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A COMMON FIXED POINT THEOREM FOR COMPATIBLE MAPPINGS

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We obtain necessary and sufficient conditions for a pair of continuous mappings to possess a unique common fixed point.

1. INTRODUCTION

Several authors have obtained necessary and sufficient conditions for two or three commuting, continuous maps to possess a unique common fixed point. We list four of them.

Theorem A (Jungck[2]) — Let f be a continuous selfmap of a complete metric space (X, d). Then f has a fixed point in X if and only if there exists an $\alpha \in (0, 1)$ and a mapping $g : X \to X$ which commutes with f and satisfies $g(X) \subset f(X)$ and $d(gx, gy) \leq \alpha d(x, y)$ for all $x, y \in X$. Indeed, f and g have a unique common fixed point.

Theorem B (Fisher[1]) — Let S and T be continuous selfmaps of a complete metric space (X, d). Then S and T have a common fixed point in X if and only if there exists a continuous mapping A of X into $SX \cap TX$ such that $AS = SA$, $AT = TA$ and $d(Ax, Ay) \leq \alpha d(Sx, Ty)$ for all $x, y \in X$ and $0 < \alpha < 1$. Indeed, S, T and A have a unique common fixed point.

Recently Koparde and Waghmode[4] established a similar result, for a different contractive definition, in Hilbert spaces. But their theorem is also true in complete metric spaces, and is listed below.

Theorem C — Let S and T be continuous selfmaps of a Hilbert space X. Then S and T have a common fixed point in X if and only if there exists a continuous mapping A of X into $SX \bigcap TX$ which commutes with S and T and satisfies the inequality

$$\| Ax - Ay \| \leq \alpha \| Ax - Sx \| + \beta \| Ay - Ty \| + \gamma \| Sx - Ty \|$$

for all x, y in X, where $\alpha, \beta, \gamma \geq 0$ with $0 < \alpha + \beta + \gamma < 1$. Indeed S, T and A then have a unqiue common fixed point.

Let f be a continuous selfmap of a metric space X. A selfmap g of X is said to be f-contractive if $d(gx, gy) < d(fx, fy)$ for each x, y in X for which $gx \neq gy$.

Theorem D (Park[5]) — A continuous selfmap f of a metric space X has a fixed point if and only if there exists an f-contractive map g, which commutes with f, a subset $M \subset X$, and a point $x_0 \in M$ such that

$$d(fx, fx_0) - d(gx, gx_0) \geq 2d(fx_0, gx_0)$$

for every $x \in X \setminus M$, and g maps M into a compact subset of X. Indeed, f and g have a unique common fixed point.

From Jungck[3] let S and T be a pair of selfmaps of a complete metric space (X, d). Then S and T are said to be compatible if $\lim d(STx_n, TSx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim Sx_n = \lim Tx_n = t$ for some $t \in X$.

Let $\mathbb{R}^+$ denote the set of nonnegative reals, and $w : \mathbb{R}^+ \to \mathbb{R}^+$ a continuous function such that $0 < \omega(r) < r$ for all $r > 0$.

2. Theorem and Corollaries

Theorem — Let f and g be continuous selfmaps of a complete metric space (X, d). Then f and g have a common fixed point in X if and only if there exists a continuous map $h : X \to f(X) \bigcap g(X)$ which is compatible with f and g and which satisfies

$$\begin{aligned} d(hx, hy) \leq \max \{&d(hx, fx), d(hy, gy), d(fx, gy), \\ &[d(hx, gy) + d(hy, fx)]/2\} \\ &- w \max \{d(hx, fx), d(hy, gy), d(fx, gy), [d(hx, gy) \\ &+ d(hy, fx)]/2\}) \ldots \quad (2.1) \end{aligned}$$

for all $x, y \in X$. Indeed, f, g and h have a unique common fixed point.

Proof : We shall first show that the condition is sufficient. Let x_0 be any point of X. Since $h(X) \subset f(X)$, there exists a point $x_1 \in X$ such that $hx_0 = fx_1$. Since $x_1 \in X$ and $h(X) \subset g(X)$, there exists a point $x_2 \in X$ such that $hx_1 = gx_2$. In this way a sequence $\{x_n\}$ is constructed so that $hx_{2n} = fx_{2n+1}$ and $hx_{2n+1} = gx_{2n+2}$, $n = 0, 1, \ldots$. Define $d_n = d(hx_n, hx_{n+1})$. From (2.1),

$$d_{2n} = d(hx_{2n}, hx_{2n+1}) = d(hx_{2n+1}, hx_{2n})$$

$$\leq \max \{d(hx_{2n+1}, fx_{2n+1}), d(hx_{2n}, gx_{2n}), d(fx_{2n+1}, gx_{2n}), [d(hx_{2n+1}, gx_{2n}) + d(hx_{2n}, fx_{2n+1})]/2\}$$
$$- w(\max \{d(hx_{2n+1}, fx_{2n+1}), d(hx_{2n}, gx_{2n}), d(fx_{2n+1}, gx_{2n}), [d(hx_{2n+1}, gx_{2n}) + d(hx_{2n}, fx_{2n+1})]/2\})$$
$$= \max \{d(hx_{2n+1}, hx_{2n}), d(hx_{2n}, hx_{2n-1}), d(hx_{2n}, hx_{2n-1}), d(hx_{2n+1}, hx_{2n-1})/2\}$$
$$- w(\max \{d(hx_{2n+1}, hx_{2n}), d(hx_{2n}, hx_{2n-1}), d(hx_{2n+1}, hx_{2n-1})/2\})$$
$$= \max \{d_{2n}, d_{2n-1}, [d_{2n-1} + d_{2n}]/2\} - w(\max \{d_{2n}, d_{2n-1}, d_{2n}, [d_{2n-1} + d_{2n}]/2\}).$$

If $d_{2n} > d_{2n-1}$ for any n, then $d_{2n} \leq d_{2n} - w(d_{2n}) < d_{2n}$, a contradiction. Therefore $d_{2n} \leq d_{2n-1} - w(d_{2n-1})$. Similarly, it can be shown that $d_{2n+1} \leq d_{2n} - w(d_{2n})$, so that, for each n, $d_{n+1} \leq d_n - w(d_n)$, which implies that $\sum_{i=0}^{n} w(d_i) \leq d_0 - d_{n+1} \leq d_0$. Therefore the series converges and $\lim w(d_n) = 0$.

Since $\{d_n\}$ is a decreasing sequence of nonnegative terms, it converges. Call the limit p. Suppose that $p > 0$. Then, since w is continuous, $\lim w(d_n) = w(p) = 0$, a contradiction. Therefore $p = 0$.

We now wish to show that $\{hx_n\}$ is Cauchy sequence. Assume that it is not Cauchy. Then, for every positive number ε and for every positive integer k there exist two positive integers $2m(k)$ and $2n(k)$ such that $2m(k) > 2n(k) > k$ and $d(hx_{2m(k)}, hx_{2n(k)}) > \varepsilon$. Further, let $2m(k)$ denote the smallest even integer for which $2m(k) > 2n(k) > k$, $d(hx_{2m(k)}, hx_{2n(k)}) > \varepsilon$ and $d(hx_{2m(k)-2}, hx_{2n(k)}) \leq \varepsilon$.

Then

$$\varepsilon < d(hx_{2n(k)}, hx_{2m(k)}) \leq d(h_{2n(k)}, hx_{2m(k)-2}) + d_{2m(k)-2} + d_{2m(k)-1}$$
$$\leq \varepsilon + d_{2m(k)-2} + d_{2m(k)-1}.$$

Taking the limit as $n \to \infty$ yields

$$\lim d(hx_{2m(k)}, hx_{2n(k)}) = \varepsilon. \qquad \text{... (2.2)}$$

Using the triangular inequality,

$$| d(hx_{2m(k)}, hx_{2n(k)+1}) - d(hx_{2m(k)}, hx_{2n(k)}) | \leq d_{2n(k)},$$

$$| d(hx_{2m(k)+1}, hx_{2n(k)+1}) - d(hx_{2m(k)}, hx_{2n(k)+1}) | \leq d_{2m(k)},$$

and

$$| d(hx_{2m(k)+1}, hx_{2n(k)+2}) - d(hx_{2m(k)+1}, hx_{2n(k)+1}) | \leq d_{2n(k)+1}.$$

From (2.2) and the above inequalities

$$\varepsilon = \lim_k d(hx_{2m(k)}, hx_{2n(k)+1})$$

$$= \lim_k d(hx_{2m(k)+1}, hx_{2n(k)+1}) = \lim_k d(hx_{2m(k)+1}, hx_{2n(k)+2}).$$

From (2.1),

$$d(hx_{2m(k)+1}, hx_{2n(k)+2})$$

$$\leq \max \{d(hx_{2m(k)+1}, fx_{2m(k)+1}), d(hx_{2n(k)+2}, gx_{2n(k)+2}),$$

$$d(fx_{2m(k)+1}, gx_{2n(k)+2}), [d(hx_{2m(k)+1}, g(x_{2n(k)+2})$$

$$+ d(hx_{2n(k)+2}, fx_{2m(k)+1})]/2\} - w(\max \{d(hx_{2m(k)+1}, fx_{2m(k)+1}),$$

$$d(hx_{2n(k)+2}, gx_{2n(k)+2}), d(fx_{2m(k)+1}, gx_{2n(k)+2}),$$

$$[d(hx_{2m(k)+1}, g(x_{2n(k)+2}), d(hx_{2n(k)+2}, fx_{2m(k)+1})]/2\})$$

$$= \max \{d(hx_{2m(k)+1}, hx_{2m(k)}), d(hx_{2n(k)+2}, hx_{2n(k)+1}),$$

$$d(hx_{2m(k)}, hx_{2n(k)+1}), [d(hx_{2m(k)+1}, hx_{2n(k)+1})$$

$$+ d(hx_{2n(k)+2}, hx_{2m(k)})]/2\} - w \max \{d(hx_{2m(k)+1}, hx_{2m(k)}),$$

$$d(hx_{2n(k)+2}, hx_{2n(k)+1}), d(hx_{2m(k)}, hx_{2n(k)+1}), [d(hx_{2m(k)+1}, hx_{2n(k)+1}),$$

$$+ [d(hx_{2n(k)+2}, hx_{2m(k)})]/2\})$$

$$= \max \{d_{2n(k)+1}, d_{2m(k)}, d(hx_{2m(k)}, hx_{2n(k)+1}),$$

$$[d(hx_{2m(k)+1}, hx_{2n(k)+1}) + d_{2n(k)} + d_{2n(k)+1} + d(hx_{2m(k)}, hx_{2n(k)+1})]/2\}$$

$$- w (\max \{d_{2m(k)}, d_{2n(k)+1}, d(hx_{2m(k)}, hx_{2n(k)+1}),$$

$$[d(hx_{2m(k)+1}, hx_{2n(k)+1}) + d_{2n(k)} + d_{2n(k)+1} + d(hx_{2m(k)}, hx_{2n(k)+1})]/2\}).$$

Taking the limit as $n \to \infty$ we get $\varepsilon \leq \varepsilon - w(\varepsilon)$, which gives $w(\varepsilon) \leq 0$, a contradiction, so $\{hx_n\}$ is Cauchy, hence convergent. Call the limit u. Thus lim hx_{2n+1} = lim $fx_{2n+1} = u$. Since f and h are compatible,

$$\lim d(fhx_{2n+1}, hfx_{2n+1}) = 0. \qquad \text{... (2.3)}$$

Since also $\lim hx_{2n+2} = \lim gx_{2n+2} = u$,

$$\lim d(hgx_{2n+2}, ghx_{2n+2}) = 0. \qquad \text{... (2.4)}$$

The continuity of f, g and h imply that $fu = gu = hu$.

From the triangular inequality,

$$d(fx_{2n+1}, ghx_{2n+2}) \le d(fhx_{2n+1}, hfx_{2n+1}) + d(hfx_{2n+1}, hgx_{2n+2}) + d(hgx_{2n+2}, ghx_{2n+2}).$$

Taking the limit as $n \to \infty$, using (2.3), (2.4), and the continuity of f and g, we have

$$d(fu, gu) \le \lim d(hfx_{2n+1}, hgx_{2n+2}).$$

From (2.1),

$$\begin{aligned} d(hfx_{2n+1}, hgx_{2n+2}) \le \max \{&d(hfx_{2n+1}, ffx_{2n+1}),\ d(hgx_{2n+2}, ggx_{2n+2}), \\ &d(ffx_{2n+1}, ggx_{2n+2}),\ [d(hfx_{2n+1}, ggx_{2n+2}) \\ &+ d(hgx_{2n+2}, ffx_{2n+1})]/2\} \\ &- w(\max\{d(hfx_{2n+1}, ffx_{2n+1}), \\ &d(hgx_{2n+2}, ggx_{2n+2}), \\ &d(ffx_{2n+1}, ggx_{2n+2}),\ [d(hfx_{2n+1}, ggx_{2n+2}) \\ &+ d(hgx_{2n+2}, ffx_{2n+1})]/2\}). \qquad \ldots (2.5) \end{aligned}$$

From (2.3) and the continuity of f,

$$\lim d(hfx_{2n+1}, ffx_{2n+1}) \le \lim d(hfx_{2n+1}, fhx_{2n+1}) + \lim d(fhx_{2n+1}, ffx_{2n+1}) = 0.$$

From (2.4) and the continuity of g,

$$\lim d(hgx_{2n+2}, ggx_{2n+2}) \le \lim d(hgx_{2n+2}, ghx_{2n+2}) + \lim d(ghx_{2n+2}, ggx_{2n+2}) = 0.$$

From the continuity of f and g, $\lim d(ffx_{2n+1}, ggx_{2n+2}) = d(fu, gu)$.

Using (2.3), (2.4), and the continuity of f and g,

$$\begin{aligned} &\lim[d(hfx_{2n+1}, ggx_{2n+2}) + d(hgx_{2n+2}, ffx_{2n+1})]/2 \\ &\le \lim [d(hfx_{2n+1}, fhx_{2n+1}) + (fhx_{2n+1}, ggx_{2n+2}) \\ &\quad + d(hgx_{2n+2}, ghx_{2n+2}) + (ghx_{2n+2}, ffx_{2n+1})]/2 \\ &= d(fu, gu). \end{aligned}$$

Taking the limit of (2.5) as $n \to \infty$ yields

$$d(fu, gu) \le d(fu, gu) - w(d(fu, gu)),$$

which implies that $fu = gu$.

In a similar manner it can be shown that $fu = hu$.

Using (2.1) and the continuity of f and g,

$$d(fhx_{2n+1}, hx_{2n+2}) \leq d(fhx_{2n+1}, hfx_{2n+1}) + (hfx_{2n+1}, hx_{2n+2}).$$

$$d(hfx_{2n+1}, hx_{2n+2}) \leq \max\{d(hfx_{2n+1}, ffx_{2n+2}), d(hx_{2n+2}, gx_{2n+2}), d(ffx_{2n+1}, gx_{2n+2}), [d(hfx_{2n+1}, gx_{2n+2}) + d(hx_{2n+2}, ffx_{2n+1})]/2\} - w(\max\{d(hfx_{2n+1}, ffx_{2n+2}), d(hx_{2n+2}, gx_{2n+2}), d(ffx_{2n+1}, gx_{2n+2}), [d(hfx_{2n+1}, gx_{2n+2}) + d(hx_{2n+2}, ffx_{2n+1})]/2\}).$$

Taking the limit as $n \to \infty$ yields

$$d(fu, u) \leq d(fu, u) - w(d(fu, u)),$$

which implies that $fu = u$ and u is a common fixed point of f, g and h.

Let v be another common fixed point. Then, from (2.1),

$$\begin{aligned} d(u, v) = d(hu, hv) &\leq \max\{d(hu, fu), d(hv, gv), d(fu, gv), [d(hu, gv) + d(hv, fu)]/2\} \\ &\quad - w(\max\{d(hu, fu), d(hv, gv), d(fu, gv), [d(hu, gv) + d(hv, fu)]/2\}) \\ &= d(u, v) - w(d(u, v)), \end{aligned}$$

which implies that $u = v$.

To prove the condition necessary, let $fz = gz = z$ for some $z \in X$, and define h by $hx = z$ for all $x \in X$. Then h is continuous from X to $f(X) \cap g(X)$. Moreover, for $x \in X$, $hfx = z$, $fhx = fz = z$, and $hgx = z$, $ghx = z$, $ghx = gz = z$, so h commutes with f and g, and therefore the maps are compatible.

Further, h satisfies (2.1).

We have the following Corollaries.

Corollary 1 — Let f and g be continuous selfmaps of a complete metric space (X, d). Then f and g have a unique common fixed point if and only if

$$d(x, y) \leq \max\{d(x, fx), d(y, gy), d(fx, gy), [d(x, gy) + d(y, fx)]/2\} - w(\max\{d(x, fx), d(y, gy), d(fx, gy), [d(x, gy) + d(y, fx)]/2\})$$

for all $x, y \in X$.

PROOF : Set $h = I$, the identity map, in the Theorem.

Corollary 2 — Let f be a continuous selfmap of a complete metric space (X, d). Then f has a unique fixed point if and only if

$$d(x, y) \leq \max \{d(x, fx), d(y, fy), d(x, y), [d(x, fy) + d(y, fx)]/2\}$$

$$- w (\max \{d(x, fx), d(y, fy), d(x, y), [d(x, fy) + d(y, fx)]/2\})$$

for all $x, y \in X$.

PROOF : Let $f = g$ in Corollary 1.

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