Indian J. pure appl. Math., 26(1) : 1-19, January 1995

NEW H-KKM THEOREMS AND THEIR APPLICATIONS TO GEOMETRIC PROPERTY, COINCIDENCE THEOREMS, MINIMAX INEQUALITY AND MAXIMAL ELEMENTS*

XIE PING DING

Department of Mathematics, Sichuan Normal University, Chengdu, Sichuan, People's Republic of China 610066

(*Received* 30 *June* 1993; *after revision* 15 *July* 1994; *accepted* 6 *October* 1994)

In this paper, we introduce the notions of compact closure and compact interior for sets in topological spaces and the notions of transfer compact closedness and transfer compact openness for mappings. By using these notions, we prove some new H-KKM type theorems which generalize the most of known H-KKM and F-KKM type theorems in the literature. As applications, we obtain some new generalizations of the Ky Fan type geometric property of H-spaces, coincidence theorems and minimax inequality in H-spaces. We also give some new existence theorems of maximal elements for preference relations in H-spaces which generalize the recent results of Tian.

1. INTRODUCTION

The classical Knaster-Kuratowski-Mazurkiewicz (KKM) theorem[34] is a basic result which is equivalent to many basic theorems such as Sperner Lemma, Brouwer fixed point theorem and Fan's minimax inequality. Since KKM theorem was given, the theorem has been generalized in various directions and has become an important and fundamental tool in treating many sophisticated nonlinear problems. The most important generalization is the Fan-Knaster-Kuratowski-Mazurkiewicz (FKKM) theorem which was obtained by Ky Fan[19, 21, 22] and can be used to prove many existence theorems for various nonlinear problems. Recently, Horvath[26,27], Ding[11, 12], Ding and Tan[17], Chang and Ma[10] and Tarafdar[55] have generalized the FKKM theorem to H-spaces and have given many applications in various fields.

Shioji[47] and Park[43] have established some new FKKM theorems involving an upper semicontinuous set-valued mapping with acyclic values in a Hausdorff topological vector space. These theorems generalize the most of known results in the literature. Ding[13] has obtained a new generalization of H-FKKM type theorems which unifies and generalizes the corresponding results mentioned above.

*This project is supported by the National Natural Science Foundation of P.R. China.

Recently, by introducing the notions of transfer closedness and transfer continuity for mappings, Tian[57] offered a further generalization of FKKM theorem of Fan[21,22] and by applying his FKKM theorem, he obtained some generalizations of Ky Fan minimax inequality and some existence theorems on the maximal elements of binary relations, price equilibrium and complementarity problem by relaxing the compactness and convexity of sets.

In this paper, we shall introduce the notions of compact closure and compact interior for sets in topological spaces and the notions of transfer compact closedness and transfer compact openness for mappings. By using these notions and our H-FKKM theorem, we obtain a new H-FKKM type theorem which unifies and generalizes the most of known H-KKM theorems and FKKM theorems in the literature. As applications, we obtain some new generalizations of the Ky Fan type geometric property of H-spaces, coincidence theorems in H-spaces. We also obtain some new existence theorems of maximal elements for preference relations in H-spaces which generalize the recent results of Tian[57].

2. Preliminaries

Let X and Y be nonempty sets, we shall denote by $\mathcal{F}(X)$ the family of all nonempty finite subsets of X and by 2^Y the family of all subsets of Y. Let $F : X \to 2^Y$ be a set-valued mapping. For $A \subset X$ and $y \in Y$, let

$$F(A) = \bigcup \{F(x) : x \in A\} \text{ and } F^{-1}(y) = \{x \in X : y \in F(x)\}.$$

For $B \subset Y$, the upper inverse of B under F is defined by

$$F^{+}(B) = \{x \in X : \phi \neq F(x) \subset B\}.$$

A subset A of a topological space X is said to be compactly closed (resp., compactly open) in X if for each nonempty compact subset K of X, $A \bigcap K$ is closed (resp., open) in K. A subset A of X is called a k-text set if it is compactly closed in X. A topological space X is called a k-space if each k-test set is closed (or equivalently, a subset B of X is open in X if and only if B is compactly open in X, e.g. see Wilansky[58] (p.142) or Dugundji[18] (p. 248). However the topological vector space $\mathbf{R}^{\mathbf{R}}$ is not a k-space, e.g. see Kelley[31] (p. 240) or Wilansky[58] (p. 143)). Therefore the notions of compact openness and compact closedness for subsets of a topological space is a true generalization of the notions of openness and closedness for subsets of a topological space.

Let A be a subset of a topological space X, we define the compact closure of A, denoted by $ccl(A)$, as

$$ccl(A) = \bigcap \{B \subset X : A \subset B \text{ and } B \text{ is compact closed in } X\}.$$

Define the compact interior of A, denoted by $cint(A)$, as

$$cint(A) = \bigcup \{B \subset X : B \subset A \text{ and } B \text{ is compact open in } X\}.$$

It is easy to see that $ccl(A)$ is the smallest compactly closed subset containing A and $cint(A)$ is the largest compactly open subset which is contained in A. Clearly, we have $ccl(A) \bigcap K = cl_K(A \bigcap K)$ for each nonempty compact subset K of X where $cl_K(A \bigcap K)$ denotes the closure of $A \bigcap K$ in K. We also have $cint(A) \bigcap K = int_K(A \bigcap K)$ for each nonempty compact subset K of X where $int_K(A \bigcap K)$ denotes the interior of $A \bigcap K$ in K. A subset A of X is compactly closed (resp., compactly open) if and only if $ccl(A) = A$ (resp., $cint(A) = A$). If X and Y are topological spaces, a mapping $F : X \to 2^Y$ is said to be upper semicontinuous (in short, u.s.c.) in X if the set $F^+(V)$ is open in X for each open subset V of Y. An extended real-valued function $f : X \to R \bigcup \{\pm\infty\}$ is lower semicontinuous (in short, l.s.c.) on X if the set $\{x \in X : f(x) > r\}$ is open in X for each $r \in R \bigcup \{\pm\infty\}$; f is called u.s.c. on X if $-f$ is l.s.c. on X.

The following notions were introduced by Bardaro and Ceppitelli[2, 3].
A pair $(X, \{\Gamma_A\})$ is said to be an H-space if X is a topological space and $\{\Gamma_A\}$ is a family of contractible subsets of X indexed by $A \in \mathcal{F}(X)$ such that $\Gamma_A \subset \Gamma_{A'}$, whenever $A \subset A'$. Clearly, each topological vector space and its convex subsets are all H-spaces with $\Gamma_A = co(A)$ for each $A \in \mathcal{F}(X)$ where $co(A)$ is the convex hull of A. A subset D of an H-space $(X, \{\Gamma_A\})$ is said to be (i) H-convex if $\Gamma_A \subset D$ for each $A \in \mathcal{F}(D)$; (ii) weakly H-convex if $\Gamma_A \bigcap D$ is contractible for each $A \in \mathcal{F}(D)$; (iii) H-compact in X if for each $A \in \mathcal{F}(X)$, there exists a compact weakly H-convex subset D_A of X such that $D \bigcup A \subset D_A$.

Following Tarafdar[54], for a nonempty subset D of an H-space $(X, \{\Gamma_A\})$, we define the H-convex hull of D, denoted by $H - co(D)$, as

$$H - co(D) = \bigcap \{B \subset X : D \subset B \text{ and } B \text{ is } H\text{-convex}\}.$$

Obviously $H - co(D)$ is the smallest H-convex subset containing D and by Lemma 1 of Tarafdar[54], we have

$$H - co(D) = \bigcup \{H - co(A) : A \in \mathcal{F}(D)\}.$$

A mapping $F : D \to 2^X$ is called H-KKM if for each $A \in \mathcal{F}(D)$, $H - co(A) \subset F(A)$.

Recall that a topological space is acyclic if it is nonempty, connected and has vanishing rational homology in all positive dimensions, see McClendon[37] and Park[43]. In particular, any contractible space is acyclic, and hence any convex or star-shaped set in topological vector space is acyclic. For a topological space Y, we shall denote by $ka(Y)$ the family of all compact acyclic subsets of Y.

Let $(X, \{\Gamma_A\})$ be an H-space. For each $N \in \mathcal{F}(X)$, $H - co(N)$ is said to be a polytope in X. $(X\ \{\Gamma_A\})$ is called an H-space with compact polytopes if each polytopes in X is compact. If X is a convex subset of vector space with finite topology, then X is a convex space (see, Lassonde[35]) and it is easy to see that $(X, \{\Gamma_A\})$ with $\Gamma_A = co(A)$ for each $A \in \mathcal{F}(X)$ is also an H-space with compact polytopes.

In our recent paper[13], by using Lemma 1 of Ding and Tan[17] and Lemma 1 of Shioji[47], we have proved the following H-KKM type theorem which generalizes many known FKKM and H-FKKM type theorems.

Theorem 2.1 — Let D be a nonempty subset of an H-space $(X, \{\Gamma_A\})$ with compact polytopes, Y be a Hausdorff topological space, $G : D \to 2^Y$ and $T : H-co(D) \to 2^Y$ be such that

(1) for each $A \in \mathcal{F}(D)$, $T|_Z : Z \to ka(Y)$ is u.s.c., where $Z = H - co(A)$,

(2) for each $A \in \mathcal{F}(D)$, $T(H - co(A)) \subset G(A)$,

(3) for each $A \in \mathcal{F}(D)$ and for each $x \in A$, $G(x) \bigcap T(Z)$ is relatively closed in $T(Z)$ where $Z = H - co(A)$.

Then for any $A \in \mathcal{F}(D)$,

$$T(H - co(A)) \bigcap \bigcap_{x \in A} G(x) \neq \phi.$$

Remark 2.1 : If $X = Y$ and T is the identity mapping, the condition (2) implies G is an H-KKM mapping and hence the condition (2) is an H-KKM type condition. If each $G(x)$ is compact closed, the condition (3) holds trivially. Hence Theorem 2.1 generalizes Theorem 1 of Shioji[47] to H-spaces. In our recent paper[13], by using Theorem 2.1, we have obtained some new H-KKM type theorems and have given some applications to the Ky Fan type geometric properties of H-spaces, minimax inequalities and coincidence theorems.

3. Generalizations of H-KKM Type Theorems

In order to further generalize our H-KKM type theorem in Ding[13], by relaxing the compact closedness and H-KKM type condition of $G(x)$, we need to introduce the following new notions.

Definition 3.1 (Transfer compact closedness) — Let X and Y be two topological spaces. A mapping $G : X \to 2^Y$ is said to be transfer compactly closed-valued on X if for each $x \in X$ and for each nonempty compact subset K of Y, $y \notin G(x) \bigcap K$ implies that there exist a point $x' \in X$ such that $y \notin cl_K(G(x') \bigcap K)$.

Definition 3.2 (Transfer compact openness) — Let X and Y be two topological spaces. A mapping $G : X \to 2^Y$ is said to be transfer compactly open-valued on X if for each $x \in X$ and for each nonempty compact subset K of Y, $y \in G(x) \bigcap K$ implies that there exists a point $x' \in X$ such that $y \in int_K(G(x') \bigcap K)$.

Remark 3.1 : Clearly each closed-valued (resp., open-valued) mapping $G : X \to 2^Y$ is transfer closed-valued (resp., transfer open-valued) (see, the Definitions 6 and 7 of Tian[57]) and is also compactly closed-valued (resp., compactly open-valued), i.e., for each $x \in X$, $G(x)$ is compactly closed (resp., compactly open) in X. Each transfer closed-valued (resp., transfer open-valued) mapping is transfer compactly closed-valued (resp., transfer compactly open-valued). Since the compact closedness of sets is a true generalization of closedness of sets in topological spaces, the notion of transfer compact closedness (resp., transfer compact openness) of a mapping is also a true generalization of the notion of transfer closedness (resp., transfer openness) of a mapping in Tian[57].

Theorem 3.1 — Let D be a nonempty subset of an H-space $(X, \{\Gamma_A\})$ with compact polytopes, Y be a Hausdorff topological space, $G : D \to 2^Y$ and $T : H-co(D) \to 2^Y$ be such that

(1) G is transfer compactly closed-valued in D,

(2) for each $A \in \mathcal{F}(D)$, $T(H-co(A)) \subset (ccl\, G)(A) = \bigcup \{ccl(G(x)) : x \in A\}$,

(3) for each $A \in \mathcal{F}(D)$, $T|_Z : Z \to ka(Y)$ is u.s.c., where $Z = H-co(A)$,

(4) there exists a nonempty compact subset K of Y such that for each $A \in \mathcal{F}(D)$, $T(H-co(A)) \subset K$.

Then $K \bigcap \bigcap_{x \in D} G(x) \neq \phi$.

PROOF : By the definition of compact closure of a set, we have that for each $x \in D$, $ccl(G(x))$ is compactly closed in Y. It follows from Theorem 2.1 and the condition (4) that the family $\{ccl(G(x)) \bigcap K : x \in D\}$ has the finite intersection property. Since K is compact, $K \bigcap \bigcap_{x \in D} ccl\,(G(x)) \neq \phi$ and hence

$$\bigcap_{x \in D} cl_K\,(G(x) \bigcap K) = \bigcap_{x \in D} (ccl\,(G(x)) \bigcap K) = K \bigcap \bigcap_{x \in D} ccl(G(x)) \neq \phi.$$

Now we prove that for any nonempty compact subset K of Y,

$$\bigcap_{x \in D} cl_K\,(G(x) \bigcap K) = \bigcap_{x \in D} (G(x) \bigcap K) = K \bigcap_{x \in D} G(x).$$

Clearly, $\bigcap_{x \in D} (G(x) \bigcap K) \subset \bigcap_{x \in D} cl_K\,(G(x) \bigcap K)$. So we only need to show $\bigcap_{x \in D} cl_K\,(G(x) \bigcap K) \subset \bigcap_{x \in D} (G(x) \bigcap K)$. Suppose, by the way of contradiction, that there is some $y \in \bigcap_{x \in D} cl_K\,(G(x) \bigcap K)$ such that $y \notin \bigcap_{x \in D} (G(x) \bigcap K)$. Then for some $x \in D$, $y \notin G(x) \bigcap K$. By the condition (1) and Definition 3.1, there exists a point $x' \in D$ such that $y \notin cl_K\,(G(x') \bigcap K)$ which is a contradiction. Hence we must have

$$K \bigcap \bigcap_{x \in D} G(x) \neq \phi.$$

Remark 3.2 : If G is compactly closed-valued on X, then G is transfer compactly closed-valued on X and $ccl(G(x)) = G(x)$ for each $x \in X$ by the definition of compact closure of sets. The examples 1 and 2 of Tian[57] can be used to show that Theorem 3.1 is a true generalization of Theorem 3.2 of Ding[13]. Theorem 3.1 also generalizes Theorem 2 of Shioji[47] to H-spaces.

Theorem 3.2 — Let D be a nonempty subset of an H-space $(X, \{\Gamma_A\})$ with compact polytopes, Y be a Hausdorff topological space, $G : D \to 2^Y$. Let $T : H-co(D) \to ka(Y)$ be u.s.c. such that

(1) G is transfer compactly closed-valued in D,

(2) for each $A \in \mathcal{F}(D)$, $T(H - co(A)) \subset (ccl\, G)(A)$.

Furthermore suppose that one of the following conditions is satisfied :

(3) $H - co(D)$ is compact in Y, or

(4) there exist an H-compact subset L of $H - co(D)$ and a nonempty compact subset K of Y such that for each $N \in \mathcal{F}(D)$,

$$T(L_N) \bigcap \bigcap \{ccl\,(G(x)) : x \in L_N \bigcap D\} \subset K.$$

Then
$$cl_Y\, T\,(H - co(D)) \bigcap K \bigcap_{x \in D} G(x) \neq \phi.$$

PROOF : First suppose that the condition (3) is satisfied. Since $H - co(D)$ is compact and $T : H - co(D) \to ka(Y)$ is u.s.c., it follows from Proposition 3.1.11 of Aubin-Ekeland[1] that $T(H - co(D))$ is compact in Y. Letting $K = T(H - co(D))$ the conclusion holds from Theorem 3.1. Next suppose the condition (4) is satisfied. Note that for each $N \in \mathcal{F}(D)$, $(L_N, \{\Gamma_A \bigcap L_N\})$ is a compact H-space and $L \bigcup N \subset L_N$. By applying Theorem 3.1 with (D, X, Y, K, G, T) instead of $(L_N \bigcap D, L_N, Y, T(L_N), G|_{L_N \bigcap D}, T|_{L_N})$, we have

$$T(L_N) \bigcap \bigcap_{x \in L_N \bigcap D} G(x) \neq \phi,$$

and hence, by the condition (4),

$$T(L_N) \bigcap K \bigcap \bigcap_{x \in L_N \bigcap D} G(x) \neq \phi.$$

Since $N \subset L_N$, it follows that

$$cl_Y\, T(H - co(D)) \bigcap K \bigcap \bigcap_{x \in N} G(x) \neq \phi.$$

This show that the family $\{cl_Y\, T(H - co(D)) \bigcap K \bigcap G(x) : x \in D\}$ has the finite intersection property. Since $cl_Y\, T(H - co(D)) \bigcap K$ is compact in Y, therefore the conclusion must hold.

Remark 3.3 : Theorem 3.2 generalizes Theorem 3.3 of Ding[13], Theorem 3 of Shioji[47] and the corresponding results in Bardaro and Ceppitelli[2], Horvath[26], Park[41, 42], Chang[9], Lassonde[35], Ky Fan[19, 21, 22].

Theorem 3.3 — Let D be a nonempty subset of an H-space $(X, \{\Gamma_A\})$ with compact polytopes, Y be a Hausdorff topological space and $G : X \to 2^Y$. Suppose that $T : H - co(D) \to ka(Y)$ is u.s.c. such that

(1) G is transfer compactly closed-valued in D,

(2) for each $A \in \mathcal{F}(D)$, $T(H-co(A)) \subset (ccl\,G)(A)$,

(3) there exist an H-compact subset L of $H-co(D)$ and a nonemtpy compact subset K of Y such that for each $N \in \mathcal{F}(D)$, $X \in L_N \setminus T^+(K)$ implies

$$\bigcap \{ccl(G(z)) : z \in L_N \bigcap D\} \subset Y \setminus T(x).$$

Then
$$cl_Y T(H-co(D)) \bigcap K \bigcap \bigcap \{G(x) : x \in D\} \neq \phi.$$

PROOF : To prove the conclusion, it suffices to show that the condition (3) implies the condition (4) of Theorem 3.2. In fact, if the condition (4) of Theorem 3.2 does not hold, then for any H-compact subset L of $H-co(D)$ and for any nonempty compact subset K of Y, there exist $N \in \mathcal{F}(D)$ and $y \in T(L_N) \bigcap \bigcap_{x \in L_N \bigcap D} ccl\,(G(x))$ such that $y \notin K$. Hence there is a point $x \in L_N$ such that $y \in T(x) \setminus K$ and so $x \in L_N \setminus T^+(K)$ and

$$\bigcap \{ccl(G(z)) : z \in L_N \bigcap D\} \not\subset Y \setminus T(x).$$

This shows that the condition (3) is not satisfied and completes the proof.

Remark 3.4 : Theorem 3.3 extends Theorem 3.4 of Ding[13], Theorem 3 of Park[43] and many known H-FKKM and FKKM type theorems in the literature, see the particular forms of Theorem 3 of Park[43].

Corollary 3.1 — Let X be a nonempty convex subset of a Hausdorff topological vector space and $\phi \neq D \subset X$. Let $G : D \to 2^X$ be a set-valued mapping such that

(1) G is transfer compactly closed-valued in X,

(2) for each $A \in \mathcal{F}(D)$, $co(A) \subset (ccl\ G)(A)$,

(3) there is a nonempty subset D_0 of D such that the intersection $\bigcap_{x \in D_0} ccl\,(G(x))$ is compact and D_0 is contained in a compact convex subset X_0 of X.

Then $co(D) \bigcap \bigcap \{G(x) : x \in D\} \neq \phi$.

PROOF : For each $A \in \mathcal{F}(X)$, let $\Gamma_A = co(A)$. Then $(X, \{\Gamma_A\})$ is an H-space with compact polytopes. Let $X = Y$ and $T : co(D) \to ka(X)$ is defined by $T(x) = \{x\}$ for each $x \in co(D)$. Then T is u.s.c. and the conditions (1) and (2) of Theorem 3.2 are satisfied by the assumptions (1) and (2). Now let $L = X_0$ and $K = \bigcap \{ccl\,(G(x)) : x \in D_0\}$, then K is a nonempty compact subset of X and it is easy to see that $L = X_0$ is a nonempty H-compact subset of X. For each $N \in \mathcal{F}(D)$, let $L_N = co(X_0 \bigcup N)$. Since $D_0 \subset X_0 \bigcap D \subset L_N \bigcap D$, we must have

$$\bigcap_{x \in L_N \cap D} \subset ccl\,(G(x)) \bigcap_{x \in D_0} ccl\,(G(x)) = K$$

and hence the condition (4) of Theorem 3.2 is satisfied. By Theorem 3.2,

$$co(D) \bigcap \bigcap \{G(x) : x \in D\} \neq \phi.$$

Remark 3.5 : Corollary 3.1 improves Theorem 2 of Tian[57].

Corollary 3.2 — Let X be a nonempty convex subset of a Hausdorff topological vector space and $\phi \neq D \subset X$. Let $G : D \to 2^X$ be a set-valued mapping such that

(1) G is transfer compactly closed-valued in D,

(2) for each $A \in \mathcal{F}(D)$, $co(A) \subset (ccl\,G)\,(A)$,

(3) there exists a nonempty subset D_0 of D such that for each $y \in X \setminus D_0$ there is a point $x \in D_0$ with $y \notin ccl\,(G(x))$ and D_0 is contained in a compact convex subset X_0 of X.

Then $$D \bigcap \bigcap \{G(x) : x \in D\} \neq \phi.$$

PROOF : By (3), we have $\bigcap \{ccl\,(G(x)) : x \in D_0\} \subset D_0$ and D_0 is contained in a compact convex subset X_0 of X. Thus we have

$$\begin{aligned} \{ccl\,(G(x)) : x \in D_0\} &= X_0 \bigcap \bigcap \{ccl\,(G(x)) : x \in D_0\} \\ &= \bigcap \{ccl\,(G(x)) \bigcap X_0 : x \in D_0\} \\ &= \bigcap \{cl_{X_0}(G(x) \bigcap X_0) : x \in D_0\} \subset X_0 \end{aligned}$$

and hence $\bigcap \{ccl\,(G(x)) : x \in D_0\}$ is compact. By Corollary 3.1,

$$co(D) \bigcap \bigcap \{G(x) : x \in D\} \neq \phi.$$

Now for any $y \in \bigcap \{G(x) : x \in D\}$, we must have $y \in \bigcap \{ccl\,(G(x)) : x \in D_0\} \subset D_0$ by the condition (3). Hence $y \in D$ and $D \bigcap \bigcap \{G(x) : x \in D\} \neq \phi$.

Remark 3.6 : Corollary 3.2 improves Theorem 3 of Tian[57].

4. Some Applications

In this section, we shall give some applications of our H-KKM type theorems to the geometric properties of H-spaces, coincidence theorems and minimax inequalities in H-spaces.

Theorem 4.1 — Let D be a nonempty subset of an H-space $(X, \{\Gamma_A\})$ with compact polytopes, Y be a Hausdorff topological space, $A \subset B \subset H - co(D) \times Y$ and $C \subset H - co(D) \times Y$ such that A is nonempty closed in $H - co(D) \times Y$. Suppose that

(1) for each $x \in D$, $\{y \in Y : (x, y) \in B\} \subset ccl\,(\{y \in Y : (x, y) \in C\})$,

(2) the mapping $G : D \to 2^Y$ defined by $G(x) = \{y \in Y : (x, y) \in C\}$ is transfer compactly closed-valued in D,

(3) for each $y \in Y$, $\{x \in D : (x, y) \notin B\}$ is empty or H-convex,

(4) there exist a nonempty compact subset K of Y such that for each $x \in H-co(D)$, the set $\{y \in K : (x, y) \in A\}$ is acyclic.

Then there exists a point $y_0 \in K$ such that $D \times \{y_0\} \subset C$.

PROOF : Define the mappings $H : D \to 2^Y$ and $T : H-co(D) \to 2^Y$ by $H(x) = \{y \in Y : (x, y) \in B\}$ and $T(x) = \{y \in K : (x, y) \in A\}$. By (2), G is transfer compactly closed-valued in D. Since A is closed in $H - co(D) \times Y$, each $T(x)$ is closed and the graph of T is closed. From the Corollary 3.19 of Aubin and Ekeland[1] it follows that T is u.s.c. and hence $T : H - co(D) \to ka(Y)$ is u.s.c. by (4). We claim that for each $N \in \mathcal{F}(D)$, $T(H - co(N)) \subset H(N)$. If it is not true, then there exists $N \in \mathcal{F}(D)$ and $y \in T(H-co(N))$ such that $y \notin H(N)$ and hence we have $N \subset \{x \in D : (x, y) \notin B\}$. By (3), we have

$$\begin{aligned} H - co(N) &\subset \{x \in D : (x, y) \notin B\} \\ &\subset \{x \in H-co(D) : (x, y) \notin A\} \\ &= H - co(D) \setminus \{x \in H-co(D) : (x, y) \in A\} \\ &= H - co(D) \setminus T^{-1}(y). \end{aligned}$$

It follows that $y \notin T(H-co(N))$ which is a contradiction. Hence for each $N \in \mathcal{F}(D)$, $T(H-co(N)) \subset H(N) \subset (ccl\, G)(N)$ by (1). The condition (4) implies $T(H-co(D)) \subset K$. By applying Theorem 3.1, $K \bigcap \bigcap_{x \in D} G(x) \neq \phi$. This implies that there exists a point $y_0 \in K$ such that $D \times \{y_0\} \subset C$.

Remark 4.1 : If $B \subset C$, the condition (1) is satisfied trivially and if the set $\{y \in Y : (x, y) \in C\}$ is compactly closed for each $x \in D$, then the condition (2) is satisfied trivially. Hence Theorem 4.1 improves and generalizes Theorem 4.1 of Ding[13], Theorem 10 and Corollary 10.1 of Park[43] and the corresponding results in Shioji[47], Ha[24] and Ky Fan[19].

Theorem 4.2 — Let D be a nonempty subset of an H-space $(X, \{\Gamma_A\})$ with compact polytopes, Y be a Hausdorff topological space, $F : D \to 2^Y$ and $T, S : H - co(D) \to 2^Y$ be such that

(1) for each $A \in \mathcal{F}(D)$, $T|_Z : Z \to ka(Y)$ is u.s.c. where $Z = H - co(A)$,

(2) for each $A \in \mathcal{F}(D)$ and for each $y \in T(H-co(A))$, $y \in \bigcap_{x \in A} cint\,(F(x))$ implies $y \in \bigcap_{x \in H-co(A)} G(x)$,

(3) there exists an $N \in \mathcal{F}(D)$ such that $T(H - co(N)) \subset \bigcup_{x \in N} cint\,(F(x))$.

Then there exists a point $x_0 \in D$ such that $T(x_0) \bigcap S(x_0) \neq \phi$.

PROOF : Define a mapping $G : D \to 2^Y$ by $G(x) = Y \setminus cint\ (F(x))$ for each $x \in D$. Since $cint(F(x))$ is compactly open for each $x \in D$, $G(x)$ is compactly closed for each $x \in D$ and hence the condition (3) of Theorem 2.1 is satisfied. By the assumption (3), there exists $N \in \mathcal{F}(D)$ such that

$$T(H-co(N)) \subset \bigcup_{x \in N} cint\ (F(x)) = \bigcup_{x \in A} (Y \setminus G(x)) = Y \setminus \bigcap_{x \in N} G(x).$$

It follows that $T(H-co(N)) \bigcap \bigcap_{x \in N} G(x) = \phi$ and hence the conclusion of Theorem 2.1 does not hold. Therefore the condition (2) of Theorem 2.1 must not hold. Thus there exists an $A \in \mathcal{F}(D)$ such that $T(H-co(A)) \not\subset G(A)$, that is, there exist $y \in T(H-co(A))$ and $x_0 \in H-co(A)$ such that $y \in T(x_0)$ and

$$y \notin G(A) = \bigcup_{x \in A} (Y \setminus cint\ (F(x))) = Y \setminus \bigcap_{x \in A} cint\ (F(x))$$

and hence $y \in \bigcap_{x \in A} cint\ (F(x))$. By (2), $y \in \bigcap_{x \in H-co(A)} S(x)$. Note $x_0 \in H-co(A)$, we must have $y \in T(x_0) \bigcap S(x_0)$. This completes the proof.

Remark 4.2 : By the condition (1), for each $N \in \mathcal{F}(D)$, $T(H-co(N))$ is compact in Y. If for each $A \in \mathcal{F}(D)$, $F(x) \bigcap T(H-co(A))$ is relatively open in $T(H-co(A))$, then $T(H-co(A)) \subset \bigcup_{x \in A} cint\ (F(x))$ is equivalent to $T(H-co(A)) \subset \bigcup_{x \in A} F(x)$. Clearly, $T(H-co(A)) \subset \bigcup_{x \in A} cint\ (F(x))$ implies $T(H-co(A)) \subset \bigcup_{x \in A} F(x)$. Conversely, if $T(H-co(A)) \subset \bigcup_{x \in A} F(x)$, then we have

$$\begin{aligned} T(H-co(A)) &\subset \bigcup_{x \in A} (T(H-co(A)) \bigcap F(x)) \\ &= \bigcup_{x \in A} int_{T(H-co(A))} (T(H-co(A)) \bigcap F(x)) \\ &= \bigcup_{x \in A} cint\ (F(x)) \bigcap T(H-co(A)) \\ &\subset \bigcup_{x \in A} cint\ (F(x)). \end{aligned}$$

Hence Theorem 4.2 improves Theorem 4.2 of Ding[13].

Theorem 4.3 — Let D be a nonempty subset of an H-space $(X, \{\Gamma_A\})$ with compact polytopes, Y be a Hausdorff topological space, $F : D \to 2^Y$ and $S : H-co(D) \to 2^Y$. Let $T : H-co(D) \to ka(Y)$ be u.s.c. and K be a nonempty compact subset of Y such that

(1) F if transfer compactly open-valued in D,

(2) for each $A \in \mathcal{F}(D)$ and for each $y \in T(H-co(A))$, $y \in \bigcap_{x \in A} cint\ (F(x))$

implies $y \in \bigcap_{x \in H-co(A)} S(x)$,

(3) $cl_Y T(H-co(D)) \bigcap K \subset F(D)$,

(4) there exists an H-compact subset L of $H-co(D)$ such that for each $N \in \mathcal{F}(D)$, $x \in L_N \setminus T^+(K)$ implies $T(x) \subset \bigcup_{x \in L_N \bigcap D} cint\,(F(x))$.

Then there exists a point $x_0 \in D$ such that $T(x_0) \bigcap S(x_0) \neq \phi$.

PROOF : Define a mapping $G : D \to 2^Y$ by $G(x) = Y \setminus F(x)$ for each $x \in D$. By (1), G is transfer compactly closed-valued in D. By (4), there exist an H-compact subset L of $H - co(D)$ such that for each $N \in \mathcal{F}(D)$ and for each $y \in T(L_N) \setminus K$, there is $x* \in L_N$ such that $y \in T(x*) \setminus K$ and hence $x* \in L_N \setminus T^+(K)$. By (4), we have

$$y \in T(x*) \subset \bigcup_{x \in L_N \bigcap D} cint\,(F(x)).$$

It follows that

$$y \in \bigcup_{x \in L_N \bigcap D} (cint\,(F(x)) \bigcap T(L_N))$$

$$= \bigcup_{x \in L_N \bigcap D} int_{T(L_N)}\,(F(x) \bigcap T(L_N))$$

and hence there is an $x \in L_N \bigcap D$ such that $y \in \mathrm{int}_{T(L_N)}\,(F(x) \bigcap T(L_N))$. It follows that

$$y \notin cl_{T(L_N)}\,(T(L_N) \setminus F(x))$$

$$= cl_{T(L_N)}\,(T(L_N) \bigcap (Y \setminus F(x)))$$

$$= cl_{T(L_N)}\,(T(L_N) \bigcap G(x)).$$

Hence we have that for each $N \in \mathcal{F}(D)$,

$$T(L_N) \bigcap \bigcap_{x \in L_N \bigcap D} ccl\,(G(x)) \subset K$$

and the condition (4) of Theorem 3.2 is satisfied. It is easy to see that the condition (3) implies

$$cl_Y T(H-co(D)) \bigcap K \bigcap \bigcap_{x \in D} G(x) = \phi.$$

and hence the conclusion of Theorem 3.2 does not hold. It follows from Theorem 3.2 that there exists an $A \in \mathcal{F}(D)$ such that $T(H - co(A)) \not\subset (ccl\,G)\,(A)$. Thus there exist $y_0 \in T(H-co(A))$ and $x_0 \in H-co(A)$ such that $y_0 \in T(x_0)$ and $y \notin \bigcup_{x \in A} ccl\,(G(x))$. Note $T(x_0)$ is compact, we have

$$
\begin{aligned}
y \notin \bigcup_{x\in A} & (ccl\,(G(x)) \bigcap T(x_0)) \\
&= \bigcup_{x\in A} cl_{T(x_0)}\,(T(x_0) \bigcap G(x)) \\
&= \bigcup_{x\in A} cl_{T(x_0)}\,(T(x_0) \bigcap (Y \setminus F(x))) \\
&= \bigcup_{x\in A} cl_{T(x_0)}\,(T(x_0) \setminus (T(x_0) \bigcap F(x))) \\
&= T(x_0) \setminus \bigcap_{x\in A} int_{T(x_0)}\,(T(x_0) \bigcap F(x)).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
y \in \bigcap_{x\in A} & int_{T(x_0)}\,(T(x_0) \bigcap F(x)) \\
&= \bigcap_{x\in A} (T(x_0) \bigcap cint\,(F(x))) \\
&\subset \bigcap_{x\in A} cint\,(F(x)).
\end{aligned}
$$

By the condition (2), we have

$$
y \in \bigcap_{x\in H-co(A)} S(x).
$$

Since $x_0 \in H-co(A)$, we must have $y \in T(x_0) \bigcap S(x_0)$. This completes our proof.

Remark 4.3 : Since $T(x)$ is compact for each $x \in H-co(D)$, if for each $x \in D$, $F(x)$ is compact open, then it is easy to see that $T(x) \subset F(L_N \bigcap D)$ is equivalent to $T(x) \subset \bigcup_{x\in L_N \bigcap D} cint\,(F(x))$. Hence Theorem 4.3 improves and generalizes Theorem 4.3 of Ding[13], Theorem 1 of Park[43] and the corresponding results given in references [4-11], [13, 28, 29, 30], [32, 33, 35], [38-43], [45-48], [50-56] and [59].

Theorem 4.4 — Let D be a nonempty subset of an H-space $(X, \{\Gamma_A\})$ with compact polytopes, Y be a Hausdorff topological space and $T : H-co(D) \to ka(Y)$ be u.s.c. Let M and Q be subsets of a set Z, $g : D \times Y \to Z$ and $f : H-co(D) \times Y \to Z$ be such that

(1) the mapping $F : D \to 2^Y$ defined by $F(x) = \{y \in Y : g(x, y) \in M\}$ is transfer compactly open-valued in D,

(2) for each $A \in \mathcal{F}(D)$ and for each $y \in T(H-co(A))$, $y \in \bigcap_{x\in A} cint\,(F(x))$ implies $y \in \bigcap_{x\in H-co(A)} \{z \in Y : f(x, z) \in Q\}$,

(3) there exist an H-compact subset L of $H-co(D)$ and a nonempty compact

subset K of Y such that for each $N \in \mathcal{F}(D)$, $x \in L_N \setminus T^+(K)$ and $y \in T(x)$, there exists an $x_1 \in L_N \bigcap D$ satisfying $y \in cint(F(x_1))$.

Then either (a) there exists an $y^* \in cl_Y T(H-co(D)) \bigcap K$ such that $g(x,y) \notin M$ for all $x \in D$, or (b) there exist $x^* \in D$ and $y^* \in T(x^*)$ such that $f(x^*, y^*) \in Q$.

PROOF : Define a mapping $S : H - co(D) \to 2^Y$ by $S(x) = \{y \in Y : f(x, y) \in Q\}$, for each $x \in H - co(D)$. It is easy to see that the conditions (1), (2) and (4) of Theorem 4.3 are satisfied by the assumptions (1), (2) and (3). Suppose the conclusion (a) does not hold, then $cl_Y T\ (H - co(D)) \bigcap K \subset F(D)$ and the condition (3) of Theorem 4.3 is also satisfied. By Theorem 4.3, there exists a point $x^* \in D$ such that $T(x^*) \bigcap S(x^*) \neq \phi$, that is there is an $y^* \in T(x^*)$ such that $f(x^*, y^*) \in Q$.

Remark 4.4 : Theorem 4.4 generalizes Theorem 4.4 of Ding[13], Theorem 2.4 of Ding[11], Theorem 5 of Park[43] and many known results in literature, see the particular forms of Theorem 5 of Park[43].

Definition 4.1 — Let X and Y be two topological spaces. A function $g : X \times Y \to \mathbf{R} \bigcup \{\pm\infty\}$ is said to be γ-transfer compactly lower semicontinuous in y if for each nonempty compact subset K of Y and for all $x \in X$ and $y \in K$, $f(x, y) > \gamma$ implies that there exist some point $x' \in X$ and some relatively open neighbourhood $\mathcal{N}(y)$ of y in K such that $f(x', z) > \gamma$ for all $z \in \mathcal{N}(y)$.

It is clear that γ-transfer compact lower semicontinuity is a generalization of the γ-transfer lower semicontinuity introduced by Tian[57].

Theorem 4.5 — Let D be a nonempty subset of an H-space $(X, \{\Gamma_A\})$ with compact polytopes, Y be a Hausdorff topological space and $T : H - co(D) \to ka(Y)$ be u.s.c. Suppose that the functions $g : D \times Y \to \mathbf{R} \bigcup \{\pm\infty\}$ and $f : H - co(D) \times Y \to \mathbf{R} \bigcup \{\pm\infty\}$ satisfy the following conditions :

(1) $g(x, y)$ is γ-transfer compactly lower semicontinuous in y where $\gamma = \sup\{f(x,y) : (x,y) \in Gr(T)\}$ and $Gr(T) = \{(x, y) \in H - co\ (D) \times Y : y \in T(x)\}$ is the graph of T,

(2) for each $A \in \mathcal{F}(D)$ and for each $y \in T(H-co(A))$, $y \in \bigcap_{x \in A} cint\ (\{y \in Y : g(x, y) > \gamma\})$ implies $y \in \bigcap_{x \in H-co(A)} \{y \in Y : f(x, y) > \gamma\}$,

(3) there exist an H-compact subset L of $H - co(D)$ and a nonempty compact subset K of Y such that for each $N \in \mathcal{F}(D)$, $x \in L_N \setminus T^+(K)$ and $y \in T(x)$, there is an $x_1 \in L_N \bigcap D$ satisfying $y \in cint(\{z \in Y : g(x_1, z) > \gamma\}$.

Then (a) there exists a point $y^* \in cl_Y T(H-co(D)) \bigcap K$ such that

$$\sup_{x \in D} g(x, y^*) \leq \sup_{(x,y) \in Gr(T)} f(x, y),$$

and (b) the following minimax inequality holds :

$$\inf_{y\in K}\sup_{x\in D} g(x,y)\le \sup_{(x,y)\in Gr(T)} f(x,y).$$

PROOF : It is clear that the conclusion (a) implies the conclusion (b). In order to show the conclusion (a) we may assume that $\gamma = \sup\{f(x,y):(x,y)\in Gr(T)\}$ is finite. In Theorem 4.4, put $Z = \mathbf{R}\bigcup\{\pm\infty\}$, $M = Q = (\gamma,+\infty]$. Then by (1), (2) and (3), it is easy to check that all hypotheses of Theorem 4.4 are satisfied. Obviously, the conclusion (b) of Theorem 4.4 does not hold. Hence we conclude that the conclusion (a) of Theorem 4.4 holds, that is there exists a point $y^*\in cl_Y T(H-co(D))\bigcap K$ such that

$$\sup_{x\in D} g(x,y^*)\le \sup_{(x,y)\in Gr(T)} f(x,y).$$

Remark 4.5 : Theorem 4.5 improves and generalizes Theorem 4.5 of Ding[13], Theorem 9 of Park[43], Theorem 1 of Ha[25], Theorem 1 of Fan[20] and many known minimax inequalities in the literature, see the particular forms of Theorem 9 of Park[43].

Corollary 4.1 — Let D be a nonempty subset of a Hausdorff H-space $(X,\{\Gamma_A\})$ with compact polytopes, $\gamma\in\mathbf{R}$ and $g : D\times X\to\mathbf{R}\bigcup\{\pm\infty\}$ be such that

(1) $g(x,y)$ is γ-transfer compactly lower semicontinuous in y,

(2) for each $A\in\mathcal{F}(D)$, $H-co(D)\subset\bigcup_{x\in A} ccl\,(\{y\in X: g(x,y)\le\gamma\})$,

(3) there exist an H-compact subset L of $H-co(D)$ and a nonempty compact subset K of X such that for each $Y\in L_N\setminus K$ there is an $x\in L_N\bigcap D$ satisfying $y\in cint(\{z\in X: g(x,z)>\gamma\}$.

Then there exists a point $y^*\in cl_X(H-co(D))\bigcap K$ such that $g(x,y^*)\le\gamma$ for all $x\in D$.

PROOF : In Theorem 4.5, let $Y = X$, $f(x,y) = \gamma$ for all $(x,y)\in H-co(D)\times X$ and T be the identity mapping. In this case, we have $\bigcap_{x\in H-co(D)}\{y\in X: f(x,y)>\gamma\} = \phi$ and hence the condition (2) of Theorem 4.5 reduces to that for each $A\in\mathcal{F}(D)$ and for each $y\in H-co(A)$, $y\notin\bigcap_{x\in A} cint\,(\{y\in X: g(x,y)>\gamma\})$. Hence we have

$$\begin{aligned} y &\in X\setminus\bigcap_{x\in A} cint\,(\{y\in X: g(x,y)>\gamma\})\\ &= \bigcup_{x\in A}(X\setminus cint\,(\{y\in X: g(x,y)>\gamma\})\\ &= \bigcup_{x\in A} ccl\,(\{y\in X: g(x,y)\le\gamma\}).\end{aligned}$$

Hence, by applying Theorem 4.5 to the special case, there exists a point $y^*\in cl_X(H-co(D))\bigcap K$ such that $g(x,y^*)\le\gamma$ for all $x\in D$.

Remark 4.6 : Corollary 4.1 improves and generalizes Theorem 4 of Tian[57] to H-spaces.

5. Existence of Maximal Elements of Binary Relations

In the literature, there are two ways to nontransitive and nontotal preference theory : one through reflexive preferences (see, e.g., Sonnenschien[49] and Shafer and Sonnenschein[46]), and the other through pirreflexive references (see, e.g., Schmeidler[44], Mas-Colell[36], Gale and Mas-Colell[23], Yannelis and Prabhakar[59], Ding-Kim-Tan[14] and Ding and Tan[15,16]. In this section, we shall generalize the existence theorem of maximal elements for both types of preferences by relaxing the compactness and convexity conditions and the consumption spaces. Recall that a binary relation on a nonempty set X is a nonempty subset of $X \times X$. A binary relation $\succsim$ (resp.,$\succ$) defined on X is said to be a weak (resp., strict) preference relation if $\succsim$ is reflexive (resp., $\succ$ is irreflexive), where X may be considered as a consumption space. Let $\succsim$ be a weak preference relation on X. An element $(x, y) \in \succsim$ is written as $x \succsim y$ and read as "x is at least as good as y". Let $\succ$ be a strict preference relation on X. An element $(x, y) \in \succ$ is written as $x \succ y$ and read as "x is (strictly) preferred to y". For each $x \in X$, the weakly upper, weakly lower, strictly upper, and strictly lower contour sets (sections) of x are denoted by

$$U_w(x) = \{y \in X : y \succsim x\},$$

$$L_w(x) = U_w^{-1}(x) = \{y \in X : x \rangle \succsim\},$$

$$U_s(x) = \{y \in X : y \succ x\},$$

$$L_s(x) = U_s^{-1}(x) = \{y \in X : x \succ y\},$$

respectively.

In some cases, not all points in X can be chosen, so let $D \subset X$ be a choice set, which may be considered as, say, the budget set of feasible sèt.

Definition 5.1 — A weak preference relation $\succsim$ is said to have a greatest element on the subset D of X if there exists a point $x^* \in D$ such that $x^* \succsim x$ for all $x \in D$, or equivalently $\bigcap_{x \in D} U_w(x) \neq \phi$ on D.

Definition 5.2 — A strict preference relation $\succ$ is said to be have a maximal element on the subset D of X if there exists a point $x^* \in D$ such that $x \not\succ x^*$ for all $x \in D$, i.e., $U_s(x^*) = \phi$ on D, where $x \not\succ x^*$ means that (x, x^*) is not in $\succ$.

Definition 5.3 — Let D be a nonempty subset of an H-space $(X, \{\Gamma_A\})$. A correspondence $F : X \rightarrow 2^X$ is said to be generalized SS-convex on D if for each $A \in \mathcal{F}(D)$ and $x \in H-co(A)$, there exists a point $x_0 \in A$ such that $x_0 \notin F(x)$.

By the way, we point out that, in Definition 11 of Tian[57], "$F : X \rightarrow 2^Y$" must be replaced by "$F : Y \rightarrow 2^Y$". Otherwise $x_0 \in co\ \{x_1, ..., x_m\}$ may not be in the domain of F since X may not be convex.

In order to prove our main results in this section, we need the following consequence of Theorem 3.2.

Theorem 5.1 — Let D be a nonempty subset of a Hausdorff H-space $(X, \{\Gamma_A\})$ with compact polytopes and $G : D \to 2^X$. Let $T : H-co(D) \to ka(X)$ be u.s.c. such that

(1) G is transfer compact closed-valued in D,

(2) for each $A \in \mathcal{F}(D)$, $T(H-co(A)) \subset \bigcup_{x \in A} ccl\,(G(x))$,

(3) there exists an H-compact subset L of D such that for each $N \in \mathcal{F}(D)$ and for each $y \in T(L_N) \setminus L$, there is a point $x \in L$ with $y \notin T(L_N) \bigcap ccl\,(G(x))$.

Then $\quad D \bigcap \bigcap_{x \in D} G(x) \neq \phi.$

PROOF : In Theorem 3.2, let $X = Y$. By (3), we have $\bigcap_{x \in L} (T(L_N) \bigcap ccl\,(G(x))) = \bigcap_{x \in L} cl_{T(L_N)}(T(L_N) \bigcap G(x)) \subset L \subset L_N$ for any $N \in \mathcal{F}(D)$. Since L_N and $T(L_N)$ are compact, $K = \bigcap_{x \in L} (T(L_N) \bigcap ccl(G(x)))$ is also a compact subset of X and for each $y \in T(L_N) \setminus K$, there is a point $x \in L \subset L_N \bigcap D$ with $y \notin T(L_N) \bigcap ccl\,(G(x))$. Hence for each $N \in \mathcal{F}(D)$,

$$T(L_N) \bigcap \bigcap_{x \in L_N \cap D} ccl\,(G(x)) \subset K$$

and the condition (4) of Theorem 3.2 is satisfied. By Theorem 3.2, $K \bigcap \bigcap_{x \in D} G(x) \neq \phi$. Since $K \subset L \subset D$, we must have $D \bigcap \bigcap_{x \in D} G(x) \neq \phi$.

Remark 5.1 : Theorem 5.1 improves and generalized Theorem 3 of Tian[57] in several aspects.

Theorem 5.2 — Let D be a nonempty subset of a Hausdorff H-space $(X, \{\Gamma_A\})$ with compact polytopes, $T : H-co(D) \to ka(X)$ be u.s.c. and $\succeq$ defined on X be a weak preference relation such that

(1) U_w is transfer compactly closed-valued in D,

(2) for each $A \in \mathcal{F}(D)$, $T(H-co(A)) \subset \bigcup_{x \in A} ccl\,(U_w(x))$,

(3) there exists an H-compact subset L of D such that for each $N \in \mathcal{F}(D)$ and for each $y \in T(L_N) \setminus L$, there is a point $x \in L$ with $y \notin T(L_N) \bigcap ccl\,(U_w(x))$.

Then $\succeq$ has the greatest element in D.

PROOF : For each $x \in D$, let $G(x) = U_w(x)$. Then all assumptions of Theorem 5.1 are satisfied. By Theorem 5.1, $D \bigcap \bigcap_{x \in D} U_w(x) \neq \phi$. Thus there exists a point $x_* \in D \bigcap \bigcap_{x \in D} U_w(x)$ which means $x_* \succeq x$ for all $x \in D$.

Remark 5.2 : Theorem 5.2 improves and generalizes Theorem 5 of Tian[57] in several aspects.

Theorem 5.3 — Let D be a nonempty subset of a Hausdorff H-space $(X, \{\Gamma_A\})$ with compact polytopes, $T: H-co(D) \rightarrow ka(X)$ be u.s.c. and $\succ$ defined on X be a strict preference relation such that

(1) L_s is transfer compact open-valued in D,

(2) for each $A \in \mathcal{F}(D)$ and for each $y \in T(H-co(A)$, there exists $x \in A$ such that $y \in T(H-co(A)) \bigcap cint(L_s(x))$,

(3) there exists an H-compact subset L of D such that for each $N \in \mathcal{F}(D)$ and for each $y \in T(L_N) \setminus L$, there is a point $x \in L$ with $y \in T(L_N) \bigcap cint(L_s(x))$.

Then $\succ$ has a maximal element on D.

PROOF : For each $x \in D$, let $G(x) = X \setminus L_s(x)$. Then G is transfer compactly closed-valued in D. Since $T(L_N)$ is compact, we have that

$$\begin{aligned} y &\in T(L_N) \bigcap cint(L_s(x)) \\ &= int_{T(L_N)} (T(L_N) \bigcap L_s(x)) \end{aligned}$$

which implies

$$\begin{aligned} y &\notin T(L_N) \setminus int_{T(L_N)} (T(L_N) \bigcap L_s(x)) \\ &= cl_{T(L_N)} (T(L_N) \setminus (T(L_N) \bigcap L_s(x))) \\ &= cl_{T(L_N)} (T(L_N) \bigcap (X \setminus L_s(x))) \\ &= cl_{T(L_N)} (T(L_N) \bigcap G(x)) \\ &= T(L_N) \bigcap ccl(G(x)) \end{aligned}$$

and hence the condition (3) of Theorem 5.1 is satisfied. By (2), for each $A \in \mathcal{F}(D)$, we have

$$\begin{aligned} T(H-co(A)) &\subset \bigcup_{x \in A} (T(H-co(A)) \bigcap cint\,(L_s\,(x))) \\ &= \bigcup_{x \in A} int_{T(H-co(A))} (T(H-co(A)) \bigcap L_s\,(x)) \\ &= \bigcup_{x \in A} int_{T(H-co(A))} (T(H-co(A)) \bigcap (X \setminus G(x))) \\ &= \bigcup_{x \in A} int_{T(H-co(A))} (T(H-co(A)) \setminus (T(H-co(A)) \bigcap G(x))) \\ &= \bigcup_{x \in A} cl_{T(H-co(A))} (T(H-co(A)) \bigcap G(x)) \end{aligned}$$

$$= \bigcup_{x \in A} (T(H - co(A)) \bigcap ccl\,(G(x))$$

$$\subset \bigcup_{x \in A} ccl\,(G(x)).$$

Thus, the condition (2) of Theorem 5.1 is also satisfied. By Theorem 5.1, $D \bigcap \bigcap_{x \in D} G(x) \neq \phi$. So there exists a point $x^* \in D$ such that $U_s(x^*) = \phi$, i.e., x^* is a maximal element of $\rangle$ on D.

Remark 5.3 : Theorem 5.3 improves and generalizes Theorem 6 of Tian[57] in several aspects.

References

1. J. P. Aubin and I. Ekeland, *Applied Nonlinear Analysis*, John Wiley and Sons, New York, 1984.
2. C. Bardaro and R. Ceppitelli, *J. Math. Anal. Appl.* **132**(1988), 484-90.
3. C. Bardaro and R. Ceppitelli, *J. Math. Anal. Appl.* **137**(1989), 46-58.
4. H. Ben-El-Mechaiekh, P. Deguire and A. Granas, *C. R. Acad. Sci. Paris* **295**(1982), 257-59.
5. H. Ben-El-Mechaiekh, P. Deguire and A. Granas, *C. R. Acad. Sci. Paris* **259**(1982), I, 337-40; II, 281-384.
6. F. E. Browder, *Math. Ann.* **177**(1968), 283-301.
7. F. E. Browder, *Proc. Nat Acad. Sci. USA* **74**(1977), 4749-51.
8. F. E. Browder, *Contemp. Math.* **26**(1984), 67-80.
9. S. Y. Chang, *Soochow J. Math.* **15**(1989), 7-17.
10. S. S. Chang and Y. H. Ma, *J. Math. Anal. Appl.* **163**(1992), 406-21.
11. X. P. Ding, *J. Sichuan Normal Univ.* **14**(2)(1991), 27-32.
12. X. P. Ding, *Acta Math. Sinica* **36**(2)(1993), 273-79.
13. X. P. Ding, *Bull. Austral. Math. Soc.* **48**(1993), 451-64.
14. X. P. Ding, W. K. Kim and K. K. Tan, *J. Math. Anal. Appl.* **162**(1992), 508-17.
15. X. P. Ding and K. K. Tan, in : *Fixed Point Theory and Applications* (Ed. K. K. Tan), World Sci, Singapore, London, 1992, pp. 80-96.
16. X. P. Ding and K. K. Tan, *J. Math. Anal. Appl.* **177**(1993), 226-38.
17. X. P. Ding and K. K. Tan, *J. Austral. Math. Soc. (ser. A)* **49**(1990), 111-28.
18. J. Dugundji, *Topology*, Allyn and Bacon, Inc., Boston, 1966.
19. Ky Fan, *Math. Ann.* **142**(1961), 305-10.
20. Ky Fan, in : *Inequalities III* (Ed. O. Shisha), Acad. Press, New York, 1972, pp. 103-13.
21. Ky Fan, in : *Game Theory and Related Topics* (Eds. Moeshlin and Pallaschke), North-Holland, Amsterdam 1979, pp. 151-56.
22. Ky Fan, *Math. Ann.* **226** (1984), 519-37.
23. D. Gale and A. Mas-Colell, *J. Math. Econom.* **2**(1975), 9-15.
24. C. W. Ha, *Math. Ann.* **248**(1980), 73-77.
25. C. W. Ha, *Proc. Am. Math. Soc.* **99**(1987), 680-82.
26. C. Horvath, in : *Nonlinear and Convex Analysis* (Eds. Lin and Simons), Dekker, New York, 1987, pp. 99-106.
27. C. Horvath, *J. Math. Anal. Appl.* **156** (1991), 341-57.
28. J. Jiang, *Acta Math. Sinica* **4**(1988), I, 64-71; II, 234-41.
29. J. Jiang, *Acta Math. Sinica* **4**(1988), 356-63.
30. J. Jiang, *Acta Math. Sinica* **5**(1989), 307-20.

31. J. L. Kelley, *General Topology*, Springer-Verlag, 1955.
32. H. M. Ko and K. K. Tan, *Tamkang J. Math.* **17**(1986), 37-45.
33. H. Komiya, *Proc. Am. Math. Soc.* **96**(1986), 599-602.
34. B. Knaster, C. Kuratowski and S. Mazurkiewicz, *Fund. Math.* **14**(1929), 132-37.
35. M. Lassonde, *J. Math. Anal. Appl.* **97**(1983), 151-201.
36. A. Mas-Colell, *J. Math. Econom.* **1**(1974), 237-46.
37. J. F. McClendon, *Proc. Am. Math. Soc.* **90**(1984), 149-54.
38. L. McLinden, in : *Resumes des Colloque International : Theorie du Point Fixe et Applications*", 5-9 Juin 1989, Marseille-Luminy, pp. 150-51.
39. G. Mehta, *Comm. Math. Univ. Caril.* **28**(1987), 377-85.
40. G. Mehta and E. Tarafdar, *J. Econom. Theory* **41**(1987), 333-39.
41. S. Park, *J. Math. Anal. Appl.* **141**(1989), 164-76.
42. S. Park, *J. Korean Math. Soc.* **28**(1991), 275-83.
43. S. Park, in : *Fixed Point Theory and Applications* (Ed. K. K. Tan), World Sci. Singapore, London, 1992, pp. 248-77.
44. D. Schmeidler, *Econometrica* **37**(1969), 578-85.
45. S. Sessa, *Comm. Math. Univ. Carol.* **29**(1988), 567-75.
46. W. Shafer and H. Sonnenschein, *J. Math. Econom.* **2**(1975), 345-48.
47. N. Shioji, *Proc. Am. Math. Soc.* **111**(1991), 187-95.
48. S. Simons, *Proc. Symp. Pure Math.* **45**(1986), Pt. 2, pp. 377-92.
49. H. Sonnenschein, in : *Preferences, Utility and Demand* (Eds. Chipman, Hurwicz, Richter and Sonnenschein), Harcourt Brace Jovanovich, New York, 1971.
50. W. Takahashi, *Proc. Symp. Pure Math.* **45**(1986), 419-24.
51. E. Tarafdar, *Proc. Am. Math. Soc.* **67**(1977), 95-98.
52. E. Tarafdar, *Publ. Math. Debrecen* **29**(1982), 219-26.
53. E. Tarafdar, *Indian J. Math.* **28**(1986), 229-40.
54. E. Tarafdar, *J. Math. Anal. Appl.* **63**(1987), 475-79.
55. E. Tarafdar, *Bull. Austral. Math. Soc.* **42**(1990), 133-40.
56. E. Tarafdar and T. Husain, *J. Math Anal. Appl.* **63**(1978), 371-76.
57. D. Tian, *J. Math. Anal. Appl.* **170**(1992), 457-71.
58. A. Wilansky, *Topology for Analysis*, Waltham Massachusetts, Toronto, 1972.
59. N. Yannelis and N. D. Prabhakar, *J. Math. Econom.* **12**(1983), 233-45; *Erratum* **13**(1984), 305.