Indian J. pure appl. Math., 26(5) : 441-443, May 1995

SUBSTITUTION OPERATORS ON A SPACE OF INFINITELY DIFFERENTIABLE FUNCTIONS

R. K. SINGH[1], NAND LAL[2] AND ROMESH KUMAR[3]

[1,3]*Department of Mathematics, University of Jammu, Jammu* 180 004
[2]*Department of Mathematics, Banaras Hindu University, Varanasi* 221 005

(*Received* 28 *February* 1994; *after revision* 4 *January* 1995; *accepted* 11 *January* 1995)

In this article we characterize the substitution operators on the space of infinitely differentiable functions with compact supports.

1. INTRODUCTION AND PRELIMINARIES

If $w : \mathbb{R} \to \mathbb{R}$ (or $\mathbb{C}$) is a function, then by the support of w, denoted as suppt. w, we mean the closure of the set $\{x \in \mathbb{R} : w(x) \neq 0\}$. By $C^\infty(\mathbb{R})$, we mean the space of infinitely differentiable functions on $\mathbb{R}$ and by $D(\mathbb{R})$, we denote the vector space of all C^∞-functions with compact supports in $\mathbb{R}$. Obviously, $D(\mathbb{R})$ is a subspace of $C^\infty(\mathbb{R})$. For a compact subset K of $\mathbb{R}$, let $D_K(\mathbb{R}) = \{f \in D(\mathbb{R}) : \text{suppt. } f \subset K\}$. Let $D_K(\mathbb{R})$ have the topology defined by the seminorms

$$P_{K,m}(f) = \sup \{|f^{(\alpha)}(x)| : 0 \leq \alpha \leq m, \ x \in K\},$$

where $m \geq 0$ and $f^{(\alpha)}$ denotes the derivative of order α of f. Then $(D(\mathbb{R}), D_K(\mathbb{R}) : K$ is compact in $\mathbb{R})$ is an inductive system. With the inductive limit topology $D(\mathbb{R})$ is a topological vector space[1]. A sequence $\{f_n\}$ of functions in $D(\mathbb{R})$ converges to a function $f \in D(\mathbb{R})$ if there exists a compact subset K of $\mathbb{R}$ such that

(i) suppt. $f \subset K$ and suppt. $f_n \subset K$ for all n

(ii) for each positive integer α, the sequence $\{f_n^{(\alpha)}\}$ of the derivative of order α converges uniformly to the derivative of order α of f.

A substitution operator on $D(\mathbb{R})$ is a continuous linear map from $D(\mathbb{R})$ into itself of the form $f \to f \circ \phi$, where $\phi : \mathbb{R} \to \mathbb{R}$ is a map. We denote such a map by C_ϕ and call it the substitution operator induced by ϕ. The following result, which we shall employ in the paper is well known.

Lemma 1.1 — Let K be a compact subset of $\mathbb{R}$. Then there exist a function $f \in D(\mathbb{R})$ such that $f(x) = 1$ for every $x \in K$.

For details about the space $D(\mathbb{R})$ we refer to Conway[1], Kesavan[3] and Rudin[4].

2. Characterization of Substitution Operators

In this section we find out those mappings which induce substitution operators on $D(\mathbb{R})$.

Theorem 2.1 — Let $\phi : \mathbb{R} \to \mathbb{R}$ be a mapping. Then ϕ induces a substitution operator on $D(\mathbb{R})$ if and only if ϕ is infinitely differentiable and $\phi^{-1}(K)$ is a compact subset of $\mathbb{R}$ for every compact subset K of $\mathbb{R}$.

Proof : Suppose ϕ induces a substitution operator on $D(\mathbb{R})$. Then $f \circ \phi \in D(\mathbb{R})$ for every $f \in D(\mathbb{R})$. First of all we show that ϕ is infinitely differentiable. We claim that $\phi([-k, k])$ is a bounded subset of $\mathbb{R}$ for each $k \in \mathbb{N}$. Suppose on the contrary, for some k, $\phi([-k, k])$ is unbounded. Then there is a sequence $\{x_n\}$ in $[-k, k]$ such that $x_n \to x$ in $[-k, k]$, $|\phi(x_n)| \to \infty$ and $x_n \neq x$. By Lemma 1.1 there exists a function f in $D(\mathbb{R})$ such that $f(\phi(x)) = 1$. Since $f \circ \phi \in D(\mathbb{R})$.

$$\lim_{x_n \to x} \frac{(f \circ \phi)(x_n) - (f \circ \phi)(x)}{x_n - x} \text{ must exist.}$$

Since suppt. f is compact, $\phi(x_n) \notin$ suppt. f, $n \geq N$ for some $N \in \mathbb{N}$. Thus $(f \circ \phi)(x_n) = 0$ for every $n \geq N$. Also $-f(\phi(x)) = -1$. Thus

$$\lim_{x_n \to x} \left[\frac{(f \circ \phi)(x_n) - (f \circ \phi)(x)}{x_n - x} \right] = \lim_{x_n \to x} \left[-\frac{1}{x_n - x} \right].$$

But the limit on the right hand side of the above equation does not exist. This is a contradiction. Therefore $\phi([-k, k])$ is bounded for each $k \in \mathbb{N}$. Now we show that $\phi \in C^\infty(\mathbb{R})$. If $k \in \mathbb{N}$, then $\phi([-k, k])$ is bounded and hence there exists a positive number p such that $\phi([-k, k]) \subset [-p, p]$. By Lemma 1.1, there is a function h in $D(\mathbb{R})$ such that $h(x) = 1$ for $x \in [-p, p]$. Let the function g be given by $g(x) = x$ for every $x \in \mathbb{R}$ and let $f = gh$. Then $f \in D(\mathbb{R})$ and $f(x) = x$ for every x in $[-p, p]$. Since $f \circ \phi \in D(\mathbb{R})$, we have $f(\phi(x)) = \phi(x)$ for all $x \in (-k, k)$. Therefore, ϕ is infinitely differentiable on $(-k, k)$ for each $k \in \mathbb{N}$, and hence ϕ is infinitely differentiable on $\mathbb{R}$. Next, we show that $\phi^{-1}(K)$ is a compact subset of $\mathbb{R}$ whenever K is a compact subset of $\mathbb{R}$. By Lemma 1.1 there exists an $f \in D(\mathbb{R})$ such that $f(x) = 1$ for all $x \in K$. Let E be the closure of the set $\{x \in \mathbb{R} : f(\phi(x)) \neq 0\}$. Since $f \circ \phi \in D(\mathbb{R})$, the set E is compact. Now $\phi^{-1}(K) \subset E$ and therefore, $\phi^{-1}(K)$ is compact.

Conversely, suppose ϕ is infinitely differentiable and $\phi^{-1}(K)$ is a compact subset of $\mathbb{R}$ for every compact subset K of $\mathbb{R}$. For $f \in D(\mathbb{R})$, let $F = \{x \in \mathbb{R} : f(\phi(x)) \neq 0\}$ and let K be the closure of the set $\{x \in \mathbb{R} : f(x) \neq 0\}$. Then K is compact. Now $F = (f \circ \phi)^{-1}(\mathbb{R} \setminus \{0\}) \subset \phi^{-1}(K)$. Thus $\overline{F} \subset \phi^{-1}(K)$, where $\overline{F}$ is closure of F which shows that suppt. $(f \circ \phi)$ is compact. Since f and ϕ are infinitely differentiable, it follows that $f \circ \phi \in D(\mathbb{R})$. Let $C_\phi f = f \circ \phi$ for every $f \in D(\mathbb{R})$. Then

C_ϕ is a linear transformation on $D(\mathbb{R})$. We shall establish the continuity of C_ϕ. To show this let $f_n \to f$ in $D(\mathbb{R})$. Then there exists a compact set K such that suppt. $f \subset K$ and suppt. $f_n \subset K$ for all n, and for each positive integer α, the sequence $\{f_n^{(\alpha)}\}$ of the derivatives of order α converges uniformly to the derivative $f^{(\alpha)}$ of order α. We will show that $C_\phi f_n \to C_\phi f$ in $D(\mathbb{R})$. Clearly the suppt. $(f \circ \phi) \subset \phi^{-1}(K)$ and suppt. $(f_n \circ \phi) \subset \phi^{-1}(K)$ for all n. From Kamowitz and Wortman[2], one can easily show that $(f_n \circ \phi)^{(\alpha)} \to (f \circ \phi)^{(\alpha)}$ uniformly on $\phi^{-1}(K)$ for each positive integer α. Thus C_ϕ is continuous, and hence ϕ induces a substitution operator on $D(\mathbb{R})$. This completes the proof of the theorem.

REFERENCES

1. J. B. Conway, *A Course in Functional Analysis*, Springer-Verlag, New York, 1985.
2. H. Kamowitz and D. Wortman, *Rocky Mountain J. Math.* **17** (1987), 767-82.
3. S. Kesavan, *Topics in Functional Analysis and Applications*, Wiley Eastern Limited, New Delhi, 1989.
4. W. Rudin, *Functional Analysis*, Tata McGraw-Hill, Inc., New York, 1974.
5. R. K. Singh, *Proc. Am. Math. Soc.* **59** (1976), 329-33.