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SOME PROPERTIES OF A RICCI QUARTER SYMMETRIC METRIC CONNECTION IN A RIEMANNIAN MANIFOLD

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The present paper deals with an Einstein manifold M admitting a Ricci Quarter symmetric metric connection $\bar{\nabla}$ whose torsion tensor

$$T(X, Y) = \pi(Y)\, LX - \pi(X)\, LY$$

where π is a differentiable 1-form on M and L is the (1, 1) Ricci tensor defined by

$$g(LX, Y) = S(X, Y),$$

S is the Ricci tensor of M. Some properties of the manifold which depends upon the Ricci Quarter symmetric metric connection are studied.

1. INTRODUCTION

A linear connection $\bar{\nabla}$ in a Riemannian manifold M^n is said to be a Ricci Quarter symmetric connection if its torsion tensor T

$$T(X, Y) = \pi(Y)\, LX - \pi(X)\, LY \qquad \text{... (1.1)}$$

where π is a 1-form and L is the (1, 1) Ricci tensor defined by

$$g(LX, Y) = S(X, Y), \qquad \text{... (1.2)}$$

S is the Ricci tensor of M^n.

A linear connection $\bar{\nabla}$ is called a metric connection[1] if

$$(\bar{\nabla}_X g)(Y, Z) = 0. \qquad \text{... (1.3)}$$

If ∇ is the Levi-Civita connection of the manifold then a Ricci Quarter symmetric metric connection is given by Mishra and Pandey[2]

$$\bar{\nabla}_X Y = \nabla_X Y + \pi(Y)\, LX - S(X, Y)\, \rho \qquad \text{... (1.4)}$$

$$\pi(X) = g(X, \rho).$$

The present paper deals with an Einstein manifold admitting a Ricci Quarter symmetric metric connection. In this paper we have deduced necessary and sufficient conditions for the symmetry of the Ricci tensor of a Ricci quarter symmetric metric connection. In the last section it is shown that the conformal curvature tensors of ∇ and $\overline{\nabla}$ are equal. Also it is shown that if the curvature tensor of $\overline{\nabla}$ vanishes, then the manifold is of constant curvature. Finally it is shown that if the curvature tensor $\overline{R}$ of $\overline{\nabla}$ is of the form

$$\overline{R}(X, Y)Z = k\,[g(Y, Z)\,X - g(X, Z)\,Y],$$

where k is a constant then the manifold is of constant curvature.

2. Preliminaries

Let $\overline{R}$ and R be the curvature tensors of the connection $\overline{\nabla}$ and ∇ respectively. Then it can be easily proved that[2]

$$\begin{aligned}\overline{R}(X, Y)Z = R(X, Y)\,Z - M(Y, Z)\,LX + M(X, Z)\,LY \\ - S(Y, Z)\,QX + S(X, Z)\,QY + \pi(Z)\,[(\nabla_X L)\,(Y) - (\nabla_Y L)\,(X)\,] \\ - [(\nabla_X S)\,(Y, Z) - (\nabla_Y S)\,(X, Z)]\,\rho \end{aligned} \quad \text{... (2.1)}$$

where M is a tensor of type (0, 2) defined by

$$M(X, Y) = g(QX, Y) = (\nabla_X \pi)\,(Y) - \pi(Y)\,\pi(LX) + \frac{1}{2}\,\pi(\rho)\,S(X, Y) \quad \text{... (2.2)}$$

and Q is a tensor field of type (2.1) defined by

$$QX = \nabla_X \rho - \pi\,(LX)\,\rho + \frac{1}{2}\,\pi\,(\rho)\,LX. \quad \text{... (2.3)}$$

Here we shall consider M^n to be an Einstein manifold, that is,

$$S(X, Y) = \frac{r}{n}\,g(X, Y) \quad \text{... (2.4)}$$

where r is the scalar curvature.

Considering (2.1) and (2.4) we get

$$\begin{aligned}\overline{R}(X, Y)Z = R(X, Y)\,Z - \frac{r}{n}\,[M(Y, Z)\,X - M(X, Z)\,Y \\ + g(Y, Z)\,QX - g(X, Z)\,QY]. \end{aligned} \quad \text{... (2.5)}$$

Contracting (2.5) with respect to X, we get

$$\overline{S}(Y, Z) = \frac{r}{n}\,[g(Y, Z) - \{(n-2)\,M(Y, Z) + mg(Y, Z)\}] \quad \text{... (2.6)}$$

where $\overline{S}$ is the Ricci tensor of $\overline{\nabla}$ and m is the trace of the tensor $M(Y, Z)$.

Now putting $Y = Z = e_i$, where $\{e_i\}$, $i = 1, 2, ..., n$ is an orthonormal basis of the tangent space at any point, we get by taking the sum for $1 \le i \le n$ of the relation (2.6)

$$\bar{r} = \frac{r}{n}[n - 2(n-1)m] \qquad \text{... (2.7)}$$

where $\bar{r}$ denotes the scalar curvature of $\bar{\bar{\nabla}}$.

3. Symmetry Condition of the Ricci Tensor of $\bar{\bar{\nabla}}$

From (2.6) it follows that

$$\bar{S}(Y, Z) - \bar{S}(Z, Y) = \frac{r}{n}(n-2)[M(Z, Y) - M(Y, Z)]. \qquad \text{... (3.1)}$$

From (3.1) we can state the following theorem :

Theorem 1 — If an Einstein manifold M^n $(n > 3)$ with non-zero scalar curvature admits a Ricci Quarter symmetric metric connection ∇, then a necessary and sufficient condition for the Ricci tensor of ∇ to be symmetric is that the tensor M is symmetric.

Next, we shall find another necessary and sufficient condition for which the Ricci tensor of the Quarter symmetric metric connection ∇ will be symmetric.

From (2.5) we have

$$'\bar{R}(X, Y, Z, U) = 'R(X, Y, Z, U) - \frac{r}{n}[M(Y, Z)\, g(X, U) - M(X, Z)\, g(Y, U) + g(Y, Z)\, M(X, U) - g(X, Z)\, M(Y, U)] \qquad \text{... (3.2)}$$

where

$$'R(X, Y, Z, U) = g(R(X, Y)Z, U).$$

From (3.2) we get

$$'\bar{R}(X, Y, Z, U) - '\bar{R}(Z, U, X, Y)$$
$$= \frac{r}{n}[\{M(Z, Y) - M(Y, Z)\}\, g(X, U) + \{M(Z, X) - M(X, Z)]\, g(Y, U) + \{M(U, X) - M(X, U)\}\, g(Y, Z) + \{M(U, Y) - M(Y, U)\}\, g(X, Z)\}] \qquad \text{... (3.3)}$$

for $\quad 'R(X, Y, Z, U) = 'R(Z, U, X, Y)$ and $g(X, Y) = g(Y, X)$.

Now $'R(X, Y, Z, U) = '\bar{R}(Z, U, X, Y)$ if and only if

$$g(X, U)\,[M(Z, Y) - M(Y, Z)] + g(Y, U)\,[M(Z, X) - M(X, Z)]$$
$$+\, g(Y, Z)\,[M(U, X) - M(X, U)] + g(X, Z)\,[M(U, Y) - M(Y, U)] = 0. \quad ...(3.4)$$

Transvecting (3.4) we get $(n + 2)\,[M(Z, Y) - M(Y, Z)] = 0$.

That is, $$M(Y, Z) - M(Z, Y) = 0. \quad ...\ (3.5)$$

Thus $'\bar{R}\,(X, Y, Z, U) = '\bar{R}\,(Z, U, X, Y)$ if and only if $M(Y, Z) - M(Z, Y) = 0$. That is, if and only if the Ricci tensor $\bar{S}$ for the connection $\bar{\nabla}$ is symmetric (by Theorem 1).

Again we have

$$'\bar{R}\,(X, Y, Z, U) + '\bar{R}\,(X, Z, X, U) + '\bar{R}\,(Z, X, Y, U)$$
$$= r/n\,[g(X, U)\,\{M(Y, Z) - M(Z, Y)\}$$
$$+\, g(Y, U)\,\{M(Z, X) - M(X, Z)\}$$
$$+\, g(Z, U)\,\{M(X, Y) - M(Y, X)\}]. \quad ...\ (3.6)$$

So, $$'\bar{R}\,(X, Y, Z, U) + '\bar{R}(Y, Z, X, U) + '\bar{R}(Z, X, Y, U) = 0$$

if and only if $M(Y, Z) - M(Z, Y) = 0$. That is, if and only if the Ricci tensor $\bar{S}$ of $\bar{\nabla}$ is symmetric.

Hence we can state the following theorem :

Theorem 2 — In an Einstein manifold of non-zero scalar curvature admitting a Ricci quarter symmetric metric connection a necessary and sufficient condition that the Ricci tensor of the Ricci Quarter symmetric metric connection $\bar{\nabla}$ to be symmetric is that the (0, 4) curvature tensor $'\bar{R}$ of the connection $\bar{\nabla}$ satisfies either of the following two conditions :

(i) $'\bar{R}\,(X, Y, Z, U) = '\bar{R}\,(Z, U, X, Y)$,

(ii) $'\bar{R}(X, Y, Z, U) + '\bar{R}\,(Y, Z, X, U) + '\bar{R}(Z, X, Y, U) = 0$.

4. Some Properties of a Ricci Quarter Symmetric Metric Connection in a Riemannian Manifold

Let $\bar{C}(X, Y, Z, U)$ be the covariant conformal curvature tensor of the connection $\bar{\nabla}$. Then

$$\bar{C}(X, Y, Z, U) = '\bar{R}\,(X, Y, Z, U)$$
$$- \frac{1}{n-2}\,[\bar{S}(Y, Z)\,g(X, U) - \bar{S}(X, Z)\,g(Y, U)$$
$$+\, g(Y, Z)\,\bar{S}\,(X, U) - g(X, Z)\,\bar{S}\,(Y, U)]$$
$$+\, \frac{\bar{r}}{(n-1)(n-2)}\,[g(Y, Z)\,g(X, U) - g(X, Z)\,g(Y, U)]. \quad ...\ (4.1)$$

Applying (2.5), (2.6) and (2.7) in (4.1) it follows that

$$\bar{C}(X, Y, Z, U) = C(X, Y, Z, U) \quad \text{... (4.2)}$$

where $C(X, Y, Z, U)$ is the covariant conformal curvature tensor of the Levi-Civita connection ∇.

Thus we have the following theorem :

Theorem 3 — If an Einstein manifold M^n, equipped with a Quarter symmetric metric connection then the conformal curvature of ∇ is same as the conformal curvature tensor of the manifold.

Now suppose that the Ricci tensor of the Ricci Quarter symmetric metric connection ∇ vanishes.

That is,

$$\bar{S}(X, Y) = 0 \quad \text{... (4.3)}$$

Hence $\quad \bar{r} = 0. \quad$ ... (4.4)

Applying (4.3), (4.4) in (4.1) we get

$$\bar{C}(X, Y, Z, U) = {}'\bar{R}(X, Y, Z, U). \quad \text{... (4.5)}$$

Hence from Theorem 3 and (4.5) it follows that.

$$C(X, Y, Z, U) = {}'\bar{R}(X, Y, Z, U). \quad \text{... (4.6)}$$

Hence we have the following theorem :

Theorem 4 — If an Einstein manifold admits a Ricci Quarter symmetric metric connection ∇ whose Ricci tensor vanishes, then the curvature tensor $'\bar{R}$ of $\bar{\nabla}$ is equal to the conformal curvature tensor of the manifold.

If the curvature tensor of the Ricci Quarter symmetric metric connection $\bar{\nabla}$ vanishes, then the Ricci tensor also vanishes.

From (4.6) we have

$$C(X, Y, Z, U) = {}'\bar{R}(X, Y, Z, U).$$

But by hypothesis

$$'\bar{R}(X, Y, Z, U) = 0.$$

Therefore

$$C(X, Y, Z, U) = 0.$$

But it is known that if an Einstein manifold M^n $(n > 3)$ is conformally flat, then the manifold is of constant curvature.

Hence we can state the following corollary :

Corollary — If an Einstein manifold M^n $(n > 3)$ endowed with a Ricci Quarter symmetric metric connection whose curvature tensor vanishes, then the manifold is conformally flat.

In the concluding part we suppose that the curvature tensor $'\bar{R}$ of ∇ is of the form

$$'\bar{R}\,(X, Y, Z, U) = K\,\{g(Y, Z)\,g(X, U) - g(X, Z)\,g(Y, U)\} \quad \dots (4.7)$$

where K is a constant.

From (2.5) and (4.7) it follows that

$$'\bar{R}\,(X, Y, Z, U) = K[g(Y, Z)\,g(X, U) - g(X, Z)\,g(Y, U)] + \frac{r}{n}\,[M(Y,Z)\,g(X,U) - M(X,Z)\,g(Y,U) + g(Y,Z)\,M(X,U) - g(X,Z)\,M(Y,U)]. \quad \dots (4.8)$$

Putting $X = U = e_i$ and taking sum for $1 \le i \le n$ of the relation (4.8) we get

$$S(Y, Z) = K(n-1)\,g(Y, Z) + \frac{r}{n}\,[(n-2)\,M(Y,Z) + mg(Y,Z)]. \quad \dots (4.9)$$

Again putting $Y = Z = e_i$ and taking sum for $1 \le i \le n$ of the relation (4.9) we get

$$r = kn(n-1) + \frac{r}{n} \cdot 2(n-1)\,m. \quad \dots (4.10)$$

Now using (4.8), (4.9) and (4.10) in (4.1) it follows that

$$C(X, Y, Z, U) = 0$$

that is, the manifold is conformally flat.

Hence by the known result stated earlier we can state the following theorem :

Theorem 5 — If an Einstein manifold M^n $(n > 3)$ admits a Ricci Quarter symmetric metric connection whose curvature tensor $'\bar{R}$ is of the form (4.7), then the manifolds is of constant curvature.

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