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ON COMMON FIXED POINTS WITH APPLICATIONS TO DIFFERENTIAL EQUATIONS IN ORDERED BANACH SPACES

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The aim of the present paper is to consider existence of common fixed points of operators in ordered spaces by applying a generalized monotone iteration method. No continuity hypotheses are imposed on the operators in question. As an application we shall study existence of common solutions of two discontinuous differential equations in ordered Banach spaces.

1. INTRODUCTION

Let P be a nonempty set P equipped with a 'partial ordering relation' $\leq$. Denote $x < y$ when $x \leq y$ and $x \neq y$. A mapping $G : P \to P$ is called increasing in a subset X of P if $x, y \in X$ and $x \leq y$ imply that $Gx \leq Gy$. G is said to be well-order closed if sup $G[C]$ exists in P whenever C and $G[C]$ are well-ordered chains in P.

The following lemma forms a basis to our study.

Lemma 1.1 (Heikkilä and Lakshmikantham[2], Corollary 1.2.1) — Assume that $G : P \to P$ is well-order closed.

(a) If $y \leq Gy$ for each $y \in P$, then G has a fixed point.

(b) If $a \leq Ga$ and G is increasing in $[a)$, then G has the least fixed point in $[a) = \{x \in P \mid a \leq x\}$.

The assertions can be proved by using a generalized monotone iteration method developed in Heikkilä and Lakshmikantham[2]. In both cases the fixed point in question is obtained as the maximum of the well-ordered chain C in P satisfying

$$a = \min C \text{ and } a < x \in C \text{ if and only if } x = \sup G\{y \in C \mid y < x\},$$

where a in case (a) is an arbitrary element of P.

As a consequence of Lemma 1.1 we have :

Proposition 1.1 — Mappings A, B : $P \to P$ have a common fixed point if AB is well-order closed, and if

(a) $y \leq By \leq ABy$ for all $y \in P$.

PROOF : Condition (a) implies that $y \leq ABy$ whenever $y \in P$. Because AB is well-order closed, then AB has by Lemma 1.1 a fixed point x. Applying condition (a) we obtain $x \leq Bx \leq ABx = x$, so that $x = Bx$. Hence also $x = ABx = Ax$, whence x is a common fixed point of A and B. □

Corollary 1.1 — Let A, B : $P \to P$ be two mappings such that AB is well-order closed. If $x \leq Ax$ and $x \leq Bx$ for every $x \in P$, then A and B have a common fixed point.

If B : $P \to P$ is increasing and A : $P \to P$ is well-order closed, then AB is well-order closed, since the image $B[C]$ of every well-ordered chain C is a well-ordered chain if B is increasing. Thus we have

Corollary 1.2 — Let A, B : $P \to P$ be two mappings such that B : $P \to P$ is increasing, A : $P \to P$ is well-order closed and conditions (a) of Proposition 1.1 hold. Then A and B have a common fixed point.

Proposition 1.2 — Let A, B : $P \to P$ be two mappings such that

(i) $AB[P]$ has a lower bound a in P, AB is increasing in $[a)$ and AB is well-order closed;

(ii) $By \leq ABy \leq BABy$ for all $y \in P$.

Then A and B have the least common fixed point.

PROOF : From (i) it follows that the hypotheses of Lemma 1.1 (b) hold for $G = AB$, whence AB has the least fixed point x_* in $[a)$. Condition (ii) implies that $x_* \leq Bx_*$ and $Bx_* \leq ABx_* = x_*$, hence $Bx_* = x_*$. Thus $Ax_* = ABx_* = x_*$, so that x_* is a common fixed point of A and B. If x is another common fixed point of A and B, then it is also a fixed point of AB, and $a \leq ABx = x$, whence $x \in [a)$. Then $x_* \leq x$, since x_* is the least fixed point of AB. □

2. Common Fixed Point Results in Ordered Banach Spaces

Let E be a vector space. A subset K of E is a positive cone in E if $K + K \subseteq K$, $-K \cap K = \{0\}$ and $\lambda K \subseteq K$ whenever $\lambda \geq 0$. The relation

$$x \leq y \text{ if and only if } y - x \in K,$$

defines a partial ordering in E. We say that E is ordered by K, and that K is the order cone of E. Note that $K = \{u \in E \mid 0 \leq u\}$. By an ordered Banach space we mean a Banach space $E = (E, \|\cdot\|)$ equipped with a partial ordering $\leq$ induced by a closed cone K in E. We say that a sequence $(x_n)_{n=0}^{\infty}$ of E is increasing (resp. decreasing) if $x_n \leq x_m$ (resp. $x_m \leq x_n$) whenever $n \leq m$. The next result follows from Proposition 1.1.5 of Heikkilä and Lakshmikantham[2].

Lemma 2.1 — Let C be a chain in an ordered Banach space E. If each increasing sequence of C has a subsequential limit, then C contains an increasing sequence which converges to $\sup C$.

The order cone K of an ordered Banach space E is called regular if each increasing and order bounded sequence of K is convergent in E. If every increasing and norm bounded sequence of K is convergent in E, we say that K is fully regular. It can be shown that if K is fully regular then it is also regular. Converse holds if E is reflexive (cf. Guo and Lakshmikantham[1]).

Proposition 2.1 — Let E be an ordered Banach space with an order cone K, P a closed subset of E and $A, B : P \to P$ two mappings such that

(1) $By \le ABy \le BABy$ whenever $y \in P$;

(2) AB is increasing and $AB[P]$ has a lower bound in P.

Then A and B have the least common fixed point in the following cases :

(a) $AB[C]$ is relatively compact whenever C is a well-ordered chain in P;

(b) K is regular and P is order bounded;

(c) K is fully regular and P is norm bounded;

(d) E is reflexive and P is norm bounded and convex.

PROOF : Each of the hypotheses (a)-(c) imply by Lemma 2.1 that AB is well-order closed. The same holds also in case (d) by Proposition 1.3.6 of Heikkilä and Lakshmikantham[2]. Thus the assertions follow from Proposition 1.2. □

3. Applications to Differential Equations

Let $J = [0, T]$, $T > 0$, and $(E, \|\cdot\|)$ an ordered Banach space with order cone K. Denote by $AC(J, E)$ the space of all absolutely continuous functions $x : J \to E$. A partial ordering in $C(J, E)$ can be defined by

$$x \le y \text{ if and only if } x(t) \le y(t) \text{ for every } t \in J.$$

Consider now two initial value problems (IVP)

$$x' = f(t, x) \quad x(0) = x_0 \qquad \text{... (3.1)}$$

and

$$x' = g(t, x) \quad x(0) = x_0, \qquad \text{... (3.2)}$$

where $f, g : J \times E \to E$ and $x_0 \in E$. A function $x \in AC(J, E)$ is called an upper solution of (3.1) if $x'(t) \ge f(t, x(t))$ for almost all $t \in J$, and $x(0) \ge x_0$, and a lower solution if the reversed inequalities are satisfied. If equalities hold, we say that y is a solution of (3.1).

Denote by $\mathbb{R}_+$ the set of nonnegative real numbers, and assume that the functions $f, g : J \times E \to E$ satisfy for given $x_0 \in E$ the following conditions :

(A0) $f(\cdot, x(\cdot))$ and $g(\cdot, x(\cdot))$ are strongly measurable whenever $x \in AC(J, E)$;

(A1) $\max \{\| f(t, x) \|, \| g(t, x) \|\} \le h(t, \| x \|)$ for all $x \in E$ and for almost all $t \in J$ where $h : J \times \mathbb{R}_+ \to \mathbb{R}_+$ is increasing in its second variable and the IVP

$$u' = h(t, u), \; u(0) = \|x_0\| \qquad \ldots (3.3)$$

has an upper solution.

Lemma 3.1 — Let the hypotheses (A0) and (A1) hold, and denote

$$P = \{x \in AC(J, E) \mid \|x(t)\| \le u(t) \text{ for all } t \in J\}, \qquad \ldots (3.4)$$

where u is an upper solution of (3.3). Then the equations

$$Ax(t) = x_0 + \int_0^t f(s, x(s))\, ds \text{ and } Bx(t) = x_0 + \int_0^t g(s, x(s))\, ds \qquad \ldots (3.5)$$

define mappings $A, B : P \to P$.

PROOF : Clearly $x(t) \equiv x_0 \in P$ so that P is nonempty. By condition (A1) we have

$$\max\{\|f(s, x(s))\|, \|g(s, x(s))\|\} \le h(s, \|x(s)\|) \le h(s, u(s)) \le u'(s) \qquad \ldots (a)$$

for all $s \in J$ and $x \in P$. This and condition (A0) ensure that A and B are defined in P. Applying (3.5) and (a) we obtain

$$\|Ax(t)\| \le \|x_0\| + \int_0^t \|f(s, x(s))\|\, ds \le \|x_0\| + \int_0^t u'(s)\, ds \le u(t)$$

for all $t \in J$, whenever $x \in P$. Similarly one can show that $\|Bx(t)\| \le u(t)$ for all $t \in J$ and $x \in P$, so that $A[P] \subseteq P$ and $B[P] \subseteq P$. □

Proposition 3.1 — The IVPs (3.1) and (3.2) have a common solution if f, g satisfy hypotheses (A0), (A1) and

(A2) $y(t) \le x_0 + \int_0^t g(s, y(s))\, ds$ and $g(t, y(t)) \le f(t, x_0 + \int_0^t g(s, y(s))\, ds)$ for all $t \in J$ and $y \in P$,

and if E is one of the following spaces :

(a) an ordered normed space with fully regular order cone,

(b) a reflexive Banach space, ordered by any closed cone or via continuous embedding in an ordered Banach space with fully regular order cone,

(c) an ordered Hilbert space,

(d) $\mathbb{R}^m$ with any norm and ordered by any closed cone,

(e) The space $B(\Omega, Y)$ of bounded functions $u : \Omega \to Y$, with pointwise ordering and supremum norm, where Ω is a nonempty set and Y is an ordered Banach space which is reflexive or its order cone is fully regular,

(f) $L^p(\Omega, Y)$, with a.e. pointwise ordering, where Y is an ordered Banach space with fully regular order cone, $1 \leq p \leq \infty$, and $(\Omega, \mathcal{A}, \mu)$ is a measure space with σ-finite μ if $p = \infty$,

(g) $L^p([a, b], Y)$, with a.e. pointwise ordering, where Y is an ordered reflexive Banach space and $1 \leq p < \infty$,

(h) sequence space $l^p(Y)$, with componentwise ordering, where Y is an ordered Banach space with fully regular order cone and $1 \leq p \leq \infty$,

(i) sequence space $l^p(Y)$, with componentwise ordering, where Y is an ordered reflexive Banach space and $1 < p < \infty$,

(j) Orlicz space $L_M(\Omega)$, with a.e. pointwise ordering, where Ω is a bounded domain in $\mathbb{R}^m$ and M satisfies the Δ_2-condition,

(k) Sobolev space $W^{k,p}(\Omega)$, with a.e. pointwise ordering, where Ω is an open subset of $\mathbb{R}^m$, $k \in \mathbb{N}$ and $1 < p < \infty$,

(l) Orlicz-Sobolev space $W^k L_M(\Omega)$, with a.e. pointwise ordering, where Ω is a bounded domain in $\mathbb{R}^m$, $k \in \mathbb{N}$, and both M and $\overline{M}$ satisfy the Δ_2-condition.

PROOF : Applying (A2) and (3.4) we obtain for all $y \in P$ and $t \in J$

$$y(t) \leq By(t) = x_0 + \int_0^t g(s, y(s))\, ds \leq x_0 + \int_0^t f(s, x_0 + \int_0^s g(\tau, y(\tau))\, d\tau)\, ds$$

$$= x_0 + \int_0^t f(s, By(s))\, ds = ABy(t),$$

whence $y \leq By \leq ABy$ for each $y \in P$. If $t, \bar{t} \in J$, and $x \in P$, then

$$\| ABx(t) - ABx(\bar{t}) \| = \| \int_{\bar{t}}^t f(s, Bx(s))\, ds \| \leq | \int_{\bar{t}}^t \| f(s, Bx(s)) \|\, ds |$$

$$\leq | \int_{\bar{t}}^t u'(s))\, ds | = | u(t) - u(\bar{t}) | \text{ for all } x \in P.$$

This implies that if C is a well-ordered chain in P, then the chain $AB[C]$ is equicontinuous. Moreover, $AB[C] \subseteq P$, so that $AB[C]$ is pointwise norm bounded in E. From Theorems 5.8.2 and 5.8.3 of Heikkilä and Lakshmikantham[2] it then follows that each one of the hypotheses (a)-(l) imply the existence of sup $AB[C]$ in P. Thus all the hypotheses of Proposition 1.1 hold, and A and B have a common fixed point x in P. This means that x is a common solution of the integral equations

$$x(t) = x_0 + \int_0^t f(s, x(s))\, ds \text{ and } x(t) = x_0 + \int_0^t g(s, x(s))\, ds$$

in P, or, equivalently, x is a common solution of (3.1) and (3.2) in P. □

As a consequence of Proposition 1.2 we obtain :

Proposition 3.2 — The IVPs (3.1) and (3.2) have the least common solution in P if $f : J \times E \to K$ and $g : J \times E \to E$ satisfy hypotheses (A0), (A1) and

(A3) $f(t, \cdot)$ and $g(t, \cdot)$ are increasing in $[x_0)$ for almost all $t \in J$;

(A4) $$f(t, x(t)) \leq g(t, x_0 + \int_0^t f(s, x(s))\, ds \quad \text{and} \quad g(t, x(t)) \leq f(t, x_0 + \int_0^t g(s, x(s))\, ds)$$
for $t \in J$ and $x \in P$,

and if E is one of the spaces listed in Proposition 3.1(a)-(l).

PROOF : Let $A, B : P \to P$ be defined by (3.5). Since f is K-valued, then $a(t) \equiv x_0$ is a lower bound of $AB[P]$ in P. From (A3) it follows that AB is increasing in $[a)$, and the proof of Proposition 3.1 implies that AB is well-order closed. Applying condition (A4) we obtain (cf. the proof of Proposition 3.1) $By \leq ABy$ and $Ay \leq BAy$ for each $y \in P$. Thus A and B satisfy the hypotheses of Proposition 1.2, whence they have the least common fixed ppint x_*. By (3.5), x_* is then the least common solution of the IVPs (3.1) and (3.2) in P. □

From Corollary 1.1 it follows :

Proposition 3.3 — The IVPs (3.1) and (3.2) have a common solution if E is one of the spaces listed in Proposition 3.1, and if f, g satisfy hypotheses (A0), (A1) and

(A5) $$x(t) \leq x_0 + \int_0^t f(s, x(s))\, ds \text{ and } x(t) \leq x_0 + \int_0^t g(s, x(s))\, ds \text{ for } t \in J, x \in P.$$

As a consequence of Lemma 1.1 we obtain the following result concerning the solvability of the IVP (3.1).

Proposition 3.4 — Let E be one of the spaces listed in Proposition 3.1, and let $f : J \times E \to E$ satisfy conditions given in (A0) and (A1) for f. If

(f0) $$x(t) \leq x_0 + \int_0^t f(s, x(s))\, ds \text{ for all } t \in J \text{ and } x \in P,$$

then the IVP (3.1) has a solution in P. If (f0) is replaced by

(f1) f is K-valued and $f(t, \cdot)$ is increasing in $[x_0)$ for almost all $t \in J$, then (3.1) has the least solution in P.

Remark 3.1 : No continuity hypotheses are imposed on mappings A and B in sections 1 and 2, and on functions f and g in section 3.

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