}} \int_T^t (t-s)^{k-1} a(s)\, ds = -\infty. \qquad \ldots (16)$$

Finally, for every $t \geq T$, we obtain

$$\int_{t_0}^t (t-s)^{k-1} a(s)\, ds \leq \int_{t_0}^T (t-s)^{k-1} |a(s)|\, ds + \int_T^t (t-s)^{k-1} a(s)\, ds$$

$$\leq (t-t_0)^{k-1} \int_{t_0}^T |a(s)|\, ds + \int_T^t (t-s)^{k-1} a(s)\, ds.$$

Thus, from (16) it follows that

$$\liminf_{t\to\infty} \frac{1}{t^{k-1}} \int_{t_0}^t (t-s)^{k-1} a(s)\, ds = -\infty,$$

which contradicts (5). The proof of theorem is now complete.

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REFERENCES

1. G. J. Butler, *Canad. J. Math.* **29** (1977), 472-79.
2. I. V. Kamenev, *Math. Zamethi* **23** (1978), 249-51.
3. J. S. W. Wong, *Canad. Math. Soc.* **8** (1987), 299-302.
4. J. S. W. Wong, *Canad. J. Math.* **45** (1993), 1094-1103.
5. J. S. W. Wong, *Proc. Am. Math. Soc.* **98** (1986), 109-12.
6. J. S. W. Wong, *Diff. int. Eq.* **6** (1993), 83-91.