Indian J. pure appl. Math., 28(6) : 775-789, June 1997

EXISTENCE OF SOLUTIONS OF GENERAL NONLINEAR STOCHASTIC INTEGRAL EQUATIONS

R. SUBRAMANIAM AND K. BALACHANDRAN

Department of Mathematics, Bharathiar University, Coimbatore 641 046

(*Received* 30 *April* 1996; *after revision* 3 *October* 1996; *accepted* 20 *December* 1996)

In this paper we establish the existence of solution of a more general class of nonlinear stochastic integral equations. The main tools used here is the measure of noncompactness and the fixed point theorem of Darbo type. The results generalize the results of Szynal and Wedrychowicz[9].

1. INTRODUCTION

The mathematical description of various processes in physical, biological and engineering sciences give rise to random or stochastic integral equations. Theoretical treatments of such problems can be found in earlier papers[3-7, 10].

The most important problem examined up to now is the one concerning the existence of solutions of considered equations. It is solved mostly by the Banach fixed point principle, the Schauder fixed point theorem and successive approximations[3, 5, 10]. In this paper we use the notion of measure of noncompactness in a Banach space and the fixed point theorem of Darbo type, (cf. Banas and Goebel[2] and Szynal and Wedrychowicz[9]). This approach allows us to weaken the conditions of Padgett and Tsokos[5], Rao and Tsokos[6], Rao[8] and Tsokos and Padgett[10]. We replace the Lipschitz type conditions by those with sublinear functions.

The aim of this paper is to prove an existence theorem for a more general stochastic integral equation of the following form :

$$x(t; w) = h(t, x(t; w)) + \sum_{i=1}^{m} \int_0^t a_i(t, \tau; w)\, f_i(\tau, x(\tau; w))\, d\tau$$

$$+ \sum_{j=1}^{n} \int_0^t b_j(t, \tau; w)\, g_j(\tau, x(\tau; w))\, d\tau,$$

where $t > 0$ and

(i) $w \in \Omega$, the supporting set of the complete probability measure space (Ω, A, P)

(ii) $x(t; w)$ is the unknown random function for $t \in R_+$

(iii) h is a scalar function, $h : R_+ \times R \to R$

(iv) $a_i(t, \tau; w)$, $i = 1, 2, ..., m$ and $b_j(t, \tau; w)$, $j = 1, 2, ..., n$ are stochastic kernels defined for $0 \leq \tau \leq t < \infty$;

(v) $f_i(t, x)$, $i = 1, 2, ..., m$ and $g_j(t, x)$, $j = 1, 2, ..., n$ are maps from $R_+ \times R_+$ into R;

(vi) $t \in R_+$ and $\beta(t; w)$ is a stochastic process.

The integral in the first part of the stochastic system (1) is an ordinary Lebesgue integral with probabilistic characterization, while the integral in the second part is an Ito-Doob integral.

2. MATHEMATICAL PRELIMINARIES

With respect to the random process $\beta(t; w)$ we shall assume that for each $t \in R_+$, a minimal σ-algebra A_t, $A_t \subset A$, is such that $\beta(t; w)$ is measurable with respect to A_t. In addition, we shall assume that the minimal σ-algebra A_t is an increasing family such that

(i) the random process $\{\beta(t; w), A_t : t \in R_+\}$ is a real martingale and

(ii) there is a real continuous non-decreasing function, $F(t)$, such that for $s < t$ we have

$E\{|\beta(t; w) - \beta(s; w)|^2\} = E\{|\beta(t; w) - \beta(s; w)|^2 | A_t\} = F(t) - F(s)$ P–a.s. where E denotes the expected value of the random process.

We denote by $L^2(\Omega, A, P)$, the space of A-measurable square integrable maps $x(t; w)$ with

$$\| x(t) \|_{L^2} = \left[\int_\Omega | x(t; w) |^2 \, dP(w) \right]^{1/2}.$$

We now give the following definitions.

Definition 1 — We shall call $x(t; w)$ a random solution of the stochastic integral equation (1) if for every fixed $t \in R_+$, $x(t; w) \in L^2(\Omega, A, P)$ and satisfies (1) P-a.e.

Throughout this paper X will denote an infinite dimensional real Banach space with norm $\| \ \|$ and the zero element 0. $V(x, r)$ stands for the closed ball centered at x of radius r. Denote by M_X the family of all nonempty bounded subsets of X and by N_X the family of all relatively compact and nonempty subset of X.

The following axioms defining a measure of noncompactness are taken from Banas and Goebel[2].

Definition 2 — A nonempty family $B \subset N_X$ is said to be kernel (of measure of noncompactness), provided it satisfies the following conditions :

(a) $U \in B \Rightarrow \overline{U} \in B$

(b) $U \in B,\ V \subset U,\ V \neq \phi \Rightarrow V \in B$;

(c) $U, V \in B \Rightarrow \alpha U + (1-\alpha)V \in B,\ \alpha \in [0, 1]$;

(d) $U \in B \Rightarrow \text{Conv } U \in B$;

(e) B^C (the subfamily of B consisting of all closed sets) is closed in N^C with respect to the topology generated by Hausdorff metric.

Definition 3 — The function $\mu : M_X \to [0, +\infty)$ is said to be a measure of noncompactness with kernel B ($\ker \mu = B$) if it satisfies the following conditions :

(i) $\mu(U) = 0 \Leftrightarrow U \in B$;

(ii) $\mu(U) = \mu(\overline{U})$;

(iii) $\mu(\text{Conv } U) = \mu(U)$;

(iv) $U \subset V \Rightarrow \mu(U) \leq \mu(V)$;

(v) $\mu(\alpha U + (1-\alpha)V) \leq \alpha\mu(U) + (1-\alpha)\mu(V),\ \alpha \in [0, 1]$;

(vi) if $U_n \in M_X$, $\overline{U}_n = U_n$ and $U_{n+1} \subset U_n$, $n = 1, 2, \ldots$, and if $\lim_{n\to\infty} \mu(U_n) = 0$, then $U = \bigcap_{n=1}^{\infty} U_n \neq \phi$.

If a measure of noncompactness μ satisfies in addition the following two conditions :

(vii) $\mu(U + V) \leq \mu(U) + \mu(V)$;

(viii) $\mu(\alpha U) = |\alpha| \mu(U),\ \alpha \in R$;

it will be sublinear.

Let $M \subset X$ be a nonempty set and let μ be a measure of noncompactness on X.

Definition 4 — We say that a continuous mapping $T : M \to X$ is a contraction with respect to μ (μ-contraction) if for any set $U \in M_X$ its image $TU \in M_X$, and there exists a constant $k \in [0, 1)$ such that $\mu(TU) \leq k\mu(U)$.

We shall use the following modified version of the fixed-point theorem of Darbo type.

Theorem 1 — *Let C be a nonempty, closed, convex and bounded set of X and let $T : C \to C$ be an arbitrary μ-contraction. Then T has at least one fixed point in C and the set Fix $T = \{x \in C : Tx = x\}$ of all fixed points of T belongs to ker μ.*

Let $p(\ .\ )$ be a positive continuous function defined on $[0, \infty)$ such that $\lim_{T\to\infty} \sup_{t \geq T} p(t) = c$.

Let C_p denote the space of all continuous maps $x(t;\ .)$ from R_+ into $L^2(\Omega, A, P)$ with the topology defined by the norm

$$\| x \|_p = \sup \{p(t) \| x(t) \|_{L^2} : t \geq 0\} < \infty.$$

The space C_p with norm $\| \ \|_p$ is a real Banach space (see Banas[1]).

Now for $x \in C_p$, $U \in M_{C_p}$, $T > 0$ and $\varepsilon > 0$, we put

$$\beta^T(x, \varepsilon) = \sup \{\| p(t) x(t) - p(s) x(s) \|_{L^2} : t, s \in [0, T], |t - s| \leq \varepsilon\};$$

$$\beta_0^T(U, \varepsilon) = \sup \{\beta^T(x, \varepsilon) : x \in U\};$$

$$\beta_0^T(U) = \lim_{\varepsilon \to 0} \beta^T(U, \varepsilon);$$

$$\beta_0(U) = \lim_{T \to \infty} \beta_0^T(U);$$

$$a(U) = \lim_{T \to \infty} \sup_{x \in U} \sup_{t \geq T} \| x(t) \|_{L^2} p(t);$$

$$b(U) = \lim_{T \to \infty} \sup_{x \in U} \sup_{s, t \geq T} \{\| p(t) x(t) - p(s) x(s) \|_{L^2}\};$$

$$\mu_0(U) = \beta_0(U) + a(U) + \sup \{p(t) m(U(t)) : t \geq 0\};$$

$$\mu_1(U) = \beta_0(U) + b(U) + \sup \{p(t) m(U(t)) : t \geq 0\};$$

where m is a sublinear measure of noncompactness on $M_{L^2(\Omega, A, P)}$ and $U(t) = \{x(t) \cdot \in L^2(\Omega, A, P) : x \in U\}$.

The functions μ_0 and μ_1 define sublinear measure of noncompactness on M_{C_p}. It is also known that ker μ_0 is the set of all sets $U \in M_{C_p}$ such that the functions belonging to U are equicontinuous on any compact subset of R_+ and

$$\lim_{t \to \infty} p(t) \| x(t) \|_{L^2} = 0$$

uniformly with respect to $x \in U$. Further properties of μ_0 and μ_1 can be found in Banas[1] and Banas and Goebel[2].

3. Main Results

We shall make the following assumptions concerning eqn. (1)

Let $t \in R_+$ be fixed. The stochastic kernels $a_i(t, \tau; w)$, $i = 1, 2, \ldots, m$ and $b_j(t, \tau; w)$, $j = 1, 2, \ldots, n$ are assumed to be measurable for each t and τ such that $0 \leq \tau \leq t < \infty$, P-essentially bounded and continuous as maps from the set $\Gamma = \{(t, \tau) : 0 \leq \tau \leq t < \infty\}$ into $L_\infty(\Omega, A, P)$.

We define for $0 \leq \tau < t < \infty$,

$$\| |a_i(t, \tau)| \| = P\text{-ess sup}_{w \in \Omega} |a_i(t, \tau; w)|, \quad i = 1, 2, \ldots, m$$

$$\| |b_j(t, \tau)| \| = P\text{-ess sup}_{w \in \Omega} |b_j(t, \tau; w)|, \quad j = 1, 2, \ldots, n.$$

The above assumptions imply that if $x \in C_p$ then for each $t \in R_+$

$$\| a_i(t, \tau) x(\tau) \|_{L^2} \leq \| |a_i(t, \tau)| \| \| x(\tau) \|_{L^2}, \quad i = 1, 2, \ldots, m \qquad \ldots (2)$$

$$\| b_j(t, \tau)\, x(\tau) \|_{L^2} \leq \| \, | b_j(t, \tau) | \, \| \, \| x(\tau) \|_{L^2}, \quad j = 1, 2, \ldots, n. \qquad \ldots (3)$$

For the functions b_j, we have the following estimate

$$\left\| \left| \int_0^t b_j(t, \tau)\, g_j(\tau, x(\tau))\, d\beta(\tau) \right| \right\|_{L^2}$$

$$\leq \left[\int_0^t \| \, | b_j(t, \tau) | \, \|^2 \, (\| g_j(\tau, x(\tau)) \|_{L^2})^2 \, dF(\tau) \right]^{1/2} \qquad \ldots (4)$$

Assumptions — Suppose that the functions f_i, $i = 1, 2, \ldots, m$, g_j, $j = 1, 2, \ldots, n$ and h in stochastic integral equation (1) satisfy the following conditions :

(i) functions f_i, $i = 1, 2, \ldots, m$ and g_j, $j = 1, 2, \ldots, n$ are sublinear, i.e. there exist nonnegative continuous functions u_i, v_i, u_j' and v_j' such that

$$| f_i(t, x(t; w)) | \leq u_i(t) \, | x(t; w) | + v_i(t) \; P\text{-a.s.},$$

$$| g_j(t, x(t; w)) | \leq u_j'(t) \, | x(t; w) | + v_j'(t) \; P\text{-a.s.}$$

Let us denote

$$A_i = \sup \left\{ p(t) \int_0^t \| \, | a_i(t, \tau) | \, \| \, (u_i(\tau)/p(\tau))\, d\tau : t \in R_+ \right\}, \quad i = 1, 2, \ldots, m$$

$$A_j' = \sup \left\{ p(t) \left[2 \int_0^t \| \, | b_j(t, \tau) | \, \|^2 \, (u_j'(\tau)/p(\tau))^2 \, dF(\tau) \right]^{1/2} : t \in R_+ \right\}$$

for $j = 1, 2, \ldots, n$

$$B_i(t) = p(t) \left[| h(t, 0) | /m + \int_0^t \| \, | a_i(t, \tau) | \, \| \, v_i(\tau)\, d\tau \right], \quad t \in R_+$$

for $i = 1, 2, \ldots, m$

$$B_j'(t) = p(t) \sqrt{2} \left[\int_0^t \| \, | b_j(t, \tau) | \, \|^2 \, (v_j'(\tau))^2 \, dF(\tau) \right], \quad t \in R_+$$

for $j = 1, 2, \ldots, n$

$B_i = \sup \{ B_i(t) : t \in R_+ \}$, $B_j' = \sup \{ B_j'(t) : t \in R_+ \}$ and

$$B = \sum_{i=1}^{m} B_i + \sum_{j=1}^{m} B_j'$$

(ii) $\lim_{t \to \infty} B_i(t) = 0$, $i = 1, 2, ..., m$ and $\lim_{t \to \infty} B_j'(t) = 0$, $j = 1, 2, ..., n$

(iii) $| h(t, x(t; w)) - h(t, y(t; w)) | \leq k | x(t; w) - y(t; w) |$ P-a.s. for $k \in [0, 1)$;

(iv) $M = k + \sum_{i=1}^{m} A_i + \sum_{j=1}^{n} A_j' < 1$

(v) for any given but fixed $T > 0$

$\limsup_{\varepsilon \to 0} \{ \| h(t, x(t; w)) - h(s, x(s; w)) \|_{L^2} : t, s \in [0, T], | t - s | \leq \varepsilon \} = 0$ uniformly w.r.t. $x \in U \subset V(0, r)$ where $r = B/(1 - M)$

(vi) the mappings $x(t; w) \to f_i(t, x(t; w))$, $i = 1, 2, ..., m$ and $x(t; w) \to g_j(t, x(t; w))$, $j = 1, 2, ..., n$ from $C_p \to C_p$ are continuous in the topology generated by $\| \cdot \|_p$.

(vii) $\lim_{t \to \infty} p(t) \| f_i(t, x(t)) - f_i(t, y(t)) \|_{L^2} = 0$, $i = 1, 2, ..., m$,

$$\lim_{t \to \infty} p(t) \| g_j(t, x(t)) - g_j(t, y(t)) \|_{L^2} = 0, \quad j = 1, 2, ..., n,$$

uniformly w.r.t. x and $y \in V(0, r)$, $r = B/(1 - M)$.

(viii) there exist L_{1i}, $i = 1, 2, ..., m$, L_{2j}, $j = 1, 2, ..., n$ and L_3 satisfying $0 \leq L < 1$, where

$$L = \sum_{i=1}^{m} L_{1i} + \sum_{j=1}^{n} L_{2j} + L_3, \text{ such that}$$

$$m\left(\int_0^t a_i(t, \tau; w) f_i(\tau, U(\tau)) \, d\tau \leq L_{1i} \, m(U(t)) \right) \leq L_{1i} \, m(U(t)),$$

$$i = 1, 2, ..., m$$

$$m\left(\int_0^t b_j(t, \tau; w) \, g_j(\tau, U(\tau)) \, d\beta(\tau; w) \leq L_{2j} \, m(U(t)) \right) \leq L_{2j} \, m(U(t)),$$

$$j = 1, 2, ..., n$$

$$m(h(t, U(t))) \leq L_3 \, m(U(t))$$

$$U(t) = \{ x(t) \in L^2(\Omega, A, P) : x \in U \, V(0, r) \}, \quad t \geq 0,$$

where $r = B/(1 - M)$.

Lemma 1 — Let assumptions (i), (iii) and (iv) hold. Define H on C_p by

$$(Hx)(t; w) = h(t, x(t; w)) + \sum_{i=1}^{m} \int_0^t a_i(t, \tau; w) \, f_i(\tau, x(\tau; w)) \, d\tau$$

$$+ \sum_{j=1}^{n} \int_0^t b_j(t, \tau; w) \, g_j(\tau, x(\tau; w)) \, d\beta(\tau; w). \quad \ldots (5)$$

Then H maps $V(0, r)$ into $V(0, r)$ where $V(0, r) \subset C_p$ and $r = B/(1 - M)$.

PROOF : Under the assumption (i), (4) takes the form

$$\left\| \int_0^t b_j(t, \tau)\, g_j(\tau, x(\tau))\, d\beta(\tau) \right\|_{L^2}$$

$$\leq \sqrt{2} \left[\| x \|_p \left\{ \int_0^t \| \, | b_j(t, \tau) | \, \|^2 \, (u_j'(\tau)/p(\tau))^2 \, dF(\tau) \right\}^{1/2} \right.$$

$$\left. + \left\{ \int_0^t \| \, | b_j(t, \tau) | \, \|^2 \, (v_j'(\tau))^2 \, dF(\tau) \right\}^{1/2} \right]. \quad \ldots (6)$$

By (i), (iii), (2), (3), (4) and (6) we get for $x \in C_p$

$$p(t) \| H(x)(t) \|_{L^2}$$

$$\leq p(t) \left[k \| x(t) \|_{L^2} + | h(t, 0) | + \sum_{i=1}^{m} \int_0^t \| \, | a_i(t, \tau) | \, \| \, \| f_i(\tau, x(\tau)) \|_{L^2} \, d\tau \right.$$

$$\left. + \sum_{j=1}^{n} \left\{ \int_0^t \| \, | b_j(t, \tau) | \, \|^2 \, (\| g_j(\tau, x(\tau)) \|_{L^2})^2 \, dF(\tau) \right\}^{1/2} \right]$$

$$\leq k \| x \|_p + p(t) \left[| h(t, 0) | + \sum_{i=1}^{m} \int_0^t \| \, | a_i(t, \tau) | \, \| \, u_i(\tau) \| x(t) \|_{L^2} \, d\tau \right.$$

$$\left. + \sum_{i=1}^{m} \int_0^t \| \, | a_i(t, \tau) | \, \| \, v_i(\tau)\, d\tau \right]$$

$$+ \sqrt{2} \| x \|_p \sum_{j=1}^{n} \left\{ \int_0^t \| \, | b_j(t, \tau) | \, \|^2 \, (u_j'(\tau)/p(\tau))^2 \, dF(\tau) \right\}^{1/2}$$

$$+ p(t) \sqrt{2} \sum_{j=1}^{n} \left\{ \int_0^t \| \, | b_j(t, \tau) | \, \|^2 \, (v_j'(\tau))^2 \, dF(\tau) \right\}^{1/2}$$

$$\leq p(t) \left[| h(t, 0) | + \sum_{i=1}^{m} \int_0^t \| \, | a_i(t, \tau) | \, \| \, v_i(\tau)\, d\tau \right.$$

$$\left. + \sqrt{2} \sum_{j=1}^{m} \left\{ \int_0^t \| \, | b_j(t, \tau) | \, \|^2 \, (v_j'(\tau))^2 \, dF(\tau) \right\}^{1/2} \right]$$

$$+ \| x \|_p \left[k + \sum_{i=1}^{m} \int_0^t \| a_i(t, \tau) \| \| (u_i(\tau)/p(\tau))^2 \, d\tau \right.$$

$$\left. + \sqrt{2} \sum_{j=1}^{n} \left\{ \int_0^t \| b_j(t, \tau) \|^2 (u_j'(\tau)/p(\tau))^2 \, dF(\tau) \right\}^{1/2} \right]$$

We have now,

$$\| Hx \|_p \leq M \| x \|_p + B. \qquad \ldots (7)$$

Therefore H maps C_p into C_p. Let $r = B/(1 - M)$. Then H maps $V(0, r)$ into $V(0, r)$.

Lemma 2 — Let the assumptions (iii), (iv), (vi) and (vii) hold. Then H is continuous in the ball $V(0, r)$.

PROOF : Let $x, y \in V(0, r)$. By (vii) for any given $\varepsilon_{1i} > 0$, $i = 1, 2, \ldots, m$, and $\varepsilon_{2j} > 0$, $j = 1, 2, \ldots, n$ we can choose $T > 0$ such that

$$p(\tau) \| f_i(\tau, x(\tau)) - f_i(\tau, y(\tau)) \|_{L^2} < \varepsilon_{1i}, \text{ when } \tau > T \qquad \ldots (8)$$

and

$$p(\tau) \| g_j(\tau, x(\tau)) - g_j(\tau, y(\tau)) \|_{L^2} < \varepsilon_{2j}, \text{ when } \tau > T. \qquad \ldots (9)$$

On the other hand, by (vi) for any given $\varepsilon_{3i} > 0$, $i = 1, 2, \ldots, m$, there exists $\delta_{3i} > 0$ such that for all $\tau \in [0, T]$,

$$p(\tau) \| f_i(\tau, x(\tau)) - f_i(\tau, y(\tau)) \|_{L^2} < \varepsilon_{3i}, \quad i = 1, 2, \ldots, m \qquad \ldots (10)$$

whenever $\| x - y \|_p < \delta_{3i}$, $i = 1, 2, \ldots, m$.

Also for any given $\varepsilon_{4j} > 0$, $j = 1, 2, \ldots, n$,

there exists $\delta_{4j} > 0$ such that for all $\tau \in [0, T]$,

$$p(\tau) \| g_j(\tau, x(\tau)) - g_j(\tau, y(\tau)) \|_{L^2} < \varepsilon_{4j}, \; j = 1, 2, \ldots, n \qquad \ldots (11)$$

whenever $\| x - y \|_p < \delta_{4j}$, $j = 1, 2, \ldots, n$.

By (iii) for any given $\varepsilon_5 > 0$ there exists $\delta_5 > 0$ such that

$$p(t) \| h(t, x(t)) - h(t, y(t)) \|_{L^2} < \varepsilon_5 \text{ whenever } \| x - y \|_p < \delta_5. \qquad \ldots (12)$$

Furthermore, we can assume without loss of generality that there exists $T > 0$ such that $u_i(t) \geq 1$, $i = 1, 2, \ldots, m$, and $u_j'(t) \geq 1$, $j = 1, 2, \ldots, n$, where $t \geq T$ and

$$(u_i)^T = \min_{0 \leq \tau \leq T} \{u_i(\tau)\} > 0 \text{ and } (u_j')^T = \min_{0 \leq \tau \leq T} \{u_j(\tau)\} > 0.$$

Put $p_T = \max \{p(\tau) : 0 \leq \tau \leq T\}$.

Now, for $\tau \geq T$, using (12) we have

$$p(t) \| (Hx)(t) - (Hy)(t) \|_{L^2}$$

$$\leq p(t)\Bigg[\| h(t, x(t)) - h(t, y(t)) \|_{L^2}$$

$$+ \sum_{i=1}^{m} \int_0^t \| \| a_i(t, \tau) \| \| f_i(\tau, x(\tau)) - f_i(\tau, y(\tau)) \|_{L^2} \, d\tau$$

$$+ \sum_{j=1}^{n} \left\{ \int_0^t \| \| b_j(t, \tau) \|^2 \| g_j(\tau, x(\tau)) - g_j(\tau, y(\tau)) \|_{L^2} \, dF(\tau) \right\}^{1/2} \Bigg]$$

$$< \varepsilon_5 + p(t)\Bigg[\sum_{i=1}^{m} (p_T/(u_i)^T)$$

$$\times \int_0^T \| \| a_i(t, \tau) \| \| (u_i(\tau)/p(\tau)) \, p(\tau) \| f_i(\tau, x(\tau)) - f_i(\tau, y(\tau)) \|_{L^2} \, d\tau$$

$$+ \sum_{i=1}^{m} \int_T^t \| \| a_i(t, \tau) \| \| (u_i(\tau)/p(\tau))^2 \, p(\tau) \| f_i(\tau, x(\tau)) - f_i(\tau, y(\tau)) \|_{L^2} \, d\tau \Bigg]$$

$$+ \sum_{j=1}^{n} (p_T/(u_j')^T) \left\{ \int_0^T \| \| b_j(t, \tau) \|^2 \, (u_j'(\tau)/p(\tau))^2 \, (p(\tau))^2 \right.$$

$$\times \left(\| g_j(\tau, x(\tau)) - g_j(\tau, y(\tau)) \|_{L^2} \right)^2 dF(\tau) \Big\}^{1/2}$$

$$+ \sum_{j=1}^{n} \left\{ \int_T^t \| \| b_j(t, \tau) \|^2 \, (u_j'(\tau)/p(\tau))^2 \, (p(\tau))^2 \right.$$

$$\times \left(\| g_j(\tau, x(\tau)) - g_j(\tau, y(\tau)) \|_{L^2} \right)^2 dF(\tau) \Big\}^{1/2} \qquad \ldots (13)$$

By (iv) and from (8)-(13), we get

$$\sup_{t \geq T} p(t) \| H(x)(t) - (Hy)(t) \|_{L^2}$$

$$\leq \varepsilon_5 + M\left[p_T \sum_{i=1}^{m} \varepsilon_{3i}/(u_i)^T + \sum_{i=1}^{m} \varepsilon_{1i} + p_T \sum_{j=1}^{n} \varepsilon_{4j}/(u_j')^T + \sum_{j=1}^{n} \varepsilon_{2j} \right] \qquad \ldots (14)$$

Moreover, it can be seen that for any given $\varepsilon_6 > 0$ one has

$$\sup_{0 \leq t \leq T} p(t) \| (Hx)(t) - (Hy)(t) \|_{L^2} < \varepsilon_6 \text{ whenever } \| x - y \|_p < \delta_6 \qquad \ldots (15)$$

Hence by (14) and (15), for given $\varepsilon > 0$, $\| Hx - Hy \|_p < \varepsilon$ whenever $\| x - y \|_p < \delta$, $x, y \in V(0, r)$.

Therefore H is continuous.

Lemma 3 — Let assumptions (i), (iii) and (v) hold. Then

$$\beta_0(HU) = 0.$$

PROOF : Let now be given $\varepsilon > 0$, $T > 0$ and $t, s \in [0, T]$, $|t-s| < \varepsilon$. By (6) for $0 \le s \le t$ and $x \in U \subset V(0, r)$, we have

$$\begin{aligned}
&\| p(t)(Hx)(t) - p(s)(Hx)(s) \|_{L^2} \\
&\le |p(t)-p(s)| \, \| h(t, x(t)) \|_{L^2} + p(s) \, \| h(t, x(t)) - h(s, x(s)) \|_{L^2} \\
&\quad + |p(t)-p(s)| \sum_{i=1}^{m} \left\| \int_0^t \| a_i(t, \tau) \| f_i(\tau, x(\tau)) \, d\tau \right\|_{L^2} \\
&\quad + p(s) \sum_{i=1}^{m} \left\| \int_0^s (a_i(t, \tau) - a_i(s, \tau)) f_i(\tau, x(\tau)) \, d\tau \right\|_{L^2} \\
&\quad + p(s) \sum_{i=1}^{m} \left\| \int_s^t a_i(t, \tau) f_i(\tau, x(\tau)) \, d\tau \right\|_{L^2} \\
&\quad + |p(t)-p(s)| \sum_{j=1}^{n} \left\| \int_0^t \| b_j(t, \tau) \| g_j(\tau, x(\tau)) \, d\beta(\tau) \right\|_{L^2} \\
&\quad + p(s) \sum_{j=1}^{n} \left\| \int_0^s (b_j(t, \tau) - b_j(s, \tau)) g_j(\tau, x(\tau)) \, d\beta(\tau) \right\|_{L^2} \\
&\quad + p(s) \sum_{j=1}^{n} \left\| \int_s^t b_j(t, \tau) g_j(\tau, x(\tau)) \, d\beta(\tau) \right\|_{L^2} \qquad \ldots (16)
\end{aligned}$$

By using (2) with $x(\tau)$ replaced by $f_i(\tau, x(\tau))$, we obtain

$$\begin{aligned}
&|p(t)-p(s)| \sum_{i=1}^{m} \left\| \int_0^t a_i(t, \tau) f_i(\tau, x(\tau)) \, d\tau \right\|_{L^2} \\
&\le |p(t)-p(s)| \sum_{i=1}^{m} \int_0^t \| a_i(t, \tau) \| \, \| u_i(\tau) |x(\tau)| + v_i(\tau) \|_{L^2} \, d\tau \\
&\le T |p(t)-p(s)| \sum_{i=1}^{m} \left[r \max_{0 \le \tau \le T} \{ \| a_i(t, \tau) \| \, (u_i(\tau)/p(\tau)) \} \right. \\
&\qquad \left. + \max_{0 \le \tau \le T} \{ \| a_i(t, \tau) \| \, v_i(\tau) \} \right] \qquad \ldots (17)
\end{aligned}$$

Similarly we get

$$p(s)\sum_{i=1}^{m}\left\|\int_0^s (a_i(t,\tau)-a_i(s,\tau))f_i(\tau,x(\tau))\,d\tau\right\|_{L^2}$$

$$\le T\,r\,p(s)\sum_{i=1}^{m}\max\{\|\,a_i(t,\tau)-a_i(s,\tau)\,\|\,(u_i(\tau)/p(\tau))^{\cdot}:\ 0\le\tau\le T\}$$

$$+\ T\,p(s)\sum_{i=1}^{m}\max\{\|\,a_i(t,\tau)-a_i(s,\tau)\,\|\,v_i(\tau)\ :\ 0\le\tau\le T\}\quad\ldots\ (18)$$

and

$$p(s)\sum_{i=1}^{m}\left\|\int_s^t a_i(t,\tau)f_i(\tau,x(\tau))\,d\tau\right\|_{L^2}$$

$$\le|\,t-s\,|\,p(s)\left[\,r\sum_{i=1}^{m}\max\{\|\,a_i(t,\tau)\,\|\,(u_i(\tau)/p(\tau)):0\le\tau\le T\}\right.$$

$$\left.+\sum_{i=1}^{m}\max\{\|\,a_i(t,\tau)\,\|\,v_i(\tau):0\le\tau\le T\}\,\right].\quad\ldots\ (19)$$

Using (4) and (6), we have the following estimates

$$|\,p(t)-p(s)\,|\sum_{j=1}^{n}\left\|\int_0^t b_j(t,\tau)\,g_j(\tau,x(\tau))\,d\beta(\tau)\right\|_{L^2}$$

$$\le\sqrt{2}\,[F(T)-F(0)]^{1/2}\,|\,p(t)-p(s)\,|$$

$$\times\sum_{j=1}^{n}\,[\,r\,(\max\{\|\,b_j(t,\tau)\,\|^2\,(u_j'(\tau)/p(\tau))^2:0\le\tau\le T\})^{1/2}$$

$$+\ (\max\{\|\,b_j(t,\tau)-b_j(s,\tau)\,\|^2\,(v_j'(\tau))^2:0\le\tau\le T\})^{1/2}\,].\quad\ldots\ (20)$$

Similarly we get

$$p(s)\sum_{j=1}^{n}\left\|\int_0^s (b_j(t,\tau)-b_j(s,\tau))\,g_j(\tau,x(\tau))\,d\beta(\tau)\right\|_{L^2}$$

$$\le\sqrt{2}\,p(s)\,[F(T)-F(0)]^{1/2}$$

$$\times\sum_{j=1}^{n}\,[\,r\,(\max\{\|\,b_j(t,\tau)-b_j(s,\tau)\,\|^2\,(u_j'(\tau)/p(\tau))^2:0\le\tau\le T\})^{1/2}$$

$$+\ (\max\{\|\,b_j(t,\tau)-b_j(t,\tau)\,\|^2\,(v_j'(\tau))^2:0\le\tau\le T\})^{1/2}\,]\quad\ldots\ (21)$$

and

$$p(s) \sum_{j=1}^{n} \left\| \int_{s}^{t} b_j(t, \tau)\, g_j(\tau, x(\tau))\, d\beta(\tau) \right\|_{L^2}$$

$$\leq \sqrt{2}\, p(s)\, [F(t) - F(s)]^{1/2}$$

$$\times \sum_{j=1}^{n} \Big[r \Big(\max_{0 \leq \tau \leq T} \{ \| \, | b_j(t, \tau) | \, \|^2 (u_j(\tau)/p(\tau))^2 \} \Big)^{1/2}$$

$$+ \Big(\max_{0 \leq \tau \leq T} \{ \| \, | b_j(t, \tau) | \, \|^2 (v_j'(\tau))^2 \} \Big)^{1/2} \Big]. \qquad \ldots (22)$$

We need to recall the definition of the modulus of continuity which is defined for all real functions w as :

$$v_T(w; \varepsilon) = \sup \{ | w(t) - w(s) | : t, s \in [0, T], \ | t - s | \leq \varepsilon \}, \ \varepsilon > 0 \qquad \ldots (23)$$

From the properties of a_i, $i = 1, 2, \ldots, m$, b_j, $j = 1, 2, \ldots$ and p, we have $\lim_{\varepsilon \to 0} v_T(p; \varepsilon) = 0$, $\lim_{\varepsilon \to 0} v_T(F; \varepsilon) = 0$

$$\lim_{\varepsilon \to 0} v_T(\| \, | a_i | \, \| ; \varepsilon) = 0, \ i = 1, 2, \ldots, m,$$

$$\lim_{\varepsilon \to 0} v_T(\| \, | b_j | \, \| ; \varepsilon) = 0, \ j = 1, 2, \ldots, n. \qquad \ldots (24)$$

Moreover, by assumptions (iii) and (v) we see that

$$\lim_{\varepsilon \to 0} v_T(\| h(t, x(t)) - h(s, x(s)) \|_{L^2} ; \varepsilon) = 0 \qquad \ldots (25)$$

Therefore, by (16) to (22) and (24), (25), we get for $U \subset V(0, r)$

$$\beta_0(HU) = 0 \qquad \ldots (26)$$

Lemma 4 — Let assumptions (i), (ii) and (iii) hold. Then

$$a(HU) \leq M\, a(U) \qquad \ldots (27)$$

where $U \subset V(0, r)$, $r = B/(1 - M)$.

PROOF : Fix some i. It is clear, by the definition of the integral, that for any given $\alpha_{1i} > 0$ there exists a positive integer $n_i = n_i(\alpha_{1i})$ such that for $n \geq n_i$,

$$\left| \int_{0}^{t} \| \, | a_i(t, \tau) | \, \| \, (u_i(\tau) \| x(\tau) \|_{L^2} / p(\tau))\, p(\tau)\, d\tau \right.$$

$$\left| - \sum_{k=0}^{n-1} (t/n) \| \, | a_i(t, kt/n) | \, \| (u_i(kt/n) \| x(kt/n) \|_{L^2}/p(kt/n)) \, p(kt/n) \right|$$

$$< \alpha_{1i}$$

Let $T < t$. Put $k_i^* = \max \{k : 0 \le k \le n, \ kt/n < T\}$, $i = 1, 2, \ldots, m$.

Then we have

$$\int_0^t \| \, | a_i(t, \tau) | \, \| (u_i(\tau) \| x(\tau) \|_{L^2}/p(\tau)) \, p(\tau) \, d\tau$$

$$\le \alpha_{1i} + \sum_{k=0}^{k_i^*} (t/n) \| \, | a_i(t, kt/n) | \, \| \ (u_i(kt/n) \| x(kt/n) \|_{L^2}/p(kt/n)) \, p(kt/n)$$

$$+ \sum_{k=k_i^*+1}^{n-1} (t/n) \| \, | a_i(t, kt/n) | \, \| \ (u_i(kt/n) \| x(kt/n) \|_{L^2}/p(kt/n)) \, p(kt/n)$$

$$= \alpha_{1i} + I_{1i} + I_{2i}$$

Now for any given $\alpha_{2i} > 0$,

$$I_{1i} \le (k_i^* \, t \max_{kt/n < T} \{p(kt/n) \| x(kt/n) \|_{L^2}\} \max_{0 \le \tau \le T} \{\| \, | a_i(t, \tau) | \, \| \, | u_i(\tau)/p(\tau) | \}/n)$$

$$< \alpha_{2i}, \text{ for sufficiently large } n.$$

Similarly for any given $\alpha_{3i} > 0$,

$$I_{2i} \le \sup_{t \ge T} \{ p(t) \| x(t) \|_{L^2} \} \left(\int_0^t \| \, | a_i(t, \tau) | \, \| (u_i(\tau)/p(\tau)) \, d\tau + \alpha_{3i} \right)$$

for sufficiently large n.

Fix some j. For any given $\theta_{1j} > 0$, we have

$$\left| \int_0^t \| \, | b_j(t, \tau) | \, \|^2 (u_j(\tau) \| x(\tau) \|_{L^2}/p(\tau))^2 \, (p(\tau))^2 \, dF(\tau) \right.$$

$$\left. - \sum_{k=0}^{n-1} [F((k+1)t/n) - F(kt/n)] \right.$$

$$\left. \| \, | b_j(t, \tau) | \, \|^2 (u_j(kt/n) \| x(kt/n) \|_{L^2}/p(kt/n))^2 \, (p(kt/n))^2 \right|$$

$$< \theta_{1j}, \text{ for sufficiently large } n.$$

Now, let $T < t$ and $k_j^* = \min \{k : 0 \le k \le n, \ kt/n < T\}$.

Then for any given θ_{1j}, we have

$$\int_0^t \| \, | b_j(t, \tau) | \, \|^2 \, (u_j'(\tau) \, \| x(\tau) \, \|_{L^2}/p(\tau))^2 \, (p(\tau))^2 \, dF(\tau)$$

$$\leq \theta_{1j} + \sum_{k=0}^{k_j^*} [\, F((k+1)t/n) - F(kt/n) \,] \, \| \, | b_j(t, kt/n) | \, \|^2$$

$$(u_j'(kt/n) \, \| x(kt/n) \, \|_{L^2}/p(kt/n))^2 \, (p(kt/n))^2$$

$$+ \sum_{k=k_j^*+1}^{n-1} [\, F((k+1)t/n) - F(kt/n) \,] \, \| \, | b_j(t, kt/n) | \, \|^2$$

$$(u_j'(kt/n) \, \| x(kt/n) \, \|_{L^2}/p(kt/n))^2 \, (p(kt/n))^2$$

$$= \theta_{1j} + K_{1j} + K_{2j}, \; j = 1, 2, \ldots, n.$$

Now by the continuity of F for any given $\theta_{2j} > 0$, we have

$$K_{1j} \leq \max [F((k+1)t/n) - F(kt/n)] \; \max \{p^2(kt/n) \, \| x(t) \, \|_{L^2} : kt/n < T\}$$

$$\times \max \{\| \, | b_j(t, \tau) | \, \|^2 \, (u_j'(\tau)/p(\tau)^2 : 0 \leq \tau \leq T\}$$

$$< \theta_{2j}, \text{ for sufficiently large } n.$$

Similarly for any given $\theta_{3j} > 0$,

$$K_{2j} \leq (\sup \{p(t) \, \| x(t) \, \|_{L^2} : t > T\})^2$$

$$\times \left[\int_0^t \| \, | b_j(t, \tau) | \, \|^2 \, (u(\tau)/p(\tau))^2 \, dF(\tau) + \theta_{2j} \right]$$

for sufficiently large n. By (iii), we get

$$p(t) \, \| h(t, x(t)) \, \|_{L^2} \leq k \sup \{p(t) \, \| x(t) \, \|_{L^2} : t > T\} + p(t) \, | \, h(t, 0) \, |$$

Therefore, by the above considerations, we get

$$p(t) \, \| H(t)(t) \, \|_{L^2} \leq M \sup \{p(t) \, \| x(t) \, \|_{L^2} : t \geq T\}$$

$$+ p(t) \left[\sum_{i=1}^{m} (\alpha_{1i} + \alpha_{2i} + r\alpha_{3i}) + \sum_{j=1}^{n} (\sqrt{\theta_{1j}} + \sqrt{\theta_{2j}} + 2\sqrt{\theta_{3j}}) \right]$$

$$+ p(t) \left[| \, h(t, 0) \, | + \sum_{i=1}^{m} \int_0^t \| \, | a_i(t, \tau) | \, \| \, v_i(\tau) \, d\tau \right]$$

$$+ \sqrt{2}\, p(t) \sum_{j=1}^{n} \left[\int_0^t \| \, | b_j(t, \tau) | \, \|^2 (v_j'(\tau))^2 \, dF(\tau) \right]^{1/2}.$$

By assumptions, we obtain

$$\lim_{T \to \infty} \sup_{x \in U} \{ \sup \{ p(t) \| (Hx)(t) \|_{L^2} : t \geq T \} \}$$

$$\leq \left[\sum_{i=1}^{m} (\alpha_{1i} + \alpha_{2i} + r\alpha_{3i}) + \sum_{j=1}^{n} (\sqrt{\theta_{1j}} + \sqrt{\theta_{2j}} + 2\sqrt{\theta_{3j}}) \right] c$$

$$+ M \lim_{T \to \infty} \sup_{x \in U} \{ \sup \{ p(t) \| x(t) \|_{L^2} : t \geq T \} \}.$$

Let $\alpha_{1i} \to 0$, $\alpha_{2i} \to 0$ and $\alpha_{3i} \to 0$ for $i = 1, 2, \ldots, m$ and $\theta_{1j} \to 0$, $\theta_{2j} \to 0$ and $\theta_{3j} \to 0$ for $j = 1, 2, \ldots, n$.

Then we get (27).

Theorem 2 — *Under the assumptions (i)-(viii), there exists at least one solution* $x \in C_p$ *of equation (1) such that*

$$\lim_{t \to \infty} p(t) \| x(t) \|_{L^2} = 0.$$

PROOF : Now by Lemma 3 and Lemma 4 and the assumption (viii), we obtain $\mu_0(HU) \leq D\mu_0(U)$, where $D = \max \{L, M\}$. This proves along with Lemma 2, that H is a μ_0 contraction and thus the proof is complete.

REFERENCES

1. J. Banas, *Demonstratio Math.* **14** (1981), 127-33.
2. J. Banas and K. Goebel, Measure of Noncompactness in Banach Spaces, *Lect. Not. pure appl. Math.*, **60**, Marcel Dekker Inc., New York, 1980.
3. A. T. Bharucha-Reid, *Random Integral Equations*, Academic Press, New York, 1972.
4. J. S. Milton and C. P. Tsokos, *J. Math. phys. Sci.* **4** (1971) 361-71.
5. W. J. Padgett and C. P. Tsokos, *Proc. Am. math. Soc.* **33** (1972), 534-42.
6. A. N. V. Rao and C. P. Tsokos, *Coll. Math.* **35** (1976), 141-46.
7. V. Sree Hari Rao, *J. Math. Anal. Appl.* **74** (1980), 311-17.
8. V. Sree Hari Rao, *Pacific J. Math.* **108** (1983), 397-405.
9. D. Szynal and S. Wedrychowicz, *Ann. di Mat pura appl.*, (IV), CXLII (1985), 105-19.
10. C. P. Tsokos and W. J. Padgett, Random Integral Equations with Applications to Stochastic Systems, *Lect. Not. Math.*, Springer-Verlag, New York, 1971.