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AN h-METRICAL d-CONNECTION OF A SPECIAL MIRON SPACE*

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The purpose of the present paper is to introduce the notion of a new metric, called dual (α, β) metric, in a Miron space and to find a suitable d-connection for such a space with this metric and to study some of its characteristic features. The terminology and notation follow the papers by Miron[3,4] with a few changes.

1. INTRODUCTION

A Hamilton space[3, 4] is a pair $H^n = (M, H)$, where $H : T^*M \to R$ is a Hamilton function for which the d-tensor field

$$g^{ij}(x \cdot p) = 1/2\ \dot\partial^i \dot\partial^j H,\ (i, j = 1, \ldots, n),\ (\dot\partial^i = \partial/\partial p_i),$$

is nondegenerate. T^*M is the total space of the cotangent bundle (T^*M, π^*, M) to M. The coordinates of a point in T^*M is denoted by (x^i, p_i).

If the function $H(x, p)$ is 2-homogeneous with respect to p_i, then H^n is called a Miron space.

A d-connection on M is denoted by $H\Gamma(N) = (H^i_{jk}, C^{ijk}_i)$, where $N(= N^i_k)$ is a non-linear connection on T^*M, H^i_{jk} is a special d-object and C^{ijk}_i is a d-tensor field of type (2.1).

2. DUAL (α, β) METRIC AND d-CONNECTION

As is known[3, 5], the metric function H of the form

$$H = a^{ij}(x)\, p_i\, p_j - p_i\, b^i(x) + A(x),$$

of a class of Hamilton spaces originates from the Hamiltonian theory of electro magnetic field in the presence of gravitational field and such a space is not a Miron

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space. However, a class of Hamilton spaces for which metric function H is of the form

$$H = \lambda\, a^{ij}(x)\, p_i\, p_j,$$

is a Miron space. Such metric function originates from Hamiltonian theory of gravitational field.

On the otherhand, there is an important notion of (α, β) metric[1, 2] in the theory of Finsler spaces which has widely attracted physicist's attention. Keeping in view the above considerations, the following problem naturally rise. What corresponds to the notion of (α, β) metric in the duality tangent-cotangent bundle ? So we propose :

Definition 1 — A Miron metric function H in a C^∞-manifold M is called a dual (α, β)-metric if H is 2-homogeneous function $h(\alpha, \beta)$ of the variables

$$\alpha(x, p) = (a^{ij}\, p_i\, p_j)^{1/2}, \quad \beta(x, p) = p_i\, b^i(x),$$

where $a^{ij}(x)$ is a Riemannian metric and $b^i(x)$ is a vector field which does not depend on p_i.

In the following the subscripts α and β stand for the partial differentiations by α and β respectively. Further, we assume $h_\beta \neq 0$.

The fundamental tensor $g^{ij}(x, p)$ of the Miron space $\zeta^n = (M, h(\alpha, \beta))$ is given by

$$g^{ij} = t\, a^{ij} + t_0\, b^i\, b^j + t_{-1}\,(b^i\, P^j + b^j\, P^i) + t_{-2}\, P^i\, P^j, \qquad \ldots (2.1)$$

where

$$P^i = a^{ij}\, P_j, \quad t = h\, h_\alpha/\alpha, \quad t_0 = h\, h_{\beta\beta} + (h_\beta)^2,$$

$$t_{-1} = (h\, h_{\alpha\beta} + h_\alpha\, h_\beta)/\alpha, \quad t_{-2} = \{h\, h_{\alpha\alpha} + (h_\alpha)^2 - h\, h_\alpha/\alpha\}/\alpha^2.$$

Next, we consider the linear d-connection $H_\gamma = (\gamma^0_{jk} = \gamma^i_{jk}\, p_i,\ \gamma^i_{jk},\ 0)$ of the space ζ^n, where γ^i_{jk} are the Christoffel symbols of the tensor field a^{ij} from metric α. Let ";" denote the h-covariant differentiation with respect to H_γ. Then, if $b^i_{;k} = 0$, we have $g^{ij}_{;k} = 0$ because of $a^{ij}_{;k} = 0$ and $p_{i;k} = 0$. Further for H_γ, $T^i_{jk} = 0$, $D = 0$, and $C^{jk}_i = 0$; and as such (by the definition of Rund type d-connection[4]), we have

Proposition 2.1 — The linear d-connection H_γ of a Miron space ζ^n with dual (α, β)-metric satisfying $b^i_{;k} = 0$ is the d-connection of the Rund type of ζ^n.

It is noticed that the h-covariant differentiation with respect to the d-connection of the Rund type coincides with that with respect to the canonical d-connection $\overset{c}{CT}(N)$ as mentioned in Miron[3].

The properties of the linear d-connection H_γ leads us to the following :

Definition 2.2 — A d-connection $H\Gamma(N)$ of a Miron space ζ^n with dual (α, β) metric is called the h-metrical d-connection if it satisfies the conditions :

(i) h-deflection tensor

$$D_{ij} (= p_{i|j}) = 0,$$

(ii) $a^{ij}_{|k} = 0$,

(iii) $g^{ij}_{|k} = 0$.

Since $H\Gamma(N)$ satisfies $H_{|k} = 0$, where $H = g^{ij} p_i p_j$, $P^i_{|k} = 0$ and $\alpha_{|k} = 0$, we have from

$$H_{|k} = \alpha_{|k} H_\alpha + \beta_{|k} H_\beta = 0,$$

$$\beta_{|k} = b^i_{|k} p_i = 0. \qquad \ldots (2.2)$$

Next from (2.1) and (iii), we have

$$g^{ij}_{|k} = b^i_{|k} (t_0 b^j + t_{-1} P^j) + b^j_{|k} (t_0 b^i + t_{-1} P^i) = 0,$$

which on transvecting with p_j and noticing (2.2) gives

$$b^i_{|k} = 0, \qquad \ldots (2.3)$$

because of homogeneity and assumption $h_\beta \neq 0$, to $\beta + t_{-1} \alpha^2 = h h_\beta \neq 0$.

On the otherhand it is easy to verify (iii), if the conditions (i), (ii) and (2.3) hold.

Thus we have :

Proposition 2.2 — A d-connection $H\Gamma$ of ζ^n with dual (α, β) metric is the canonical h-metrical connection, if and only if $H\Gamma$ satisfies the conditions $D_{ij} = 0$, $a^{ij}_{|k} = 0$ and $b^i_{|k} = 0$.

3. Main Result

Let $H\Gamma(N) = (H^i_{jk}, C^{jk}_i)$ be the canonical h-metrical d-connection stated in Definition (2.2) where we put

$$H^i_{jk} (x, p) = \gamma^i_{jk} (x) + A^i_{jk} (x, p). \qquad \ldots (3.1)$$

The d-tensor field A^i_{jk} must be to satisfy (i), (ii) and (2.3).

Next (i) means $N_{ij} = H^0_{ij}$. Also (ii) and (2.3) imply

$$A^{ij}_k + A^{ji}_k = 0, \qquad \ldots (3.2)$$

and

$$b^i_{;j} = B_r A^{ir}_j, \qquad \ldots (3.3)$$

respectively where $B_r = b^h a_{hr}$. Then we find $b^i_{;j} B_i = 0$, so the length of b^i relative to metric α is constant, i.e. $|b|^2 = a_{ri} b^r b^i = \text{const.}$ ... (3.4)

Conversely, for (3.4); there exists a d-tensor A^{ij}_k that satisfies (3.2) and (3.3). Thus we have :

Theorem — A Miron space with dual (α, β)-metric admits the h-metrical d-connection, if and only if the length of the vector field b^i relative to the Riemannian metric α is constant.

A d-tensor field

$$\overset{o}{A}{}^{ij}_k = (b^i_{;k} b^j - b^j_{;k} b^i)/|b|^2 \qquad \text{... (3.5)}$$

is an example satisfying the conditions (3.2) and (3.3). It must be noticed that $\overset{o}{H}{}^i_{jk} = \gamma^i_{jk} + \overset{o}{A}{}^i_{jk}$ is a function of x only. Accordingly we must have $C_{ijk}|_{\overset{o}{|}} = 0$ (Singh[6]). So we observe that in the Miron space stated in the Theorem.

$$C_{ijk}|_{\overset{o}{|}l} = 0, \qquad \text{... (3.6)}$$

holds, where "$\overset{o}{|}$" denotes h-covariant differentiation relative to d-connection $H\,\overset{o}{\Gamma}(N) = (\overset{o}{H}{}^i_{jk},\ C^{jk}_i)$, where

$$\overset{o}{H}{}^i_{jk} = \gamma^i_{jk} + \overset{o}{A}{}^i_{jk}.$$

Finally, we consider a conformal change in α such that

$$\bar{\alpha} = (\bar{a}^{ij}(x)\, p_i p_j),\ \bar{a}^{ij} = |b|^2 a^{ij}$$

holds in a domain D of M where $\beta \neq 0$. So for the Miron space $\bar{\xi}^n(D, h(\bar{\alpha}, \beta))$, $\bar{a}_{ij} b^i b^j = 1$ i.e., the condition (3.4) is satisfied. Thus we conclude :

Proposition 3.1 — A Miron space $(M, h(\alpha, \beta))$ with dual (α, β)-metric where $\beta \neq 0$, can be modified, by a conformal change of the associated space (M, α), to satisfy the condition (3.4).

Note : There are other suitable d-connections in a Miron space with dual (α, β) metric. However, these are future problem.

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