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QUASI-RELATION ON TERNARY SEMIGROUP

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D-classes, ***H***-classes and quasi-relation on ternary semigroup have been introduced along with the several other results containing the condition for a congruence on T is idempotent separating is given. Quasi-relation coincides with ***H***-relation is also shown.

1. INTRODUCTION

Sioson[3] defined the definitions of left ideal, right ideal, lateral ideal and quasi-ideal in ternary semigroup T which are given below :

Definition 1.1 — A non-empty set T is said to be a ternary semigroup if it is closed under ternary operation [ ], and satisfies the associative law —

For each X_i in T, where i lies between 1 and 5

$$[X_1 X_2 X_3 X_4 X_5] = [[X_1 X_2 X_3] X_4 X_5] = [X_1 [X_2 X_3 X_4] X_5]$$

$$= [X_1 X_2 [X_3 X_4 X_5]].$$

Definition 1.2 — (a) A non-empty set L is said to be a left ideal of T if $[TTL] \subseteq L$.

(b) A non-empty set M is said to be a lateral ideal of T if $[TMT] \subseteq M$.

(c) A non-empty set R is said to be a right ideal of T if $[RTT] \subseteq R$.

(d) A non-empty set Q is said to be a quasi-ideal of T if

$$[TTQ] \cap \{[TQT] \cup [TTQTT]\} \cap [QTT] \subseteq Q.$$

(e) A non-empty set I is said to be an ideal of T if it is left, lateral and right ideal of T.

Definition 1.3 — Let a be a non-zero element of T then

(1) the principal left ideal generated by a is $(a)_l = \{a\} \cup [TTa]$

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(2) the principal right ideal generated by a is $(a)_r = \{a\} \cup [aTT]$.

(3) the principal lateral ideal generated by a is $(a)_m = \{a\} \cup [TaT] \cup [TTaTT]$.

With the help of these definitions, we introduced the following definition.

Definition 1.4 — Let a be a non-zero element of T then the principal quasi-ideal generated by a is $(a)_q = \{a\} \cup \{[TTa] \cap ([TaT] \cup [TTaTT]) \cap [aTT]\}$.

Lehmer[1] defined the concept of ternary group, the structure of a ternary group is as follows.

Definition 1.5 — A non-empty ternary semigroup G is said to be a ternary group if for all a, b, c in G, there exist unique x, y, z in G such that $[xab] = c$, $[ayb] = c$, $[abz] = c$.

Definition 1.6 (a) — An element a of ternary semigroup T is said to be an idempotent if $[aaa] = a$.

Definition 1.6 (b) — An idempotent element e is an identity of ternary group G if $[eaa] = a = [aea] = [aae]$ for all a in G.

Remark 1.6 (c) : (1) Clearly the identity is unique.

(2) $[eea] = [eeeaa] = [eaa] = a$

$[aee] = [aaeee] = [aae] = a$

$[eae] = [eaeae] = [eaaeeae] = [aeeae] = [aae] = a.$

Example 1.6 (d) — This example shows that a ternary semigroup has an identity element in the sense of Definition 1.6 (b)

$T = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$ is a ternary semigroup under matrix multiplication and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ plays the role of the identity in ternary semigroup.

Definition 1.7 — An element x of a ternary group G is said to be an inverse of a in G, if x is the unique element in G such that

$$[xaa] = e = [axa] = [aax].$$

With the help of these definitions, we construct the structure of Green's equivalence relation (Dixit and Dewan[10]) in ternary semigroup which is as follows :

Definition 1.8 — If a ternary semigroup T does not have the identity element then we adjoin an idempotent element e in T such that for all a, b in T

$$[eab] = [aeb] = [abe] = \begin{cases} \text{in } T \cup \{e\} \\ a \text{ if either } b = a \text{ or } b = e \\ b \text{ if either } a = b \text{ or } a = e. \end{cases}$$

In that case the new ternary semigroup is denoted by T^1 where

$$T^1 = \begin{cases} T \text{ if } T \text{ has an identity element} \\ T \cup \{e\}. \end{cases}$$

The following example shows that every ternary semigroup cannot be embedded in general in an ordinary semigroup.

Example 1.9 — $T = \{i, -i\}$ is a ternary semigroup with complex triple multiplication. It is easy to see that $T \cup \{1\}$ does not embed in an ordinary semigroup.

Remark : (a) By not embedding, in the above example, we mean that we cannot adjoin the identity according to Definition 1.8.

(b) We adjoin the identity to the ternary semigroup if it is possible.

(c) Whenever we are able to adjoin the identity in a ternary semigroup or ternary semigroup has the identity itself then only we can define Green's Equivalence Relation.

Definition 1.10 (Dixit and Dewan[10]) — Let a and b two non-zero elements of T, then

(1) $a \; \boldsymbol{L} \; b$ if a and b generate the same principal left ideal of T.

(2) $a \; \boldsymbol{M} \; b$ if a and b generate the same principal lateral ideal of T.

(3) $a \; \boldsymbol{R} \; b$ if a and b generate the same principal right ideal of T.

(4) $a \; \boldsymbol{J} \; b$ if a and b generate the same principal ideal of T.

(5) $a \; \boldsymbol{D} \; b$ if $a \; (\boldsymbol{L} \circ \boldsymbol{M} \circ \boldsymbol{R}) b$

(6) $a \; \mathbf{H} \; b$ if $a \; (\mathbf{L} \cap \mathbf{M} \cap \mathbf{R}) b$

Proposition 1.11 — $\boldsymbol{L} \circ \boldsymbol{M} \circ \boldsymbol{R}$ is an equivalence relation if the following equivalities hold :

(1) $\boldsymbol{L} \circ \boldsymbol{M} \circ \boldsymbol{R} = \boldsymbol{R} \circ \boldsymbol{M} \circ \boldsymbol{L}$

(2) $\boldsymbol{L} \circ \boldsymbol{M} = \boldsymbol{M} \circ \boldsymbol{L}$

(3) $\boldsymbol{M} \circ \boldsymbol{R} = \boldsymbol{R} \circ \boldsymbol{M}$

(4) $\boldsymbol{L} \circ \boldsymbol{R} = \boldsymbol{R} \circ \boldsymbol{L}$

The proof is trivial. Thus we conclude that

$$\boldsymbol{D} = \boldsymbol{L} \circ \boldsymbol{M} \circ \boldsymbol{R} = \boldsymbol{R} \circ \boldsymbol{M} \circ \boldsymbol{L}$$

$$\boldsymbol{H} = \boldsymbol{L} \cap \boldsymbol{M} \cap \boldsymbol{R} = \boldsymbol{R} \cap \boldsymbol{M} \cap \boldsymbol{L}.$$

Proposition 1.12 (Dixit and Dewan[10]) — Let a and b be any two elements of T then

(a) $a \; \boldsymbol{L} \; b$ if and only if $[T^1T^1a] = [T^1T^1b]$.

(b) $a \; \boldsymbol{M} \; b$ if and only if

$$[T^1aT^1] \cup [TTaTT] = [T^1bT^1] \cup [TTbTT].$$

(c) $a \; \boldsymbol{R} \; b$ if and only if $[aT^1T^1] = [bT^1T^1]$.

(d) $a \; \boldsymbol{J} \; b$ if and only if $[T^1T^1aT^1T^1] = [T^1T^1bT^1T^1]$.

or $\{a\} \cup [aTT] \cup [TTa] \cup [TaT] \cup [TTaTT]$

$$= \{b\} \cup [bTT] \cup [TTb] \cup [TbT] \cup [TTbTT].$$

Remark 1.13 (a) : For any element a in T

$$[aTT] \cup [TTa] \not\subseteq [TaT] \cup [TTaTT].$$

This is shown by the following example :

Example 1.13 (b) — $T = \{0, e, a, b\}$ is a ternary semigroup under the following table :

	0	e	a	b
0	0	0	0	0
e	0	e	a	b
a	0	b	0	0
b	0	b	0	0

$[abc] = (a(bc)) = ((ab)c)$ for all a, b, c in T

$[aTT] \cup [TTa] = \{0, a, b\}$

$[TaT] \cup [TTaTT] = \{0, b\}$

Clearly $[aTT] \cup [TTa] \not\subseteq [TaT] \cup [TTaTT]$.

Remark 1.13 (c) : $\boldsymbol{M} \neq \boldsymbol{J}$. Suppose $\boldsymbol{M} = \boldsymbol{J}$.

For any element a in T, we have $M_a = J_a$ where

$$M_a = \{a \; \boldsymbol{M} \; x : \text{for all } x \text{ in } T\}$$

$$J_a = \{a \; \boldsymbol{J} \; x : \text{for all } x \text{ in } T\}.$$

Therefore $M_a = J_a$ implies

$$[T^1aT^1] \cup [TTaTT] = [T^1T^1aT^1T^1]$$

$$\{a\} \cup [TaT] \cup [TTaTT]$$

$$= \{a\} \cup [aTT] \cup [TTa] \cup [TaT] \cup [TTaTT]$$

$$\Rightarrow \quad [aTT] \cup [TTa] \subseteq [TaT] \cup [TTaTT]$$

which is a contradiction by Remark 1.13(a) and hence $M_a \neq J_a$. Hence $\boldsymbol{M \neq J}$.

2. *D*-CLASSES IN TERNARY SEMIGROUP

Definition 2.1 (Sioson[3]) — An element a of a ternary semigroup T is said to be regular if there exist x and y in T such that $[axaya] = a$.

Definition 2.2 (Sioson[3]) — A ternary semigroup T is said to be regular if all its elements are regular.

Definition 2.3 — An idempotent element e is said to be a left (or lateral or right) identity of T if $[eaa] = a$ (or $[aea] = a$ or $[aae] = a$).

Remark 2.4 : (1) Left (or lateral or right) identity may not be unique.

(2) But if e is an identity (i.e. e plays the role of left, lateral and right identity simultaneously), then e is unique. Uniqueness clearly follows from the definition.

Remark 2.5 : (1) Let e be an idempotent element in a ternary semigroup T. If e is left (lateral, right) identity for T, then e is a left (lateral, right) identity for R_e (M_e, L_e).

(2) If e is lateral identity of T, then for any idempotent a in R_e (or L_e), e is a left (or right) identity.

Definition 2.6 (Lehmer[1]) — An element b in T (containing an identity element e) is said to be an inverse of an element a in T if $[aab] = e = [aba] = [baa]$.

Definition 2.7 — An element b in T (containing an identity element e) is said to be a left inverse if for an element a in T $[aab] = e$.

Similarly we can define lateral and right inverse of T.

Proposition 2.8 — Let a, b be any two elements in T then for b in $R_e \cap M_g \cap L_f$ is the inverse of a in $L_e \cap M_g \cap R_f$ if and only if the following equalities hold :

(1) $[aab] = e$, $[aba] = g$ and $[baa] = f$

(2) $[bba] = f$, $[bab] = g$ and $[abb] = e$

(3) $[aabaa] = a$ and $[bbabb] = b$

if there exist the lateral identities e, g and f.

PROOF : Let b be in $R_e \cap M_g \cap L_f$ and having the inverse a in $L_e \cap M_g \cap R_f$. By Remark 2.5 e is the left identity for R_e, g is the lateral identity for M_g and f is right identity for L_f then a being the inverse of b is the left, lateral and right inverse of b implies

$$[aab] = e,\ [aba] = g \text{ and } [baa] = f.$$

Similarly $[bba] = f$, $[bab] = g$ and $[abb] = e$ and $[aabaa] = [aga] = a$. In the similar manner we get $[bbabb] = b$.

The converse immediately follows from the given equalities.

Proposition 2.9 — Let a be idempotent element of a $\boldsymbol{D}$-class D in a ternary semigroup T. If an idempotent element a^* in T is the inverse of a then $a^* \in D$ and the three **H**-classes for some $c1$, $c2$, $c3$ in T. $R_a \cap M_{c1} \cap L_{a^*}$, $L_a \cap M_{c2} \cap R_{a^*}$, $L_{c3} \cap M_a \cap R_{c3}$ contain the idempotents $[aaa^*]$, $[a^*aa]$, $[aa^*a]$ respectively.

PROOF : Clearly $[aaa^*]$, $[a^*aa]$ and $[aa^*a]$ are idempotents of T. Now $[aaa^*aa] = a$ and $[a^*a\ [aaa^*]] = a^*$. Thus $a\ \boldsymbol{R}\ [aaa^*]$, $[aaa^*]\ \boldsymbol{M}\ [aaa^*]$ and $[aaa^*]\ \boldsymbol{L}\ a^*$ implies $[aaa^*]$ in $R_a \cap M_{c1} \cap L_{a^*}$ where $c1 = [aaa^*]$. The proof for the other two **H**-classes are similar.

Propositon 2.10 — Let T be a ternary semigroup containing idempotent elements e, g, f such that e, g, f are lateral identities of T. Let D be a $\boldsymbol{D}$-class containing an idempotent element a in T such that $a \in R_e \cap M_g \cap L_f$. There exists an idempotent inverse a^* of a in $R_f \cap M_g \cap L_e$ such that $[aaa^*] = e$, $[aa^*\ a] = g$ and $[a^*aa] = f$.

PROOF : By Remark 2.5. e is the left identity of R_e, g is the lateral identity of M_g and f is the right identity of L_f for any idempotent a in $R_e \cap M_g \cap L_f$.

Since a in $R_e \cap M_g \cap L_f$ then $[eaa] = a$, $[aga] = a$ and $[aaf] = a$. Moreover there exists $t_1, t_2, t_3, t_4, t_5, t_6$ in T^1 (ternary semigroup T adjoining the identity element) such that

$$[at_1\ t_2] = e,\ [t_3\ at_4] = g,\ [t_5\ t_6 a] = f.$$

Choosing $a^* = [fegfe]$ we get

$$[aaa^*] = e,\ [aa^*a] = g,\ [a^*aa] = f$$

$$[aaa^*aa] = a \text{ and } [a^*a^*aa^*a^*] = a^*.$$

Conversely
$$\begin{aligned}[a^*a^*e] &= [a^*a^*[aaa^*]] = [a^*[a^*fa^*]aaa^*] = [a^*a^*[a^*aa]a^*aaa^*] \\ &= [[a^*a^*a^*]\ [aaa^*aa^*]a^*] \\ &= [a^*aa^*] = [a^*[a^*a^*aa^*a^*]a^*] \\ &= [a^*a^*a^*] = a^*.\end{aligned}$$

Similarly $[a^*ga^*] = a^*$ and $[fa^*a^*] = a^*$ hence e is the right identity of a^*, g is the lateral identity of a^* and f is the left identity a^*

Thus a^* is an inverse of a and a^* in $L_e \cap M_g \cap R_f$.

Proposition 2.11 — An $\boldsymbol{H}$-class contains at most one inverse.

PROOF : Suppose a^* and a' are two inverses of a in $\boldsymbol{H}$-class $R_f \cap M_g \cap L_e$ such that

$$[aaa^*] = e = [aaa'],\ [aa^*a] = g = [aa'a] \text{ and}$$

$$[a^*aa] = f = [a'aa].$$

By Proposition 2.8
$$\begin{aligned} a' &= [a'a'a\ a'a'] = [a'a'f\ a\ a\ a'a'] \\ &= [a^*a^*\ a\ a\ a\ a\ e\ a'a'] \\ &= [a^*a^*\ a\ a\ a^*] \\ &= [a^*a^*\ a\ a\ a\ e\ a^*] \\ &= [a^*a^*\ a\ a^*a^*] = a^* \end{aligned}$$

Thus each element of $\boldsymbol{H}$-class has an unique inverse.

Remark 2.12 : The above proposition implies that a^*, the inverse of a obtained in Proposition 2.10 is unique.

Proposition 2.13 — Let T be a ternary semigroup containing the idempotent elements e, g, f such that e, g and f are lateral identities of T. Then $R_e \cap M_g \cap L_f$ is non-empty if and only if there exists an idempotent element a of a **D**-class D in T with an unique idempotent inverse a^* in $L_e \cap M_g \cap R_f$ such that

$$[aaa^*] = e,\ [aa^*a] = g,\ [a^*aa] = f.$$

Proof immediately follows from Proposition 2.10 and Remark 2.12 and the converse part is trivial.

Example 2.14 — $T = \{(a, b)$, such that a and b are real numbers$\}$. Then T is a ternary semigroup under the following operation :

$$\begin{aligned} [(x_1, x_2)\ (x_3, x_4)\ (x_5, x_6)] &= ((x_1, x_2) * (x_3, x_4)) * (x_5, x_6) \\ &= (x_1, x_4) * (x_5, x_6) \\ &= (x_1, x_6). \end{aligned}$$

Clearly $((x_1, x_2) * (x_3, x_4)) * (x_5, x_6) = (x_1, x_2) * ((x_3, x_4) * (x_5, x_6))$.

Then (2, 0), (2, 3) and (0, 3) are the lateral identities of T.

For any element (x, y) of T we have

$$[\ (x, y)\ (2, 0)\ (x, y)\ ] = (x, y)$$

$$[\ (x, y)\ (2, 3)\ (x, y)\ ] = (x, y)$$

$$[\ (x, y)\ (0, 3)\ (x, y)\ ] = (x, y).$$

Thus (2, 0), (2, 3) and (0, 3) are left, lateral and right identities of $R_{(2,0)}$, $M_{(2,3)}$ and $L_{(0,3)}$ respectively by Remark 2.5.

$$[\ (2, 3)\ (2, 3)\ (2, 3)\ ] = (2, 3) \text{ is an idempotent of } T$$

$$[\ (2, 0)\ (2, 3)\ (2, 3)\ ] = (2, 3)$$

$$[\ (2, 3)\ (2, 3)\ (0, 3)\ ] = (2, 3).$$

This implies $(2, 3) \in R_{(2,0)} \cap M_{(2,3)} \cap L_{(0,3)}$.

Then there exists an unique inverse $(0, 0) \in L_{(2,0)} \cap M_{(2,3)} \cap R_{(0,3)}$ where (2, 0), (2, 3) and (0, 3) are the right, lateral and left identities of $L_{(2,0)}$, $M_{(2,3)}$ and $R_{(2,3)}$ respectively by Remark 2.5.

Such that $[(2, 3)\ (2, 3)\ (0, 0)\ ] = (2, 0)$

$$[(2, 3)\ (0, 0)\ (2, 3)\ ] = (2, 3)$$

$$[(0, 0)\ (2, 3)\ (2, 3)\ ] = (0, 3).$$

Lemma 2.15 — Let a, b be $\boldsymbol{R}$-equivalent ($\boldsymbol{M}$-equivalent, $\boldsymbol{L}$-equivalent) elements in a ternary semigroup T and let x_1, x_2, y_1, y_2 are in ternary semigroup T^1 (adjoining the identity element to T) such that

$$[ax_1x_2] = b\ ([x_1ax_2] = b \text{ and } [x_1x_2a] = b) \text{ and } [by_1y_2] = a\ ([y_1by_2] = a, [y_1y_2b] = a).$$

Then the map

$$^{\rho}x_1x_2/L_a\ (^{\rho}x_1x_2/M_a,\ ^{\rho}x_1x_2/R_a) \text{ and } ^{\rho}y_1y_2/L_b\ (^{\rho}y_1y_2/M_b,\ ^{\rho}y_1y_2/R_b),$$

are mutually inverse $\boldsymbol{R}$-class ($\boldsymbol{M}$-class, $\boldsymbol{L}$-class) preserving bijections from L_a onto L_b (from M_a onto M_b, from R_a onto R_b) and from L_b onto L_a (from M_b onto M_a, from R_b onto R_a) respectively.

Proposition 2.16 — If H and K are two $\boldsymbol{H}$-classes such that $H = R_e \cap M_g \cap L_f$ and $K = L_e \cap M_g \cap R_f$ where e, g, f are the idempotents of T as well as lateral identities of T. Then H and K are isomorphic.

PROOF : Let $a \in H$ be an idempotent of T, then by Proposition 2.10 and Remark 2.12, there exists a unique inverse a^* in K such that $[aaa^*] = e$, $[aa^*a] = g$ and $[a^*aa] = f$. By applying Green's Lemma (2.15) we get $^{\rho}a/H : H \to P$ (P is the $\boldsymbol{H}$-class containing a) defined by $x \to [aaxaa]$ is a bijection from H onto P. $^{\sigma}a^*/P : P \to K : y \to [a^*ya^*]$ is a bijection from P onto K. Then the composition map $\phi : {}^{\sigma}a^* \circ {}^{\rho}a : H \to K$ defined as $\phi(\mu) = [a^*\ [aa\mu aa]a^*]$ for μ in H is a well defined bijection from H onto K with the inverse

$$\phi^*(v) = [aa[a^*va^*]aa].$$

Also for x_1, x_2, x_3 in H we have

$$[\phi(x_1)\,\phi(x_2)\,\phi(x_3)] = [[a^*aax_1\,aaa^*]\;[a^*aax_2\,aaa^*]\;[a^*aax_3\,aaa^*]]$$

$$= [a^*aa\,[[x_1ff]\,[x_2ff]\,x_3]\,aaa^*]$$

$$= [a^*aa\,[x_1x_2x_3]\,aaa^*] = \phi([x_1\,x_2\,x_3]).$$

3. H-RELATION ON TERNARY SEMIGROUP

Proposition 3.1 — Let a and b be two idempotent elements in regular ternary semigroup T. Then $(a, b) \in H$ if and only if $[aaa^*] = [bbb^*]$, $[aa^*a] = [bb^*b]$ and $[a^*aa] = [b^*bb]$.

The proof is trivial.

Remark 3.2 : The above result is true for ordinary ternary semigroups too.

Remark 3.3 : Since H-class contains at most one inverse implies if $a = b$ then $a^* = b^*$.

Definition 3.4 — Let U be a ternary subsemigroup of T. Then Green's equivalence relation on U will be defined in the same way as they are defined on T. To distinguish the relation from T we write left similar, lateral similar and right similar on U as L^U, M^U, R^U. Similarly we defined D^U, J^U and H^U. Now if (a, b) in L^U (Dixit and Dewan[10]) then there exists u_1, u_2, v_1, v_2 in U^1 (adjoining U to the identity element) such that $[\mu_1\mu_2 a] = b$ and $[v_1v_2b] = a$.

Similarly we can easily show the following results :

$$L^U \subseteq L^T \cap (U \times U) \qquad \ldots (1)$$

$$M^U \subseteq M^T \cap (U \times U) \qquad \ldots (2)$$

$$R^U \subseteq R^T \cap (U \times U) \qquad \ldots (3)$$

$$D^U \subseteq D^T \cap (U \times U) \qquad \ldots (4)$$

$$H^U \subseteq H^T \cap (U \times U) \qquad \ldots (5)$$

$$J^U \subseteq J^T \cap (U \times U). \qquad \ldots (6)$$

Proposition 3.5 — Let U be an idempotent ternary sub-semigroup of a ternary semigroup T. Then

$$L^U = L^T \cap (U \times U),\; M^U = M^T \cap (U \times U),\; R^U = R^T \cap (U \times U),$$

$$H^U = H^T \cap (U \times U).$$

*PROOF : Clearly L^U contained in $L^T \cap (U \times U)$. Conversely, let (a, b) be in $L^T \cap (U \times U)$ and a^*, b^* be the inverse of a and b respectively in D class of U. Then $([a^*aa], a)$ is in L^U contained in L^T. Thus $([a^*aa], [b^*bb])$ in L^T and this implies the following results

$$[[a^*aa]\ [a^*aa]\ [b^*bb]] = [[a^*aa]\ [a^*aa]\ e]$$
$$= [a^*aa\ a^*aa\ a^*aa] = [a^*aaa^*aaa] = [a^*aa] \quad \ldots (7)$$

where e is an adjoining idempotent of T satisfying the Definition 1.8.

Similarly we can prove the following equation.

$$[[b^*bb]\ [b^*bb]\ [a^*aa]] = [b^*bb] \quad \ldots (8)$$

(7) and (8) involve the elements of U, therefore $([a^*aa], [b^*bb])$ is in L^U. By transitivity in L^U, we get (a, b) is in L^U.

The proofs are similar for other results.

The following example shows the existence of an idempotent ternary sub-semigroup of a ternary semigroup.

Example 3.6 — Let $T \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$ be a ternary semigroup under matrix multiplication.

$U = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}$ is an idempotent ternary sub-semigroup of T.

Moreover, we have

$$L^U = M^U = R^U = H^U = D^U = J^U = I_U$$

(where I_U is the identity map from U to itself).

Remark 3.7 : Let T be a regular ternary semigroup and ρ be a congruence on T. Then T/ρ (Dixit and Dewan[10]) (the quotient ternary sub-semigroup of T) defined as $(x\rho : x$ is in $T)$ is a regular ternary semigroup.

Definition 3.8 — Let T be a ternary semigroup and E be the set of idempotents of T. Then a equivalence relation ρ on T is idempotent separating if $\rho \cap (E \times E) = 1_E$. i.e. if no ρ class contains more than one idempotent.

Proposition 3.9 — Let T be a regular ternary semigroup. If a congruence ρ on T is idempotent separating then ρ is contained in H.

PROOF : Let (a, b) in ρ, then by Dixit and Dewan[10] $([aaa^*], [baa^*])$ in ρ where a^* is an inverse of a and, $[aaa^*]\ \rho = [baa^*]\ \rho$ implies $[aaa^*]\ \rho$ is an idempotent element of T/ρ,

i.e. $[aaa^*]\ \rho = e\rho$ where $e\rho$ is an idempotent element of T/ρ and ρ is in idempotent separating congruence.

Therefore, $[aaa^*] = e$. Thus $R_a = R_{[aaa^*]} \leq R_{[baa^*]} \leq R_b$. Similarly $R_b \leq R_a$. Therefore $a \boldsymbol{R} b$. In a similar manner $a \boldsymbol{L} b$ and $a \boldsymbol{M} b$. Thus $a \boldsymbol{H} b$.

4. Quasi-relation on Ternary Semigroup

Definition 4.1 — Let a and b be two non-zero elements of T then $a \boldsymbol{Q} b$ if a and b generate the same principal quasi-ideal of T.

Proposition 4.2 — Let a and b be two non-zero elements of T then $a \boldsymbol{Q} b$ if and only if $(a)_q = (b)_q$, the proof is trivial.

Proposition 4.3 — The equivalence relation $\boldsymbol{H}$ and $\boldsymbol{Q}$ coincide in a ternary semigroup T.

PROOF : If $a \boldsymbol{H} b$ for any a, b in T, then $a(\boldsymbol{R} \cap \boldsymbol{M} \cap \boldsymbol{L})b$ implies

$$\{a\} \cup \{[aTT] \cap ([TaT] \cup [TTaTT]) \cap [TTa]\}$$

$$= \{b\} \cup \{[bTT] \cap ([TbT] \cup [TTbTT]) \cap [TTb]\}.$$

Hence $a \boldsymbol{Q} b$.

Conversely, if possible a is not $\boldsymbol{H}$-related to b implies either a is not $\boldsymbol{R}$-related to b or a is not $\boldsymbol{M}$-related to b or a is not $\boldsymbol{L}$-related to b. This implies that a is not $\boldsymbol{Q}$-related to b, which is a contradiction. Hence $a \boldsymbol{H} b$.

$$\{a\} \cup [aTT] = \{b\} \cup [bTT], \{a\} \cup [TTa] = \{b\} \cup [TTb]$$

$$\{a\} \cup [TaT] \cup [TTaTT] = \{b\} \cup [TbT] \cup [TTbTT]$$

which gives $a \boldsymbol{H} b$.

Definition 4.4 — A ternary semigroup T is said to be commutative if all the three elements commute to each other i.e. for all a, b, c in T, we have

$$[abc] = [bac] = [cba] = [cab] = [bca] = [acb].$$

Example 4.5 — Note that $T = \{0, \pm i\}$ is a commutative ternary semigroup but not a commutative binary semigroup under complex multiplication.

Proposition 4.6 — If the elements a, $[at_1\, t_2]$, $([t_1\, at_2], [t_1 t_2\, a])$ of a commutative ternary semigroup $T(t_1, t_2 \in T)$ both belong to the $\boldsymbol{H}$-class H of T then $[Ht_1t_2] = H$

$$([t_1Ht_2] = H, \quad [t_1t_2H] = H).$$

PROOF : Let h be any element of $\boldsymbol{H}$-class in T then

$$[T^1T^1h] = [T^1T^1a] = [T^1T^1\, [ht_1t_2]] = [[T^1T^1h]\, t_1t_2].$$

Since $h = [x_1x_2a]$ for some x_1, x_2 in T^1 and $\boldsymbol{R}$ is a left congruence, therefore $[[x_1x_2a]T^1T^1] = [[[x_1x_2a]t_1t_2]T^1T^1]$ implies $[hT^1T^1] = [[ht_1t_2]T^1T^1]$.

And $a \boldsymbol{R} [at_1t_2]$ implies $a = [at_1t_2t_3t_4]$ for some t_3, t_4 in T^1.

Now $[T^1hT^1] = [T^1[[x_1x_2a]t_1t_2]\,[t_3t_4T^1]]$

$\subseteq [T^1[ht_1t_2]T^1]$.

On the other hand $[T^1[ht_1t_2]T^1] \subseteq [T^1hT^1]$ implies that $[ht_1t_2] \in H$ for any h in H.

Thus implies $[Ht_1t_2] \subseteq H$.

Conversely h $\boldsymbol{R}$ $[ht_1t_2]$ implies $h = [ht_1t_2t_3t_4]$ for some t_3, t_4 in T^1.

This implies $[Ht_3t_4]$ contained in H.

Now h $\boldsymbol{H}$ $[ht_1t_2]$ implies $h = [x_1x_2ht_1t_2]$ for some x_1, x_2 in T^1.

$$h = [x_1x_2ht_1t_2] = [[x_1x_2ht_1t_2]t_3t_4t_1t_2] = [[ht_3t_4]\,t_1t_2]$$

$$H \subseteq [[Ht_3t_4]\,t_1t_2] \subseteq [Ht_1t_2] \text{ thus } H = [Ht_1t_2].$$

T being a commutative ternary semigroup implies

$$H = [t_1Ht_2] \quad \text{and} \quad H = [t_1t_2H].$$

Proposition 4.3 and Proposition 4.6 give the following result.

Proposition 4.7 — If the elements a, $[at_1t_2]$, $([t_1at_2],\ [t_1t_2a])$ (t_1, t_2 is in T) of a commutative ternary semigroup T, both generate the same principal quasi-ideal Q of T, then

$$[H_at_1t_2],\ ([t_1\,H_a\,t_2],\ [t_1t_2H_a])$$

coincides with the generator class of Q where H_a in the generator $\boldsymbol{H}$-class of a.

Proposition 4.8 — If T has an idempotent element e and Q is a minimal quasi-ideal of T, $Q = [eTeTe]$, then the $\boldsymbol{H}$-class H_e of Q is a subgroup of T, having the identity element e.

PROOF : Q being a minimal quasi-ideal of T is a ternary subgroup of T and Q coincides with H_e, the $\boldsymbol{H}$-class generated by e implies H_e is a ternary subgroup of T. e being the identity of Q is the identity of H_e.

Proposition 4.9 — If a, b and c elements of a ternary semigroup then $[abc] \in R_a \cap M_b \cap L_c$ implies $H_{[abc]} = R_a \cap M_b \cap L_c$.

The proof is trivial.

The above result immediately gives the following result.

Proposition 4.10 — If the generator elements a, b and c of the principal quasi-ideals $(a)_q, (b)_q, (c)_q$ respectively have the property $[abc] \in R_a \cap M_b \cap L_c$ then the generator class $H_{[abc]}$ of the principal quasi-ideal $([abc])_q$ coincides with the set

$$H_{[abc]} = R_a \cap M_b \cap L_c.$$

Proposition 4.11 — If the elements a and b of a ternary semigroup T are **D**-equivalent, then there exists one-to-one correspondence between principal quasi-ideal $(a)_q$ and the principal quasi-ideal $(b)_q$.

PROOF : $a \; \boldsymbol{D} \; b$ gives the existence of c in T such that

$$[at_1t_2] = c,\ [ct_3t_4] = a,\ [t_5t_6c] = b, \text{ and } [t_7t_8b] = c, \text{ for some } t_i \text{ in } T^1$$

where i lies between 1 and 8.

Define a map $\phi : (a)_q \to (b)_q$ by

$\phi(h) = [t_5t_6ht_1t_2]$ for all h in $(a)_q$ implies that

$[t_5t_6ht_1t_2]$ in $(b)_q$, which gives that ϕ is well-defined.

Conversely the same way we defined another map $\psi : (b)_q \to (a)_q$ by $\psi(k) = [t_7t_8kt_3t_4]$ for all k in $(b)_q$ which is well-defined.

Since k is in $(b)_q$, therefore

$$k = [y_1y_2b] = [y_3by_4] = [by_5y_6];\ y_i \text{ is in } T^1,$$

where i lies between 1 and 6

Consider $(\phi \circ \Psi)(k) = [t_5t_6[t_7t_8kt_3t_4]t_1t_2]$

$$= [t_5t_6cy_5y_6t_3t_4t_1t_2]$$

$$= [y_1y_2[bt_3t_4]t_1t_2]$$

$$= [y_1y_2t_5t_6c] = [y_1y_2b] = k$$

$\Rightarrow \phi \circ \Psi = I_{(b)_q}$

the same ways $(\Psi \circ \phi)(h) = h$ where h is in $(a)_q$. Thus $\Psi \circ \phi = I_{(a)_q}$.

Hence ϕ is an one-to-one correspondence between $(a)_q$ and $(b)_q$.

Proposition 4.12 — If a and b are the elements of a ternary semigroup T are $\boldsymbol{D}$-equivalent then there exists an one-to-one correspondence between H_a (the generator $\boldsymbol{H}$-class of a) and H_b (the generator $\boldsymbol{H}$-class of b).

From the above result we immediately get the following result.

Proposition 4.13 — If a and b are the elements of ternary semigroup T which are **D**-equivalent then the principal quasi-ideal $(a)_q$, $(b)_q$ and the $\boldsymbol{H}$-classes H_a, H_b respectively have the same cardinal number.

Proposition 4.14 — If e and f are two idempotent elements of T then the classes H_e and H_f are subgroups of T and are isomorphic to each other.

PROOF : Since the principal quasi-ideal generated by e and f are H_e and H_f respectively are the minimal quasi-ideal of T. All minimal quasi-ideal of T are isomorphic and a minimal quasi-ideal of T is a ternary subgroup of T which implies the classes H_e and H_f are subgroups of T and are isomorphic to each other.

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