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CONFORMALLY MINKOWSKI TYPE SPACES AND CERTAIN *d*-CONNECTIONS IN A MIRON SPACE*

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The main purpose of the present paper is to consider the problem of conformally Minkowski type spaces relative to an *h*-recurrent *d*-connection in a Miron space. The terminology and notations in the present paper are referred to Miron[3] with a few changes.

MOTIVATION TO STUDY MIRON SPACES

The Cartan spaces (nowadays such spaces are called Miron spaces) were defined by E. Cartan using the notation of a real metric but its construction is very complicated. However, these spaces can be viewed as dual of the Finsler spaces (In fact, this duality allows to obtain a number of new properties of the Cartan spaces); and as such we are allowed to determine a canonical non-linear connection (it was done by Miron[3]) and a canonical connection on which the Cartan geometry is based. Thus the Cartan spaces (Miron spaces) have the similar importance as the Finsler spaces.

1. INTRODUCTION

Let M be an n-dimensional c^∞-manifold and (T^*M, π^*, M) its cotangent bundle. For a local chart (U, ϕ) on M, the coordinates of a point $u \in \pi^{*-1}(U) \subset T^*M$ is denoted by (x^i, p_i), $(i, j, k = 1, 2, ..., n)$.

A d-tensor field of type (r, s) on M is defined by its components, satisfying the classical law of transformation relative to a transformation of canonical coordinate systems in T^*M.

A d-connection on M is denoted by $H\Gamma(N) = (N^i_k, H^i_{jk}, C^{jk}_i)$ where N^i_k is a non-linear connection on T^*M, H^i_{jk} is a special d-object and C^{jk}_i is a d-tensor field of type (2, 1). Often, we write $H\Gamma(N) = (H^i_{jk}, C^{jk}_j)$ if there is no chance of confusion.

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Let $\zeta^n = (M^n, H(x, p))$ be a Miron space, i.e., a Hamilton space equipped with a fundamental function $H(x, p)$, $p \neq 0$ which is 2-homogeneous with respect to p_i. The fundamental tensor of this space is given by $g^{ij} = (\dot\partial^i, \dot\partial^j H)/2$, $\dot\partial^i = \partial/\partial p_i$.

It is well-known that a Finsler space possesses a number of metrical connections, one of these is a canonical Finsler connection known as the Cartan connection. In the same way in a Miron space ζ^n there exists a unique (canonical) d-connection $MC^*\,\Gamma(N)$ (called hereafter Miron-Cartan connection) which is determined from the function $H(x, p)$ by the system of axioms (C^*).

$(C^*1)\ \nabla^h g = 0$, $(C^*2)\ \nabla^k g = 0$, $(C^*3)\ T = 0$, $(C^*4) = S^1 = 0$, $(C^*5)\ D = 0$, where $g = (g^{ij}(x, p))$ is the fundamental tensor.

In section 2, we shall apply the tools described in Aikou and Hashiguchi[1] to prove that a d-connection is uniquely determined from H by giving five d-tensors ∇_g^h, ..., D appearing in the system (C^*) arbitrarily. As a consequence, we shall denote by the symbol $C^*\Gamma(4, 5)$, for instance, a d-connection which is determined from $H(x, p)$ by $(C^*1, 2, 3)$ and giving S^1 and D. In section 3, we shall obtain some conformal invariants. Our main result is given in Theorem (section 4).

2. The Miron-Cartan Expression of a d-Connection

Let $H\Gamma = (N_k^i, H_{jk}^i, C_i^{jk})$ be a d-connection on a Miron space ζ^n. Then for transformation $MC^*\Gamma(\overset{c}{N}) \to H\Gamma(N)$, we can write

$$N_k^i = \overset{c}{N}{}_k^i + A_k^i\ , \qquad \text{... (2.1)}$$

$$H_{jk}^i = \overset{c}{H}{}_{jk}^i + \overset{c}{C}{}_j^{i1} A_{kl} + B_{jk}^i\ ,$$

$$C_i^{jk} = \overset{c}{C}{}_i^{jk} + D_i^{jk}\ ,$$

where (A, B, D) are arbitrary d-tensor fields of the type (0, 2), (1, 2) and (2, 1) respectively and $\overset{c}{C}{}^{ijk} = -\frac{1}{2}\dot\partial^k g^{ij}$. (Here we indicate the components of Miron-Cartan connection by putting c above them.)

If we consider Obata's operators :

$$\underset{1}{\Lambda}{}_{sj}^{ir} = (\delta_s^i\,\delta_j^r - g_{sj}\,g^{ir})/2,$$

$$\underset{2}{\Lambda}{}_{sj}^{ir} = (\delta_s^i\,\delta_j^r + g_{sj}\,g^{ir})/2,$$

then we have :

Proposition 2.1 — For any d-tensor fields U_k^{ij}, V^{ijk} and D_k^j :

$$g^{ij}_{|k} = U^{ij}_k \Leftrightarrow \underset{2}{\Lambda^{ir}_{sj}} B^s_{rk} = U^i_{jk}/2 \ ,$$

$$g^{ij}\,|^k = V^{ijk} \Leftrightarrow \underset{2}{\Lambda^{ir}_{sj}} D^{sk}_r = V^{ik}_j/2, \qquad \text{... (2.2)}$$

$$D_{jk} = N_{jk} - H^0_{jk} \Leftrightarrow A_{jk} = B^0_{jk} + D_{jk} \, . \qquad \text{... (2.3)}$$

If we put

$$W^{ij}_k = -(\dot\partial^r g^{ij}) A_{kr}, \qquad \text{... (2.4)}$$

then for any given d-connection $H\Gamma(N)$, we have (in the notation of Proposition 2.1)

$$H^i_{jk} = \overset{c}{H}{}^i_{jk} + (W^i_{jk} + U^i_{jk})/2 - \underset{1}{\Omega^{ir}_{sj}} A^s_{rk} \ , \qquad \text{... (2.5)}$$

where A^s_{rk} is some d-tensor field, which gives

$$T^i_{jk} = \sigma_{(jk)} \left\{ (W^i_{jk} + U^i_{jk})/2 - \underset{1}{\Omega^{ir}_{sj}} A^s_{rk} \right\} . \qquad \text{... (2.6)}$$

Next, if we define * operation (hereafter we call Aikou-Hashiguchi operation[1]) for any d-tensor fields K_{jhk} and K^h_{jk} as follows :

$$K^*_{jhk} = (K_{jkh} + K_{kjh} - K_{khj})/2,$$

$$K^{*h}_{jk} = (K^h_{jk} + K^h_{kj} - K^h_{kj})/2. \qquad \text{... (2.7)}$$

Then by applying Aikou-Hashiguchi * operation to (2.6) we find

$$T^{*i}_{jk} = \underset{1}{\Omega^{ir}_{sj}} (W^s_{rk} + U^s_{rk} - A^s_{rk}) \, ,$$

by which we can eliminate $\underset{1}{\Lambda^{ir}_{sk}} A^s_{rk}$ from (2.5). Similarly we can proceed with C^{jk}_i. In this way we obtain :

Proposition 2.2 — Let $H\Gamma(N) = (N^i_k, H^i_{jk}, C^{jk}_i)$ be any d-connection in a Miron space. And, if d-tensor fields A^i_k, U^{ij}_k, V^i_{jk} and W^{ij}_k are defined by (2.1), (2.2) and (2.4) respectively, then we have

$$H^i_{jk} = \overset{c}{H}{}^i_{jk} + W^{*i}_{jk} + U^{*i}_{jk} - T^{*i}_{jk} \ , \qquad \text{... (2.8)}$$

$$C^{jk}_i = \overset{c}{C}{}^{jk}_i - V^{*jk}_i + S^{*jk}_i . \qquad \text{... (2.9)}$$

Noticing second of (2.1), (2.3) and (2.8), we find

$$A_{jk} = W^{*0}_{jk} + E_{jk} \ , \qquad \text{... (2.10)}$$

where

$$E_{jk} = U^{*0}_{jk} - T^{*0}_{jk} + D_{jk}. \qquad \ldots (2.11)$$

Due to $W^{*0}_{ls} = \overset{c}{C}{}^{r}_{ls} A^{0}_{r}$, (2.10) becomes

$$A_{jk} = E_{jk} - g_{jl}\, g_{ks}\, (\dot\partial^r\, g^{ls})/2\, E^{0}_{r}. \qquad \ldots (2.12)$$

Consequently, we have :

Theorem 2.1A — Let $H\Gamma(N)$ be any d-connection in a Miron space. If U^{ij}_{k}, V^{ijk}, W^{ij}_{k}, E_{ik} and A_{ik} are given by (2.2), (2.4), (2.11) and (2.12) respectively, then coefficients of $H\Gamma(N)$ are expressed as first of (2.1), (2.8) and (2.9).

It follows that a d-connection is uniquely determined by its five d-tensors ∇^{h}_{g}, ..., D. The expression of $H\Gamma(N)$ stated in Theorem 2.1A is called the Miron-Cartan expression of $H\Gamma(N)$.

On the other hand, we have :

Theorem 2.1B — Let $U^{ij}_{k}\,(= U^{ji}_{k})$, $V^{ij}_{k}\,(= V^{ji}_{k})$, $T^{i}_{jk}\,(= -T^{i}_{kj})$, $S^{jk}_{i}\,(= -S^{kj}_{i})$ and D_{ik} be any d-tensor fields in a Miron space. Then there exists a unique d-connection $H\Gamma(N)$ in which the h- and v-covariant derivatives of g^{ij}, the d-torsion tensor fields and h-deflection tensor field are the above given d-tensor fields respectively. Moreover, if we obtain W^{ij}_{k}, E_{ik} and A_{ik} by (2.4), (2.11) and (2.12) from the given d-tensor fields, then $H\Gamma(N)$ is given by first of (2.1), (2.8) and (2.9).

The proof is similar to Aikou and Hashiguchi, so it is enough to state the result.

3. Some Distinctive Conformal Invariants

Here and hereafter we slightly change our notation for the coefficients $MC^{*}\,\Gamma(N)$ i.e, they are denoted by $\overset{c}{N}{}^{i}_{k}$, $\overset{c}{H}{}^{i}_{jk}$ and C^{ijk}_{i}, and return to the system of axioms (C^{*}) stated in section 1.

As in Miron[3], from the axioms $(C^{*}1, 3)$,

$$\overset{c}{H}_{ijk} = (\delta_j\, g_{ik} + \delta_k\, g_{ij} - \delta_i\, g_{jk})/2, \qquad \ldots (3.1)$$

where $\delta_j = \partial_j + \overset{c}{N}_{jh}\, \dot\partial^h$ and from the axioms $(C^{*}2, 4)$.

$$C^{ijk} = -(\dot\partial^j\, g^{ik} + \dot\partial^k\, g^{ij} - \dot\partial^i\, g^{jk})/2 \; = \; -\dot\partial^k\, g^{ij}/2. \qquad \ldots (3.2)$$

Consequently, we have a d-connection defined as follows :

Definition 3.1 — A d-connection $C^{*}\Gamma(N)(5)$ where the non-linear connection $N\,(= N^{i}_{k})$ is given, is called a special d-connection denoted by $\overset{s}{H}\Gamma(N)$.

From (3.1), replacing $\overset{c}{N}_{jh}$ by N_{jh}, we find

$$H_{ijk} = \gamma_{ijk} + N_{jh}\, C^h_{ik} + N_{kh}\, C^h_{ij} - N_{ih}\, C^h_{jk}, \qquad \ldots (3.3)$$

which gives

$$D_{jk} = N_{jk} - (\gamma^0_{jk} - N^0_h\, C^h_{jk}), \qquad \ldots (3.4)$$

where γ_{ijk} are the Christofel symbols constructed from $g_{ij}(x, p)$ with respect to ∂_i. Since $D_{0k} = N_{0k} - \gamma^0_{k0}$, $D \neq 0$ in general.

Next (3.4) can be written as

$$N_{jk} = (\gamma^0_{jk} - \gamma^0_{h0}\, C^h_{jk}) + (D_{jk} - D^0_h\, C^h_{jk}), \qquad \ldots (3.4a)$$

and hence N is determined by D. Moreover, $(\gamma^0_{jk} - \gamma^0_{h0}\, C^h_{jk})$ in (3.4a) is the canonical non-linear connection $\overset{c}{N}$ of the Miron space[3]. And we get an interesting d-connection as follows :

Example — If we put $D_{jk} = H\, C_j\, p_k$, $(C_j = C^h_{hj})$, then (3.4a)

$$\Rightarrow \quad N_{jk} = \overset{c}{N}_{jk} + HC_j\, p_k$$

$$\Rightarrow \quad H_{ijk} = \gamma_{ijk} + \overset{c}{N}_{jh}\, C^h_{ik} + \overset{c}{N}_{kh}\, C^h_{ij} - \overset{c}{N}_{ih}\, C^h_{jk}.$$

(by (3.3))

which is $\overset{c}{H}_{ijk}$ of MC^* $\overset{c}{\Gamma}(N)$ of Miron space. Thus we have a d-connection

$$H\Gamma(N) = (\overset{c}{N}{}^i_k + HC^i\, p_k,\ \overset{c}{H}{}^i_{jk},\ C^{jk}_i).$$

Now we consider a conformal change[2] $\zeta^n = (M^n, H(x, p)) \rightarrow \overline{\zeta}^n = (M^n, \overline{H}(x, p))$, $\overline{H}(x, p) = e^{2\sigma(x)} H(x, p)$. Next we consider the d-connections

$$\overset{s}{H}\Gamma(N) = (H^i_{jk},\ C^{jk}_i)$$

and $\overset{s}{H}\overline{\Gamma}(N) = (\overline{H}^i_{jk},\ \overline{C}^{jk}_i)$ of ξ^n and ζ^{-n} respectively. It is easy to see that Miron-Cartan tensor $C^{ijk}\left(= -\frac{1}{2}\dot{\partial}^k g^{ij}\right)$ is conformal invariant. Since,

$$\delta_k \overline{g}_{ij} = \overline{e}^{2\sigma}(\delta_k g_{ij} - 2g_{ij}\sigma_k), \quad (\sigma_k = \partial_k \sigma(x)),$$

(3.1) gives

$$\overline{H}^i_{jk} = H^i_{jk} - A^i_{jk}. \qquad \ldots (3.5)$$

where

$$A^i_{jk} = \delta^i_j \sigma_k + \delta^i_k \sigma_j - g_{jk}\sigma^i, \qquad (\sigma^i = g^{ih}\sigma_h).$$

For d-torsion tensor $P_{jk}^{\ i} = -(\dot{\partial}^i N_{jk} - H_{jk}^i)$, (3.5) yields

$$\bar{P}_{jk}^{\ i} = P_{jk}^{\ i} - \delta_j^i \sigma_k - \delta_k^i \sigma_j + g_{jk} \sigma^i, \qquad \ldots (3.6)$$

from which $P_k = P_k - n \sigma_k$. $\qquad \ldots (3.7)$

Now, in view of (3.7), (3.5) shows that

$$\overset{s}{H}{}_{jk}^i = H_{jk}^i - (\delta_j^i P_k + \delta_k^i P_j^i - g_{jk} P^i)/n, \ (P^i = g^{ih} P_h) \qquad \ldots (3.8)$$

are conformally invariant. So we have :

Proposition 3.1 — The d-connection $\overset{s_1}{H}\Gamma(N) = (\overset{s_1}{H}{}_{jk}^i, C_i^{jk})$ is conformally invariant for a given non-linear connection $N\,(= N_k^i)$.

Note : From (3.5) and (3.6), we see that $H_{jk}^i - P_{jk}^{\ i}$ are also conformally invariant. These invariants are not interesting due to $H_{jk}^i - P_{jk}^{\ i} = \dot{\partial}^i N_{jk}$.

Next, it is remarked that the d-connection $\overset{s_1}{H}\Gamma(N)$ satisfies the axioms (C 2, 3, 4), the deflection tensor $\overset{s}{D}$ does not vanish and $\nabla^h g$ is given by

$$g^{ij}_{|k} = -2\, g^{ij} P_k/n. \qquad \ldots (3.9)$$

Consequently, we have :

Proposition 3.2 — The conformal d-connection $\overset{s_1}{H}\Gamma(N)$ is a $C^*\Gamma$ (2, 3, 4). It is h recurrent with the recurrency vector $-2P_i/n$.

Note : Due to conformal invariance of $\overset{s_1}{H}\Gamma(N)$, all the torsion and curvature tensors of $\overset{s_1}{H}\Gamma(N)$ are conformally invariants. However, for the present purpose we need the torsion tensor $\overset{s_1}{P}{}_{jk}^{\ i}$, the curvature tensors, $\overset{s_1}{P}{}_{j\ k}^{\ i\ l}$ and $\overset{s_1}{R}{}_{j\ kl}^{\ i}$.

The torsion tensor $\overset{s_1}{P}{}_{jk}^{\ i}$ is given by

$$\overset{s_1}{P}{}_{jk}^{\ i} = P_{jk}^{\ i} - (\delta_j^i P_k + \delta_k^i P_j - g_{jk} P^i)/n \qquad \ldots (3.10)$$

relative to $\overset{s}{H}\Gamma(N)$.

Now we deal with $\overset{s_1}{P}{}_{j\ k}^{\ i\ l}{}^3$.

If we put

$$B_{jk}^i = (\delta_j^i P_k + \delta_k^i P_j - g_{jk} P^i)/n,$$

then (3.8) is written as $\overset{s_1}{H}{}^i_{jk} = H^i_{jk} - B^i_{jk}$, and we get

$$\overset{s_1}{P}{}^i_{j\,k}{}^l = P^i_{j\,k}{}^l - B^i_{jk}\,|\,l - B^i_{jr}\,C^{rl}_k, \qquad \text{... (3.11)}$$

relative to $\overset{s}{H}\Gamma(N)$.

Finally for the components[3] $\overset{s_1}{R}{}^i_{j\,kl}$, we find

$$\overset{s_1}{R}{}^i_{j\,kl} = R^i_{j\,kl} + \nu_{(kl)}\{B^s_{jk}B^i_{sl} - B^i_{jk\,|\,l}\}, \qquad \text{... (3.12)}$$

relative to $\overset{s}{H}\Gamma(N)$.

Thus we have :

Theorem 3.1 — In a Miron space, for a given non-linear connection N, conformal invariants $\overset{s_1}{P}{}^i_{jk}$, $\overset{s_1}{P}{}^i_{j\,k}{}^l$ and $\overset{s_1}{R}{}^i_{j\,kl}$ are defined by (3.10), (3.11) and (3.12) respectively relative to $\overset{s}{H}\Gamma(N)$.

4. On Conformally Minkowski Type Spaces

Due to Theorem 2.1, we can introduce :

Definition 4.1 — The Miron-Cartan h-recurrent connection ${}^h\overset{c}{MC}{}^*\Gamma(N)$ is a $\overset{c}{C}{}^*\Gamma(N)$ (1) where $g_{ij\,|\,k} = 2g_{ij}\,s_k$, for a given d-covector field $s_k(x, p)$.

Let $\overset{c}{MC}{}^*\Gamma(N)$ be the Miron-Cartan connection of a Miron space ζ^n. Then for conformally related spaces ζ^n, ζ^{-n}; we get in $\overset{c}{MC}{}^*\Gamma(N)$:

(i) $\overline{g}^{ij}_{|k} = 2\overline{g}^{ij}\,\sigma_k$, (ii) $\overline{g}^{ij}\,|^k = 0$.

Hence, in $\overline{\zeta}^{-n}$, the $\overset{c}{MC}{}^*\Gamma(N)$ is a Miron-Cartan h-recurrent connection with the recurrency vector $2\sigma_k(x)$.

Conversely, if $\zeta^{-n} = (M^n, \overline{H}(x, p))$ has the Miron-Cartan h-recurrent connection ${}^h\overset{c}{MC}{}^*\Gamma(N)$ with the recurrency vector $2\ \sigma_k = 2\partial_k\,\sigma(x)$, then ${}^h\overset{c}{MC}{}^*\Gamma(N)$ is the Miron-Cartan connection of a conformally changed $\zeta^n = (M^n, H(x, p)) = (e^{-2\sigma}\ \overline{H}(x, p))$, because of $g^{ij} = e^{-2\sigma(x)}\,g^{-ij}$ and $g^{ij}_{|k} = 0$. So we have :

Proposition 4.1 — The Miron-Cartan connection $\overset{c}{MC}{}^*\Gamma(N)$ of Miron space $\zeta^n = (M^n, H)$ is the Miron-Cartan h-recurrent connection ${}^h\overset{c}{MC}{}^*\Gamma(N)$ with the recurrency vector $2\sigma_i = 2\partial_i\,\sigma(x)$ of the conformally changed space $\zeta^{-n} = (\overline{M}^n, \overline{H} = e^{2\sigma}H)$, and conversely.

Finally, as is known[3], a Miron space is locally Minkowski type space if and only if

$$R^{j}_{i\,hk} = 0, \quad C_{i}^{jk}{}_{|h} = 0,$$

in $MC^{*}\,\overset{c}{\Gamma}(N)$. If we assume abcve ζ^{n} locally Minkowski type, then above discussion yields :

Theorem 4.1 — A Miron space is conformally Minkowski type, if and only if there exists a Miron-Cartan h-recurrent connection ${}^{h}MC^{*}\,\Gamma(N)$ with the recurrency vector of divergence $2\sigma_{i} = 2\partial_{i}\,\sigma(x)$ which satisfies $R^{j}_{i\,hk} = 0$, $C_{i}^{jk}{}_{|h} = 0$.

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