Indian J. pure appl. Math., **26**(4) : 329-338, April 1995

PERIODIC BOUNDARY VALUE PROBLEMS FOR THIRD ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH DELAY

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(*Received* 1 *February* 1994; *accepted* 1 *December* 1994)

We study the periodic boundary value problem

$$\dddot{x}(t) + f(\dot{x}(t))\,\ddot{x}(t) + g(t, \dot{x}(t-\tau)) + h(x(t)) = p(t)$$

$$x(0) - x(2\pi) = \dot{x}(0) - \dot{x}(2\pi) = \ddot{x}(0) - \ddot{x}(2\pi) = 0$$

under some resonant conditions on the asymptotic behaviour of $x^{-1} g(t, x)$ for $|x| \to \infty$. The uniqueness of periodic solutions is also examined.

1. INTRODUCTION

In this paper we study the periodic boundary value problem

$$\left\{\begin{array}{l} \dddot{x} + f(\dot{x})\,\ddot{x} + g(t, \dot{x}(t-\tau)) + h(x) = p(t) \\ \\ x(0) - x(2\pi) = \dot{x}(0) - \dot{x}(2\pi) = \ddot{x}(0) - \ddot{x}(2\pi) = 0 \end{array}\right\} \qquad \ldots (1.1)$$

with fixed delay $\tau \in [0, 2\pi)$, where $f : R \to R$ is continuous, $P : [0, 2\pi] \to R$ and $g : [0, 2\pi] \times R \to R$ are 2π-periodic in t and g satisfies certain Caratheodory conditions. The unknown function $x : [0, 2\pi] \to R$ is defined for $0 < t \leq \tau$ by $x(t - \tau) = [2\pi - (t - \tau)]$. We are specifically concerned with the existence of periodic solutions of eqn. (1.1) under some resonant conditions.

The differential equations

$$\dddot{x} + a\,\ddot{x} + f(x)\,\dot{x} + g(t, x(t-\tau)) = p(t)$$

$$x(0) - x(2\pi) = \dot{x}(0) - \dot{x}(2\pi),\ \ddot{x}(0) - \ddot{x}(2\pi) = 0$$

in which $a \neq 0$ is a constant and

$$\dddot{x} + f(\dot{x})\ddot{x} + b\dot{x} + g(t, x(t-\tau)) = p(t)$$

$$x(0) - x(2\pi) = \dot{x}(0) - \dot{x}(2\pi) = \ddot{x}(0) - \ddot{x}(2\pi) = 0$$

with $b < 0$ a constant were the object of a recent study[4, 8]. Results on the existence and uniqueness of 2π-period solutions were established subject to certain resonant conditions on g.

In what follows, we shall use the spaces $C([0, 2\pi])$, $C^k([0, 2\pi])$ and $L^k([0, 2\pi])$ of continuous, k times continuously differentiable or measurable real functions whose kth power of the absolute value is Lebesgue integrable. We shall also use the Sobolev space $W^{3,2}_{2\pi}$ and $H^1_{2\pi}$ respectively defined by

$$W^{3,2}_{2\pi} = \{x : [0, 2\pi] \to R \mid x, \dot{x}, \ddot{x} \text{ are absolutely continuous on } [0, 2\pi],\ \dddot{x} \in L^2_{2\pi} \text{ and } x(0) - x(2\pi) = \dot{x}(0) - \dot{x}(2\pi) = \ddot{x}(0) - \ddot{x}(2\pi) = 0\}$$

with norm

$$|x|^2_{W^{3,2}_{2\pi}} = \sum_{i=0}^{3} \int_0^{2\pi} |x^{(i)}(t)|^2\, dt$$

and

$$H^1_{2\pi} = \{x : [0, 2\pi] \to R \mid x \text{ is absolutely continuous on } [0, 2\pi] \text{ and } \dot{x} \in L^2_{2\pi}\}$$

with norm

$$|x|^2_{H^1_{2\pi}} = \left(1/2\pi \int_0^{2\pi} x(t)\, dt \right)^2 + 1/2\pi \int_0^{2\pi} |\dot{x}(t)|^2\, dt.$$

2. THE LINEAR CASE

Let us consider the equation

$$\dddot{x}(t) + a\ddot{x}(t) + b\dot{x}(t-\tau) + cx = 0 \qquad \text{... (2.1)}$$

$$x(0) - x(2\pi) = \dot{x}(0) - \dot{x}(2\pi) = \ddot{x}(0) - \ddot{x}(2\pi) = 0$$

where a, b, c are constants.

Lemma 2.1 — Suppose that

$$(n-1)^2 < b < n^2 \qquad \text{... (2.2)}$$

where $n \geq 1$ is a positive integer, then eqn. (2.1) has no nontrivial 2π-periodic solution.

PROOF : We consider a solution of the form $x(t) = e^{\lambda t}$ where $\lambda = in$ with $i^2 = -1$.

Then Lemma 2.1 will follow if

$$\psi(n, \tau) = -n^2 + b \cos n\tau \neq 0 \qquad \text{... (2.3)}$$

for all $n \geq 1$ and $\tau \in [0, 2\pi)$.
By (2.2) we get

$$\psi(n, \tau) \leq -n^2 + b < 0.$$

Therefore $\psi(n, \tau) \neq 0$ and the result follows. If $x \in L^1_{2\pi}$ we shall write

$$\bar{x} = \frac{1}{2\pi} \int_0^{2\pi} x(t)\, dt, \quad \tilde{x}(t) = x(t) - \bar{x}$$

so that

$$\int_0^{2\pi} \tilde{x}(t)\, dt = 0.$$

Our next result concerns the delay equation

$$\dddot{x} + a\ddot{x} + b(t)\, \dot{x}(t - \tau) + cx = 0 \qquad \text{... (2.4)}$$

$$x(0) - x(2\pi) = \dot{x}(0) - \dot{x}(2\pi) = \ddot{x}(0) - \ddot{x}(2\pi) = 0$$

where a, c are constants and $b \in L^2_{2\pi}$.

Theorem 2.1 — Let $c \neq 0$. Suppose that $b(t)$ satisfies

$$0 < b(t) < 1, \; t \in [0, 2\pi]. \qquad \text{... (2.5)}$$

Then for arbitrary a eqn. (2.4) admits in $W^{3,2}_{2\pi}$ only the trivial solution.

PROOF : If x is a possible solution of (2.4) then since

$$\frac{1}{2\pi} \int_0^{2\pi} -\dot{\tilde{x}}\, (a\ddot{x} + cx)\, dt = 0$$

as can be easily verified, we have from (2.5) that

$$0 = \frac{1}{2\pi} \int_0^{2\pi} -\dot{\tilde{x}}\, (\dddot{x} + a\ddot{x} + b(t)\, \dot{x}(t - \tau) + cx)\, dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} (\ddot{\tilde{x}}^2 - b(t)\, \dot{\tilde{x}}\, \dot{x}(t - \tau))\, dt.$$

Using the identity

$$-ab = \frac{[a-b]^2}{2} - \frac{a^2}{2} - \frac{b^2}{2}$$

we get

$$= \frac{1}{2\pi} \int_0^{2\pi} \left[\ddot{\tilde{x}}^2 + \frac{b(t)}{2} [\dot{\tilde{x}}(t) - \dot{\tilde{x}}(t-\tau)]^2 - \frac{b(t)}{2} \dot{\tilde{x}}^2(t) - \frac{b(t)}{2} \dot{\tilde{x}}^2(t-\tau) \right] dt$$

$$\geq \frac{1}{2\pi} \int_0^{2\pi} (\ddot{\tilde{x}}^2(t) - \frac{b(t)}{2} (\dot{\tilde{x}}^2(t) + \dot{\tilde{x}}^2(t-\tau)) \, dt.$$

Using the fact that

$$\int_0^{2\pi} \dot{\tilde{x}}^2(t) \, dt = \int_0^{2\pi} \dot{\tilde{x}}^2(t-\tau) \, dt$$

we get

$$= \frac{1}{2\pi} \int_0^{2\pi} (\ddot{\tilde{x}}^2(t) - b(t) \dot{\tilde{x}}^2(t)) \, dt$$

$$\geq \delta \, | \dot{\tilde{x}} |^2_{H^1_{2\pi}} = \delta \, | \dot{x} |^2_{H^1_{2\pi}}$$

by Lemma 1 of Mawhin and Ward[5] where $\delta > 0$ is a constant. This implies that x = constant a.e. But a constant map cannot be a solution of (2.4) since $c \neq 0$. Thus $x = 0'$.

3. The Non-linear Case

We shall now consider a preliminary Lemma which will enable us obtain *a priori* estimates required for our results.

Lemma 3.1 — Let all the conditions of Theorem 2.1 hold and let δ be related to $b(t)$ by Theorem 2.1.

Suppose that for $V \in L^2_{2\pi}$

$$0 \leq V(t) \leq b(t) + \varepsilon \text{ a.e., } t \in [0, 2\pi], \varepsilon > 0.$$

Then

$$\frac{1}{2\pi} \int_0^{2\pi} -\tilde{x}(t) (\dddot{x} + a\ddot{x} + V(t) \dot{x}(t-\tau) + cx) \, dt$$

$$\geq (\delta - \varepsilon) \, | \dot{\tilde{x}} |^2_2.$$

PROOF : Integrating by parts and using the identity

$$-ab = \frac{[a-b]^2}{2} - \frac{a^2}{2} - \frac{b^2}{2}$$

and noting that

$$\frac{1}{2\pi}\int_0^{2\pi} -\dot{\tilde{x}}(t)\,(a\ddot{x} + cx))\,dt = 0$$

we get

$$\frac{1}{2\pi}\int_{00}^{2\pi} -\tilde{x}\,(\dddot{x} + V(t)\,\dot{x}\,(t-\tau))\,dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (\ddot{\tilde{x}}^2(t) - V(t)\,\dot{\tilde{x}}^2(t))\,dt$$

$$\geq \frac{1}{2\pi}\int_0^{2\pi} (\ddot{\tilde{x}}^2(t) - b(t)\,\dot{\tilde{x}}^2)\,dt - \frac{\varepsilon}{2\pi}\int_0^{2\pi} \dot{\tilde{x}}^2(t)\,dt$$

$$\geq \delta\,|\dot{\tilde{x}}|^2_{H^1_{2\pi}} - \varepsilon\,|\dot{\tilde{x}}|^2_2$$

$$\geq \delta\,|\ddot{\tilde{x}}|^2_2 - \varepsilon\,|\ddot{\tilde{x}}|^2_2$$

$$= (\delta - \varepsilon)\,|\ddot{x}|^2_2 .$$

We shall next consider the non-linear delay equation

$$\dddot{x} + f(\dot{x})\,\ddot{x} + g(t, \dot{x}\,(t, \tau)) + h(x) = p(t) \qquad \text{... (3.1)}$$

$$x(0) - x(2\pi) = \dot{x}(0) - \dot{x}\,(2\pi) = \ddot{x}\,(0) - \ddot{x}\,(2\pi) = 0$$

where f, h : $R \to R$ are continuous functions and g : $[0, 2\pi] \times R \to R$ is such that $g(\cdot\ x)$ is measurable on $[0, 2\pi]$ for each $x \in R$ and $g(t, \cdot)$ is continuous on R for almost each $t \in [0, 2\pi]$.

We assume moreover that for each $r > 0$ there exists $\gamma_r \in L^2_{2\pi}$ such that $|g(t, y)| \leq \gamma_r(t)$ for a.e $t \in [0, 2\pi]$ and all $x \in [-r, r]$ such a g is said to satisfy Caratheodory's conditions.

Theorem 3.1 — Let g be a Caratheodory's function with respect to the space $L^2_{2\pi}$ such that

(i) There exists $s > 0$ such that

$$xg(t, x) \geq 0 \text{ for } |x| \geq s$$

(ii) $\lim\limits_{|x| \to +\infty} \sup \dfrac{g(t,x)}{x} \leq b(t)$ uniformly a.e.

for $t \in [0, 2\pi]$ with $b(t)$ satisfying $0 < b(t) < 1$.

(iii) $\lim\limits_{|x| \to +\infty} \operatorname{sign}(x)\, h(x) = +\infty$.

Suppose further that $P \in L^2_{2\pi}$, then for arbitrary continuous function f eqn. (3.1) has at least one 2π-periodic solution.

Proof : Let $\delta > 0$ be related to b by Lemma 3.1. Then by (i) and (ii) we can find $R > 0$ such that for a.e $t \in [0, 2\pi]$ and all x with $|x| \geq R$ we have

$$0 \leq \frac{g(t,x)}{x} \leq b(t) + \frac{\delta}{2} \cdot \qquad \text{... (3.2)}$$

Let us define as in Iyase[4]

$$\tilde{\gamma}(t,x) = \begin{cases} x^{-1} g(t,x), & |x| \geq R \\ R^{-1} g(t,R), & 0 < x < R \\ -R^{-1} g(t,-R), & -R < x < 0 \\ b(t), & x = 0. \end{cases}$$

Then

$$0 \leq \tilde{\gamma}(t,x) \leq b(t) + \frac{\delta}{2} \qquad \text{... (3.3)}$$

for a.e $t \in [0, 2\pi]$ and all $x \in R$.

Clearly the function

$$\tilde{g} = \tilde{\gamma}(t, \dot{x}(t-\tau))\, \dot{x}(t-\tau)$$

is a Caratheodory function. So also is g_0 defined by

$$g_0(t, \dot{x}(t-\tau)) = g(t, \dot{x}(t-\tau)) - \tilde{g}(t, \dot{x}(t-\tau)).$$

Thus there exists $\gamma_0 \in L^2_{2\pi}$ such that

$$|g_0(t, \dot{x}(t-\tau)| \leq \gamma_0(t).$$

for a.e $t \in [0, 2\pi]$ and all $x \in R$.

Problem (3.1) is thus equivalent to

$$\dddot{x} + f(\dot{x})\ddot{x} + \tilde{\gamma}(t, \dot{x}(t-\tau))\, \dot{x}(t-\tau) + g_0(t, \dot{x}(t-\tau)) + h(x) = p(t) \qquad \text{... (3.4)}$$

to which we shall apply coincidence degree theory[4].

Let $X = C^2[0, 2\pi]$, $z = L^2_{2\pi}$.

$$\text{dom}L = \{x \in X : x(0) - x(2\pi) = \dot{x}(0) - \dot{x}(2\pi) = \ddot{x}(0) - \ddot{x}(2\pi) = 0 \text{ and}$$
$$\dot{x}, \ddot{x} \text{ are absolutely continuous on } [0, 2\pi]\}.$$

Define as in Mawhin and Ward[5]

$$L : \mathrm{dom}L\ C\ x \to z,\ x \to \ddot{x}$$

$$F : x \to z,\ x \to f(\dot{x})\ddot{x}$$

$$G : x \to z,\ x \to \tilde{\gamma}(t, \dot{x}(t-\tau))\,\dot{x}(t-\tau)$$

$$H : x \to z,\ x \to h(x)$$

$$A : x \to z,\ x \to b(t)\,\dot{x}(t-\tau)$$

$$G_0 : x \to z,\ x \to g_0(t, \dot{x}(t-\tau)).$$

The proof of the theorem will follow from Theorem 4.5 of Mawhin[7] if we show that the possible solutions of the equation

$$Lx + \lambda Fx + (1-\lambda)Ax + \lambda\, Gx + \lambda\, G_0x + (1-\lambda)\,cx + \lambda Hx = \lambda p(t) \quad \text{... (3.5)}$$

where $c > 0$ are *a priori* bounded independently of $\lambda \in [0, 1]$.

For $\lambda = 0$ we get the equation

$$\ddot{x} + b(t)\,\dot{x}(t-\tau) + cx = 0$$

which by theorem (2.1) has only the trivial solution.

Observe that

$$0 \le (1-\lambda)\,b(t) + \lambda\,\tilde{\gamma}(t, \dot{x}(t-\tau)) \le b(t) + \frac{\delta}{2}.$$

Hence by Lemma 3.1 we get

$$\frac{1}{2\pi}\int_0^{2\pi} -\dddot{x}(t)\,\{\ddot{x} + \lambda f(\dot{x})\ddot{x} + [(1-\lambda)\,b(t) + \lambda\tilde{\gamma}(t, \dot{x}(t-\tau))]$$

$$\dot{x}(t-\tau) + (1-\lambda)\,cx\}$$

$$\ge \frac{\delta}{2}|\ddot{x}|_2^2.$$

Thus

$$0 = \frac{1}{2\pi}\int_0^{2\pi} -\dddot{x}(t)\,\{\ddot{x} + \lambda f(\dot{x})\ddot{x} + [(1-\lambda)\,b(t)$$

$$+ \lambda\tilde{\gamma}(t, \dot{x}(t-\tau))]\,\dot{x}(t-\tau) + (1-\lambda)\,cx$$

$$+ \lambda\, g_0(t, \dot{x}(t-\tau)) + \lambda h(x) - \lambda p(t)\}\ dt$$

$$\geq \frac{\delta}{2} |\ddot{x}|_2^2 - 2\pi (|\gamma_0|_2 + |p|_2) |\dot{x}|_2$$

and by Wirtingers inequality we get

$$\geq \frac{\delta}{2} |\ddot{x}|_2^2 - \beta |\ddot{x}|_2, \qquad \text{for some } \beta > 0.$$

Hence

$$|\ddot{x}|_2 \leq \frac{2\beta}{\delta} = \beta_1, \quad \beta_1 > 0. \qquad \ldots (3.6)$$

The inequality (3.6) implies that

$$|\dot{x}|_\infty \leq \beta_2, \quad |\dot{x}|_2 \leq \beta_3, \qquad \text{for some} \quad \beta_2, > 0, \ \beta_3 > 0.$$

By the continuity of f we derive that

$$|f(\dot{x})|_\infty \leq \beta_4 \text{ for some } \beta_4 > 0.$$

Taking the average of eqn. (3.5) on $[0, 2\pi]$ we get by the mean value theorem

$$|(1-\lambda)\, cx(t^*) + \lambda h(x(t^*))| = \left| (1-\lambda) c \frac{1}{2\pi} \int_0^{2\pi} x(t)\, dt + \lambda \frac{1}{2\pi} \int_0^{2\pi} h(x(t))\, dt \right|$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} |(1-\lambda)\, b(t) + \lambda\, \bar{Y}(t, \dot{x}(t-\tau))| \, |\dot{x}(t-\tau)|\, dt$$

$$+ \frac{1}{2\pi} \int_0^{2\pi} |g_0(t, \dot{x}(t-\tau)|\, dt + \frac{1}{2\pi} \int_0^{2\pi} |p(t)|\, dt$$

$$\leq \beta_2 \left(1 + \frac{\delta}{2} \right) + |\gamma_0|_1 + |p|_1 = \beta_5. \qquad \ldots (3.7)$$

for some $t^* \in [0, 2\pi]$. By (iii) for any $k > 0$ there is $q = q_k > 0$ such that $|h(x)| = \text{sign}(x)\, h(x) > k$ for every $|x| > \max\left\{\frac{k}{c}, q\right\}$. Hence for any $\lambda \in [0, 1]$ we have

$$|(1-\lambda)\, cx + \lambda\, h(x)| = \text{sign}\,(x)\, ((1-\lambda)\, cx + \lambda h(x)) \geq (1-\lambda) k + \lambda k = k$$

for every $|x| > \max\left\{\frac{k}{c}, q\right\}$. We now choose $k > \beta_5$, and derive that

$$|x(t^*)| \leq \max\left\{\frac{k}{c}, q\right\} = \beta_6. \qquad \ldots (3.8)$$

From (3.8) we obtain

$$x(t) = x(t^*) + \frac{1}{2\pi}\int_{t^*}^{2\pi} \dot{x}(s)\, ds.$$

Hence

$$|x|_\infty \leq \beta_6 + |\dot{x}|_\infty \leq \beta_6 + \beta_2 = \beta_7 \qquad \dots (3.9)$$

for some $\beta_7 > 0$.

From eqn. (3.5) and by continuity of h we obtain

$$|\dddot{x}|_1 \leq \beta_8 \quad \text{for some } \beta_8 > 0. \qquad \dots (4.0)$$

Now since $\dot{x}(0) = \dot{x}(2\pi)$, there exists $t_0 \in (0, 2\pi)$ such that $\ddot{x}(t_0) = 0$. Hence

$$\ddot{x}(t) = \ddot{x}(t_0) + \int_{t_0}^{2\pi} \dddot{x}(s)\, ds.$$

Therefore

$$|\ddot{x}|_\infty \leq \beta_9 \text{ for some } \beta_9 > 0.$$

Hence

$$|x|_{c^2} = |x|_\infty + |\dot{x}|_\infty + |\ddot{x}|_\infty \leq \beta_7 + \beta_2 + \beta_9 = \beta_{10}.$$

Choosing $\rho > \beta_{10} > 0$ we obtain the required *a priori* bound in $c^2[0, 2\pi]$ independently of x and λ.

4. UNIQUENESS RESULT

If in (1.1) $f(\dot{x}) = a$, $h(x) = d$ where a and d are constants, then we have the following uniqueness result.

Theorem 4.1 — Let a and d be constants with $d > 0$. Suppose g is a Caratheodory function satisfying

$$0 \leq \frac{g(t, \dot{x}_1) - g(t, \dot{x}_2)}{(\dot{x}_1 - \dot{x}_2)} \leq b(t)$$

for all $\dot{x}_1, \dot{x}_2 \in R$, $\dot{x}_1 \neq \dot{x}_2$, where $b(t) \in L^2_{2\pi}$ is scuh that $0 < b(t) < 1$. Then for all arbitrary constant a and every $\tau \in [0, 2\pi)$ the boundary value problem

$$\dddot{x} + a\ddot{x} + g(t, \dot{x}(t - \tau)) + dx = p(t) \qquad \dots (4.1)$$

$$x(0) - x(2\pi) = \dot{x}(0) - \dot{x}(2\pi) = \ddot{x}(0) - \ddot{x}(2\pi) = 0$$

has at most one solution.

PROOF : Let x_1, x_2 be any two solutions of (4.1). Set $x = x_1 - x_2$. Then x satisfies the b v p

$$\dddot{x} + a\ddot{x} + b(t)\,\dot{x}(t-\tau) + dx = 0$$

$$x(0) - x(2\pi) = \dot{x}(0) - \dot{x}(2\pi) = \ddot{x}(0) - \ddot{x}(2\pi) = 0$$

where the function $b(\cdot) \in L^2_{2\pi}$ is defined by

$$b(t) = \begin{cases} \dfrac{g(t, \dot{x}_1(t-\tau)) - g(t, \dot{x}_2(t-\tau))}{\dot{x}(t-\tau)} & \text{if } \dot{x}(t) \neq 0 \\ \dfrac{1}{2} \text{ if } \dot{x}(t) = 0 & \end{cases}$$

If $\dot{x} = 0$ on every subset of $[0, 2\pi]$ of positive measure, then x = constant = 0 since $d \neq 0$. Hence $x_1 = x_2$. Suppose that $\dot{x}(t) \neq 0$ on a certain subset of $[0, 2\pi]$ of positive measure. Then using the arguments of Theorem 2.1 we have that $x = 0$ and hence $x_1 = x_2$.

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