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ON THE ENVOLVENT THEOREM IN MULTIOBJECTIVE PROGRAMMING

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The main object of this paper is to prove some extensions of the classical envolvent theorem for the multiobjective programming. The results stated here are quite general since the programs are defined between arbitrary Banach spaces and its sensitivity is measured with respect to any parameter appearing in the restrictions.

1. INTRODUCTION

One of the reasons which makes difficult the study of the properties of a multiobjective minimization program is the fact that the solutions are minimal for the objective function and not minima in the usual sense (of the order relations). The most usual form (but not the only one) of avoiding this difficulty is to find solutions (called proper solutions) which are minima when the objective function is composed with a positive $\mathbb{R}$-valued linear functional. In Balbas *et al.*[5, 6] the solutions which are minima (and not only minimal) composing the objective function with a positive linear functional T with values in an ordered vector space (and not necessarily $\mathbb{R}$-valued) are studied, stating also (see Theorem 5 of Balbas *et al.*[5]) several conditions, easily verified in practice, which allow to assure that this functional T is a topological isomorphism. Also it is proved under quite weak conditions (see Theorems 11 and 13 and Remarks 12 and 14 of Balbas *et al.*[6]) that the set of the minimal solutions which can be transformed into minima composing the objective function with a topological isomorphism T, is dense in the efficient line and thus these solutions constitute a set which is wide enough to describe the full efficient line of the problem.

The object of this paper is to state envolvent theorems for the multiobjective programming in order to measure the sensitivity of the program in relation with any of the parameters which appear in the restriction. This is made in the context of Banach spaces (with arbitrary dimension). Thus the decision variables and the parameters belong to Banach spaces while the objective function and the restrictions are valued in ordered Banach spaces.

Theorems 6 and 7 stated in section three allow to analyze the sensitivity of the optimum value of the objective function composed with a positive linear functional T valued in a Banach lattice. Section 4 is devoted to measure the optimum value of the objective function and so it is where the main results are proved. Thus Theorem 12 states for the convex and the differential programming that the sensitivity of the program depends on the Lagrange multiplier and its sensitivity. In Remark 13, it is poined out that the classic envolvent theorem is a particular case of Theorem 12 and also other interesting particular cases are glossed.

Finally, an example about the sensitivity of a discrete time multiobjective optimal control problem is given which shows how the developed theory works in practice.

2. Notations and Preliminaries

Let X, Y, Z and P be four Banach spaces such that Y and Z are ordered vector spaces with closed convex positive cones Y_+ and Z_+ with Y_+ pointed. Consider also a subset $X_0 \subseteq X$, an open subset $P_0 \subseteq P$ and two functions $f : X_0 \to Y$ and $g : X_0 \times P_0 \to Z$. For every $p \in P_0$ let us denote by (1_p) the following optimization program :

$$\left.\begin{array}{l}\text{Mín}_{x\in X_0} \ f(x)\\ g(x,p) \le 0\end{array}\right\} (1_p) .$$

We say that $x_0 \in X$ is an optimal solution of the last program if $f(x_0)$ is a minimal element of the set $\{f(x) : x \in X_0,\ g(x,p) \le 0\}$, i.e., if there is no $x_1 \in X_0$ such that $g(x_1, p) \le 0$ and $f(x_0) - f(x_1) \in Y_+ - \{0\}$.

All along the paper, for every fixed elements $p_0 \in P_0$ and $x_0 \in X_0$, $g_{p_0} : X_0 \to Z$ and $g_{x_0} : P_0 \to Z$ will be the functions such that $g_{p_0}(x) = g(x, p_0)$ and $g_{x_0}(p) = g(x_0, p)$ for every $x \in X_0$ and $p \in P_0$, respectively. Also the program (1_p) $(p \in P_0)$ will be said to be convex (respectively differentiable) if X_0 is a convex (respectively open) subset of X and the functions f and g_p are convex (respectively continuously Fréchet differentiable).

Let W be an order complete Banach lattice ordered by a closed pointed cone W_+ (i.e., for each non empty order-bounded from below subset D of W the inf D exists and moreover $\|w_1\| \le \|w_2\|$ for every $w_1, w_2 \in W$ such that $0 \le w_1 \le w_2$) and $T : Y \to W$ a positive (i.e., $T(Y_+ - \{0\}) \subseteq W_+ - \{0\}$) linear and continuous surjective mapping such that Ker T has a topological supplement that we will denote by Y_T (recall that in the case of being Y a Hilbert space then we can take Y_T the orthogonal of Ker T). It follows now from the open-mapping theorem that the restriction T^* of T to Y_T is a topological isomorphism and as usual its inverse will be denoted by T^{*-1}.

Definition 1 — Let $p \in P_0$ and $x_p \in X_0$. We say that x_p is a T-optimal solution (or also a T-optimum) of the program (1_p) if $g_p(x_p) \le 0$ and $Tf(x_p) \le Tf(x)$ for every

$x \in X_0$ such that $g_p(x) \leq 0$ (along the paper Tf will denote the composition of f and T). Since T is a positive operator, it is clear that every T-optimal solution of (1_p) is also an optimal solution of this program.

From now on let us assume that for every $p \in P_0$ the set $\{Tf(x) : x \in X_0, g(x, p) \leq 0\}$ is order-bounded from below.

Definition 2 — Let $p \in P_0$ and $L_p : Z \to W$ be a non negative (i.e. $L_p(Z_+) \subseteq W_+$) linear and continuous operator.

2.1 If the program (1_p) is convex then we say that L_p is a Lagrange T-multiplier (of the program (1_p)) if the following equality holds :

$$\inf \{Tf(x) + L_p\, g_p(x) : x \in X_0\} = \inf \{Tf(x) : x \in X_0, g_p(x) \leq 0\}. \quad \text{... (1)}$$

2.2. If the program (1_p) is differentiable and x_p is a T-optimal solution of (1_p) then we say that L_p is a Lagrange T-multiplier (of the program (1_p)) associated to x_p if $L_p\, g_p\,(x_p) = 0$ and

$$T(df)_{x_p} = -L_p\,(dg_p)_{x_p},$$

where $(df)_{x_p}$ and $(dg_p)_{x_p}$ denote respectively the differentials of f and g_p at x_p.

The last two definitions are equivalent when the program (1_p) is convex and differentiable. Formally the following result can be stated whose proof is omitted since it is analogous to that of the corresponding known result for scalar programming:

Proposition 3 — If $p \in P_0$, the program (1_p) is convex and differentiable, x_p is a T-optimal solution of (1_p) and $L : Z \to W$ is a non-negative linear and continuous operator, then the following assertions are equivalent :

(i) L is a Lagrange T-multiplier of (1_p).

(ii) L is a Lagrange T-multiplier of (1_p) associated to x_p.

Several results about the existence of Lagrange T-multipliers can be stablished with the usual techniques. For instance, let us state one of them.

Theorem 4 — If $p \in P_0$ and (1_p) is a convex program having a T-optimal solution which satisfies the Slater condition (i.e., there exists $x_1 \in X_0$ such that $g(x, p)$ is an interior point of Z_+), then there exi s a Lagrange T-multiplier of (1_p).

PROOF : It is enough to proceed in the standard way from Th ·em 5 of Zowe[21].

Other theorems about the existence of Lagrange T-multipliers can be stated from the results obtained, for instance, in Luc[11], Singh[17] and Zowe[21].

3. THE ENVOLVENT THEOREM

This section will be devoted to study the sensitivity of the optimum value of the function Tf with respect to the changes of the parameter p. This will allow us in the next section to analize the sensitivity of the optimum value of f.

From now on along the paper let us assume that for every $p \in P_0$ there exists a T-optimal solution x_p of the program (1_p), the function g_{x_p} is differentiable at P_0 and the following two conditions hold :

$$\lim_{p \to p_0} \| (dg_{x_p})_{p_0} - (dg_{x_{p_0}})_{p_0} \| = 0 \qquad \ldots (2)$$

and

$$\lim_{p \to p_0} \frac{\| g_{x_p}(p) - g_{x_p}(p_0) - (dg_{x_p})_{p_0}(p - p_0) \|}{\| p - p_0 \|} = 0, \qquad \ldots (3)$$

where $p_0 \in P_0$.

Remark 5 : If X_0 is an open subset of X, the function g is differentiable at (x_p, p) for every $p \in P_0$ and the mapping $\gamma : P_0 \to X_0$, such that $\gamma(p) = x_p$ for every $p \in P_0$, is differentiable at p_0, then the condition (3) follows from (2).

In fact this is easily proved in an standard way remarking that for every $p \in P_0 - \{p_0\}$ we have that

$$\frac{\| g_{x_p}(p) - g_{x_p}(p_0) - (dg_{x_p})_{p_0}(p - p_0) \|}{\| p - p_0 \|}$$

$$\leq \frac{\| g_{x_p}(p) - g_{x_{p_0}}(p_0) - (dg)_{(x_{p_0}, p_0)}(p - p_0, x_p - x_{p_0}) \|}{\| p - p_0 \| + \| x_p - x_{p_0} \|} \cdot \frac{\| p - p_0 \| + \| x_p - x_{p_0} \|}{\| p - p_0 \|}$$

$$+ \frac{\| g_{x_{p_0}}(p) - g_{x_{p_0}}(p_0) - (dg)_{(x_{p_0}, p_0)}(p - p_0, 0) \|}{\| p - p_0 \|}$$

$$+ \frac{\| g_{x_p}(p_0) - g_{x_{p_0}}(p_0) - (dg)_{(x_{p_0}, p_0)}(0, x_p - x_{p_0}) \|}{\| x_p - x_{p_0} \|} \cdot \frac{\| x_p - x_{p_0} \|}{\| p - p_0 \|}$$

$$+ \frac{\| g_{x_{p_0}}(p) - g_{x_{p_0}}(p_0) - (dg_{x_{p_0}})_{p_0}(p - p_0) \|}{\| p - p_0 \|} + \| (dg_{x_p})_{p_0} - (dg_{x_{p_0}})_{p_0} \|$$

and

$$\frac{\| x_p - x_{p_0} \|}{\| p - p_0 \|} \leq \frac{\| x_p - x_{p_0} - (d\gamma)_{p_0}(p - p_0) \|}{\| p - p_0 \|} + \| (d\gamma)_{p_0} \| .$$

The following Theorems 6 (for convex programming) and 7 (for differentiable programming) are extensions (for multiobjective programming) of the classic envolvent theorem

Theorem 6 — Let $p_0 \in P_0$ and assume the following conditions :

(i) The program (1_p) is convex for every $p \in P_0$.

(ii) L_p is a Lagrange T-multiplier of (1_p) for every $p \in P_0$.

(iii) $\lim_{p \to p_0} L_p = L_{p_0}$ in the space $L(Z, W)$ of the linear and continuous function from Z into W endowed with the usual strong topology.

Then the function

$$G : P_0 \to W$$
$$p \to G(p) = Tf(x_p)$$

is differentiable at p_0 and

$$(dG)_{p_0} = L_{p_0} (dg_{x_{p_0}})_{p_0}. \quad \ldots (4)$$

PROOF : Let $p \in P_0$. Since $g(x_{p_0}, p_0) \leq 0$ we have that $L_p\, g(x_{p_0}, p_0) \leq 0$ and $Tf(x_p) \leq Tf(x_{p_0}) + L_p\, g(x_{p_0}, p) \leq Tf(x_{p_0}) + L_p\, (g(x_{p_0}, p) - g(x_{p_0}, p_0))$.

In a similar way we have

$$Tf(x_{p_0}) \leq Tf(x_p) + L_{p_0} (g(x_p, p_0) - g(x_p, p)).$$

Therefore,

$$Tf(x_{p_0}) \leq Tf(x_p) + L_{p_0} (g(x_p, p_0) - g(x_p, p))$$
$$\leq Tf(x_{p_0}) + L_p\, (g(x_{p_0}, p) - g(x_{p_0}, p_0)) - L_{p_0} (g(x_p, p) - g(x_p, p_0)),$$
$$0 \leq Tf(x_p) - Tf(x_{p_0}) - L_{p_0} (g(x_p, p) - g(x_p, p_0))$$
$$\leq L_p\, (g(x_{p_0}, p) - g(x_{p_0}, p_0)) - L_{p_0} (g(x_p, p) - g(x_p, p_0))$$

and

$$\| Tf(x_p) - Tf(x_{p_0}) - L_{p_0} (g(x_p, p) - g(x_p, p_0)) \|$$
$$\leq \| L_p\, (g(x_{p_0}, p) - g(x_{p_0}, p_0)) - L_{p_0} (g(x_p, p) - g(x_p, p_0)) \|$$
$$\leq \| L_p\, (g(x_{p_0}, p) - g(x_{p_0}, p_0) - (dg_{x_{p_0}})_{p_0} (p - p_0)) + (L_p - L_{p_0})\, (dg_{x_{p_0}})_{p_0} (p - p_0)$$
$$- L_{p_0} (g(x_p, p) - g(x_p, p_0) - (dg_{x_{p_0}})_{p_0} (p - p_0)) \|$$
$$\leq (\| L_p - L_{p_0} \| + \| L_{p_0} \|) \cdot \| g(x_{p_0}, p) - g(x_{p_0}, p_0) - (dg_{x_{p_0}})_{p_0} (p - p_0) \|$$
$$+ \| L_p - L_{p_0} \| \, \| (dg_{x_{p_0}})_{p_0} \| \; \| p - p_0 \|$$
$$+ \| L_{p_0} \| \, \| g(x_p, p) - g(x_p, p_0) - (dg_{x_p})_{p_0} (p - p_0) \|$$
$$+ \| L_{p_0} ((dg_{x_{p_0}})_{p_0} - (dg_{x_p})_{p_0})\, (p - p_0)) \|.$$

Therefore,

$$\| Tf(x_p) - Tf(x_{p_0}) - L_{p_0} ((dg_{x_{p_0}})_{p_0} (p - p_0) \|$$

$$\leq \| Tf(x_p) - Tf(x_{p_0}) - L_{p_0}(g(x_p, p) - g(x_p, p_0)) \|$$
$$+ \| L_{p_0}(g(x_p, p) - g(x_p, p_0) - (dg_{x_p})_{p_0}(p - p_0)) \|$$
$$+ \| L_{p_0}((dg_{x_p})_{p_0}(p - p_0) - (dg_{x_{p_0}})_{p_0}(p - p_0)) \|$$
$$\leq (\| L_p - L_{p_0} \| + \| L_{p_0} \|) \cdot \| g(x_{p_0}, p) - g(x_{p_0}, p_0) - (dg_{x_p})_{p_0}(p - p_0) \|$$
$$+ \| L_p - L_{p_0} \| \cdot \| (dg_{x_{p_0}})_{p_0} \| \cdot \| p - p_0 \|$$
$$+ 2 \| L_{p_0} \| \cdot \| g(x_p, p) - g(x_p, p_0) - (dg_{x_p})_{p_0}(p - p_0) \|$$
$$+ 2 \| L_{p_0} \| \cdot \| (dg_{x_p})_{p_0} - (dg_{x_{p_0}})_{p_0} \| \cdot \| p - p_0 \|,$$

whence the result trivially follows.

Theorem 7 — Let $p_0 \in P_0$ and suppose the following hypotheses :

(1) The program (1_p) is differentiable for every $p \in P_0$.

(2) The mapping $\gamma : P_0 \to X_0$ such that $\gamma(p) = x_p$ for every $p \in P_0$, is differentiable at $p_0 \in P_0$.

(3) For every $p \in P_0$ there exists a Lagrange T-multiplier L_p of the program (1_p) associated to x_p.

(4) $\lim_{p \to p_0} \dfrac{\| L_{p_0}(g(\gamma(p), p) - g(\gamma(p_0), p_0)) \|}{\| p - p_0 \|} = 0.$

Then the function $G = Tf\gamma : P_0 \to W$ is differentiable at p_0 and

$$(dG)_{p_0} = L_{p_0}(dg_{x_{p_0}})_{p_0}. \quad \ldots (5)$$

PROOF : It follows from the chain rule that

$$(dG)_{p_0} = T\,(df)_{x_0}(d\gamma)_{p_0}$$

and therefore

$$(dG)_{p_0} = -L_{p_0}(dg_{p_0})_{x_{p_0}}(d\gamma)_{p_0} = -L_{p_0}(dg_{p_0}\gamma)_{p_0}.$$

Thus to prove the theorem it is enough to see that

$$L_{p_0}(dg_{p_0}\gamma)_{p_0} = -L_{p_0}(dg_{x_{p_0}})_{p_0}.$$

So let us prove the last. In fact

$$\| L_{p_0}(g(x_p, p_0) - g(x_{p_0}, p_0) + (dg_{x_{p_0}})_{p_0}(p - p_0)) \|$$
$$\leq \| L_{p_0}(g(x_p, p) - g(x_{p_0}, p_0)) \| + \| L_{p_0} \| \cdot \| g(x_p, p) - g(x_p, p_0)$$
$$- (dg_{x_p})_{p_0}(p - p_0) \| + \| L_{p_0} \| \cdot \| (dg_{x_p})_{p_0} - (dg_{x_{p_0}})_{p_0} \| \cdot \| p - p_0 \| ,$$

from where the results follows trivially. •

Remark 8 : It follows immediately from Definition 2.2 that condition (4) of Theorem 7 is fulfilled if $L_{p_0}(g(\gamma(p), p) = 0$ for every $p \in P_0$ and in particular when $g(\gamma(p), p) = 0$ for every $p \in P_0$. This two conditions are clearly stronger than condition (4) of Theorem 7 but they are easily verified in practice.

4. Sensitivity Analysis

This section is devoted to study the sensitivity of the function $P_0 \ni p \to f(x_p) \in Y$. Let us fix an element $z \in Z - \{0\}$.

Definition 9 — Let $p \in P_0$ and $R_p \in L(Z, Y)$.

(1) If the program (1_p) is convex we say that R_p is a Lagrange multiplier of (1_p) if TR_p is a Lagrange T-multiplier of (1_p) and

$$\pi(f(x_p)) = -\pi(R_p(z)), \qquad \text{... (6)}$$

where π denotes the natural projection of Y onto Ker T.

(2) If the program (1_p) is differentiable then R_p is said to be a Lagrange multiplier of (1_p) associated to x_p if TR_p is a Lagrange T-multiplier of (1_p) associated to x_p and (6) is verified.

Proposition 10 — If $p \in P_0$, the program (1_p) is convex (respectively differentiable) and there exists a Lagrange T-multiplier of (1_p) (respectively associated to x_p), then there exists a Lagrange multiplier of (1_p) (respectively associated to x_p).

Proof : Let L_p be a Lagrange T-multiplier of (1_p) (respectively associated to (x_p) and $z' \in Z'$ such that $z'(z) = 1$, where Z' denotes the dual space of Z (recall that the existence of z' follows from the Hahn-Banach theorem). It is immediately verified that the function $R_p : Z \to Y$ such that

$$R_p(u) = T^{*-1} L_p(u) - z'(u)\, \pi f(x_p)$$

for every $u \in Z$, is a Lagrange multiplier of (1_p) (respectively, associated to x_p).

Lemma 11 — If the function $R : P_0 \to L(Z, Y)$ is differentiable at P_0 then the function $S : P_0 \to Y$ such that $S(p) = R(p)(z)$ for every $p \in P_0$, is also differentiable at p_0 and

$$(dS)_p\,(q) = (dR)_p\,(q)(z),$$

for every $p \in P_0$ and every $q \in P$.

Proof : In fact, for every $p \in P_0$ and $q \in P - \{0\}$ we have that

$$\frac{\| S(p+q) - S(p) - (dR)_p\,(q)(z) \|}{\| q \|}$$

$$= \frac{\| R(p+q)(z) - R(p)(z) - (dR)_p\,(q)(z) \|}{\| q \|}$$

$$\leq \frac{\| R(p+q) - R(p) - (dR)_p (q) \| \cdot \| z \|}{\| q \|} ,$$

whence the result follows immediately.

Theorem 12 — Suppose that for every $p \in P_0$ the program (1_p) is convex (respectively differentiable) and there exists a Lagrange multiplier R_p of (1_p) (respectively associated to x_p). Let us assume also that the conditions of Theorem 6 (respectively Theorem 7) are fulfilled taking $L_p = TR_p$. If the function $R : P_0 \to L(Z, Y)$ defined by $R(p) = R_p$ for every $p \in P_0$ is differentiable at p_0 then the function $F : P_0 \to Y$ defined by $F(p) = f(x_p)$ for every $p \in P_0$ is differentiable at P_0 and

$$(dF)_p (q) = T^{*-1} TR_p (dg_{x_p})_p (q) - \pi(dR)_p (q)(z) \qquad \text{... (7)}$$

for every $p \in P_0$ and every $q \in P$.

PROOF : It follows from Theorem 6 (respectively Theorem 7) and Lemma 11 remarking that for every $p \in P_0$ and every $q \in P$ such that $p + q \in P_0$, we have that

$$\| F(p+q) - F(p) - T^{*-1} \; TR_p (dg_{x_p})_p (q) + \pi(dR)_p (q)(z) \|$$

$$\leq \| T^{*-1} Tf(x_{p+q}) - T^{*-1} Tf(x_p) - T^{*-1} TR_p (dg_{x_p})_p (q) \|$$

$$+ \; \| \pi(f(x_{p+q})) - \pi(f(x_p)) + \pi((dR)_p (q)(z)) \|$$

$$\leq \| T^{*-1} \| \cdot \| G(p+q) - G(p) - (dG)_p (q) \|$$

$$+ \; \| \pi \| \cdot \| R_{p+q} (z) - R_p (z) - (dR)_p (q) (z) \|$$

$$\leq \| T^{*-1} \| \cdot \| G(p+q) - G(p)$$

$$- (dG)_p (q) \| + \| \pi \| \cdot \| R_{p+q} - R_p - (dR)_p (q) \| \cdot \| z \| ,$$

where the function $G : P_0 \to W$ is defined by $G(h) = Tf(x_h)$ for every $h \in P_0$.

Remark 13 : In the particular case of being $Y = W = \mathbb{R}$ we have that $T^{*-1} T$ is the identity and $\pi = 0$ and so in this case Theorem 12 coincides with the classic envolvent theorem.

In general, when T is a topological isomorphism (see Balbas *et al.*[5] or Balbas *et al.*[6]), then (7) has a more simple expression since in this case

$$(dF)_p = R_p (dg_{x_p})_p \qquad \text{... (8)}$$

for every $p \in P_0$. Other important particular case is obtained when $P = Z$ and $g(x, p) = h(x) - p$ for every $(x, p) \in X_0 \times P_0$, where h is a function from X_0 into P. In this last case (7) has the following expression :

$$(dF)_p (q) = - T^{*-1} TR_p (q) - \pi(dR)_p (q)(z). \qquad \text{... (9)}$$

5. EXAMPLE

To illustrate the results stated in Theorems 6 and 12 let us consider the non-negative and finite measure μ defined on the family of all the subsets of the natural numbers $\mathbb{N} = \{0, 1, 2, ...\}$ such that $\mu(\{t\}) = e^{-t}$ for every $t = 0, 1, 2, ...$ and the Hilbert space $L^2(\mu)$ of all sequences $(\lambda_t)_{t=0}^{\infty}$ of real numbers such that $\sum_{t=0}^{\infty} \lambda_t^2 e^{-t} < +\infty$, with the usual norm and ordered by the usual cone $\{(x_t)_{t=0}^{\infty} \in L^2(\mu) : x_t \geq 0,\ t = 0, 1, 2, ...\}$.

Following the notations used along this paper let $X = L^2(\mu) \times L^2(\mu)$ (with the usual norm), $X_0 = \{((u_t)_{t=0}^{\infty}, (v_t)_{t=0}^{\infty}) \in X : u_t, v_t \geq 0,\ t = 0, 1, 2, ...\}$, $P = Z = L^2(\mu)$, $P_0 = \{(p_t)_{t=0}^{\infty} \in P : p_t > 0,\ t = 0, 1, 2, ...\}$, $Y = \mathbb{R}^2$, $W = \mathbb{R}$, $f : X_0 \to Y$ the function defined by

$$f((u_t)_{t=0}^{\infty}, (v_t)_{t=0}^{\infty}) = -\left(\sum_{t=0}^{\infty} u_t e^{-t},\ \sum_{t=0}^{\infty} v_t e^{-2t} \right)$$

for every $((u_t)_{t=0}^{\infty}, (v_t)_{t=0}^{\infty}) \in X_0$, $g : X_0 \times P_0 \to Z$ the function defined by

$$g(((u_t)_{t=0}^{\infty}, (v_t)_{t=0}^{\infty}), (p_t)_{t=0}^{\infty}) = (u_t + v_t - p_t)_{t=0}^{\infty}$$

for every $(((u_t)_{t=0}^{\infty}, (v_t)_{t=0}^{\infty}), (p_t)_{t=0}^{\infty}) \in X_0 \times P_0$ and $T : \mathbb{R}^2 \to \mathbb{R}$ defined by $T(x, y) = \alpha x + \beta y$ (for every pair $(x, y) \in \mathbb{R}$) where

$$\alpha = \frac{1}{e^4 + 1} \quad \text{and} \quad \beta = 1 - \alpha = \frac{e^4}{e^4 + 1}.$$

Clearly Ker $T = \{\lambda(\beta, -\alpha) : \lambda \in \mathbb{R}\}$, $Y_T = \{\lambda(\alpha, \beta) : \lambda \in \mathbb{R}\}$, $\pi(a, b) = \frac{\beta a - \alpha b}{\alpha^2 + \beta^2}(\beta, -\alpha)$ for every pair $(a, b) \in \mathbb{R}^2$ and

$$T^{*-1}(r) = \left(\frac{\alpha r}{\alpha^2 + \beta^2}, \frac{\beta r}{\alpha^2 + \beta^2} \right)$$

for every $r \in \mathbb{R}$.

If for every $p = (p_t)_{t=0}^{\infty} \in P_0$ we solve with the standard technique of multiobjective programming the program

$$\left. \begin{array}{l} \text{Mín} - \sum_{t=0}^{\infty} (\alpha\, u_t e^{-t} + \beta\, v_t e^{-2t}) \\ ((u_t)_{t=0}^{\infty}, (v_t)_{t=0}^{\infty}) \in X_0 \\ g(((u_t)_{t=0}^{\infty}, (v_t)_{t=0}^{\infty}), p) \leq 0 \end{array} \right\}$$

and its dual program (see Anderson and Nash[1])

$$\left.\begin{array}{l} \text{Max} - \sum_{t=0}^{\infty} p_t \lambda_t e^{-t} \\ (\lambda_t)_{t=0}^{\infty} \in L^2(\mu) \\ (\lambda_t)_{t=0}^{\infty} \geq (\beta e^{-t})_{t=0}^{\infty} \\ \lambda_t \geq alpha, t = 0, 1, 2, \ldots \end{array}\right\}$$

we obtain that $x_p = (u(p), v(p))$ is a T-optimal solution of the program (1_p) (see Definition 1), where $u(p) = (0, 0, 0, 0, 0, p_5, p_6, p_7, ..)$ and $v(p) = (p_0, p_1, p_2, p_3, p_4, 0, 0, 0, ...)$, and $L_p = (\beta, \beta e^{-1}, \beta e^{-2}, \beta e^{-3}, \beta e^{-4}, \alpha, \alpha, \alpha, ...)$ is a Lagrange T-multiplier of (1_p).

In this conditions the hypothesis of Theorem 6 are fulfilled. Let us verify that this assertion holds. In fact for every $p = (p_t)_{t=0}^{\infty} \in P_0$ we have that $(dg_{x_p})_p(q) = -q$ for every $q \in P$ and therefore

$$L_p(dg_{x_p})_p(q) = -L_p(q) = -\left(\sum_{t=0}^{4} \beta q_t e^{-2t} + \sum_{t=5}^{\infty} \alpha q_t e^{-t}\right) \quad \ldots (10)$$

for every $q = (q_t)_{t=0}^{\infty} \in P$.

Moreover we have that

$$(dG)_p(q) = G(q) = Tf(x_q) = -T\left(\sum_{t=5}^{\infty} q_t e^{-t}, \sum_{t=0}^{4} q_t e^{-2t}\right)$$

$$= -\left(\sum_{t=5}^{\infty} \alpha q_t e^{-t} + \sum_{t=0}^{4} \beta q_t e^{-2t}\right) \quad \ldots (11)$$

for every $q = (q_t)_{t=0}^{\infty}$, and thus (10) and (11) coincides as Theorem 6 states.

Let us verify now (7) in this practical case. In fact, following the notation of Theorem 12 and proceeding as in the proof of the Proposition 10 taking any fixed element $z \in L^2(\mu)$ and $z' \in L^2(\mu)$ such that $z'(z) = 1$ [for instance $z = (0, 0, 2, 0, 0, 0, \ldots)$ and $z' = (0, 0, \frac{1}{2}e^2, 0, 0, 0, \ldots)$], we obtain that $R_p = T^{*-1} L_p - z' \pi F$ is a Lagrange multiplier of (1_p), for every $p \in P_0$. It is immediately proved that conditions of Theorem 12 hold.

Thus if $p = (p_t)_{t=0}^{\infty} \in P_0$ we have that

$$(dF)_p(q) = F(q) = -\left(\sum_{t=5}^{\infty} q_t e^{-t}, \sum_{t=0}^{4} q_t e^{-2t}\right) \quad \ldots (12)$$

for every $q = (q_t)_{t=0}^{\infty} \in P$. Moreover since $(d\,g)_p(q) = -q$ for every $q \in P$

and $\pi f(x_p) \in \operatorname{Ker} T$ we have that $T^{*-1}T(z'(q)\pi f(x_p)) = 0$ and

$$T^{*-1}TR_p(dg_{x_p})_p(q) = -T^{*-1}TR_p(q) = -T^{*-1}T(T^{*-1}L_p(q)) + T^{*-1}T(z'(q)\pi(x_p))$$

$$= -T^{*-1}(L_p(q)) = \left(\frac{-\alpha L_p(q)}{\alpha^2+\beta^2}, \frac{-\beta L_p(q)}{\alpha^2+\beta^2}\right)$$

$$= \left(\frac{-\sum_{t=0}^{4}\alpha\beta q_t e^{-2t} - \sum_{t=5}^{\infty}\alpha^2 q_t e^{-t}}{\alpha^2+\beta^2}, \frac{-\sum_{t=0}^{4}\beta^2 q_t e^{-2t} - \sum_{t=5}^{\infty}\alpha\beta q_t e^{-t}}{\alpha^2+\beta^2}\right).$$

Moreover since $L_q = (\beta, \beta e^{-1}, \beta e^{-2}, \beta e^{-3}, \beta e^{-4}, \alpha, \alpha, \alpha, \ldots)$ for $q \in P_0$ and $d(\pi F)_p = \pi F$ we have that

$$\pi(dR)_p(q)(z) = \pi(-z'(z)\pi f(x_q)) = -\pi f(x_q) = \pi\left(\sum_{t=5}^{\infty} q_t e^{-t}, \sum_{t=0}^{4} q_t e^{-2t}\right)$$

$$= \frac{-\sum_{t=0}^{4}\alpha q_t e^{-2t} + \sum_{t=5}^{\infty}\beta q_t e^{-t}}{\alpha^2+\beta^2}(\beta, -\alpha)$$

$$= \left(\frac{-\sum_{t=0}^{4}\alpha\beta q_t e^{-2t} + \sum_{t=5}^{\infty}\beta^2 q_t e^{-t}}{\alpha^2+\beta^2}, \frac{\sum_{t=0}^{4}\alpha^2 q_t e^{-2t} - \sum_{t=5}^{\infty}\alpha\beta q_t e^{-t}}{\alpha^2+\beta^2}\right)$$

for every $q = (q_t)_{t=0}^{\infty} \in P$. Therefore,

$$T^{*-1}TR_p(dg_{x_p})_p(q) - \pi(dR)_p(q)(z)$$

$$= \left(\frac{-\sum_{t=0}^{4}\alpha\beta q_t e^{-2t} - \sum_{t=5}^{\infty}\alpha^2 q_t e^{-t}}{\alpha^2+\beta^2}, \frac{-\sum_{t=0}^{4}\beta^2 q_t e^{-2t} - \sum_{t=5}^{\infty}\alpha\beta q_t e^{-t}}{\alpha^2+\beta^2}\right)$$

$$+ \left(\frac{\sum_{t=0}^{4}\alpha\beta q_t e^{-2t} - \sum_{t=5}^{\infty}\beta^2 q_t e^{-t}}{\alpha^2+\beta^2}, \frac{-\sum_{t=0}^{4}\alpha^2 q_t e^{-2t} + \sum_{t=5}^{\infty}\alpha\beta q_t e^{-t}}{\alpha^2+\beta^2}\right)$$

$$= \left(\frac{-\sum_{t=5}^{\infty}(\alpha^2+\beta^2) q_t e^{-t}}{\alpha^2+\beta^2}, \frac{-\sum_{t=0}^{4}(\alpha^2+\beta^2) q_t e^{-2t}}{\alpha^2+\beta^2}\right)$$

$$= -\left(\sum_{t=5}^{\infty} q_t e^{-t}, \sum_{t=0}^{4} q_t e^{-2t} \right) \qquad \text{... (13)}$$

for every $q = (q_t)_{t=0}^{\infty} \in P$, and thus (12) coincides with (13), as Theorem 12 asserts.

Let us verify now (8). Consider $\alpha_1, \alpha_2 \in \left(\frac{1}{e^5+1}, \frac{1}{e^4+1} \right)$, $\alpha_1 \neq \alpha_2$ and $0 \leq \beta_1, \beta_2 \leq 1$ are such that $\alpha_1 + \beta_1 = 1$ and $\alpha_2 + \beta_2 = 1$. Then

$$H = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix}$$

defines a non-negative topological isomorphism $H : \mathbb{R}^2 \to \mathbb{R}^2$, $x_p = (u(p), v(p))$ is an H-optimal solution of the program (1_p) for every $p \in P_0$ and $J_p' = (\{\beta_1, \beta_1 e^{-1}, \beta_1 e^{-2}, \beta_1 e^{-3}, \beta_1 e^{-4}, \alpha_1, \alpha_1, \alpha_1, \ldots\}, \{\beta_2, \beta_2 e^{-1}, \beta_2 e^{-2}, \beta_2 e^{-3}, \beta_2 e^{-4}, \alpha_1, \alpha_1, \alpha_1, \ldots\})$ is a Lagrangean H-multiplier of (1_p) (for every $p \in P_0$).

Since

$$H^{-1} = \frac{1}{\alpha_1 \beta_2 - \alpha_2 \beta_1} \begin{pmatrix} \beta_2 & -\alpha_2 \\ -\beta_1 & \alpha_1 \end{pmatrix},$$

it follows from the proof of Proposition 10 that for every $p \in P_0$ the function $S_p : L^2(\mu) \to \mathbb{R}^2$ such that

$$S_p(q) = H^{-1} J_p(q)$$

$$= \frac{1}{\alpha_1 \beta_2 - \alpha_2 \beta_1} \left(\sum_{t=0}^{4} \beta_1 q_t e^{-2t} + \sum_{t=5}^{\infty} \alpha_1 q_t e^{-t}, \sum_{t=0}^{4} \beta_1 q_t e^{-2t} + \sum_{t=5}^{\infty} \alpha_1 q_t e^{-t} \right) \times \begin{pmatrix} \beta_2 & -\alpha_2 \\ -\beta_1 & \alpha_1 \end{pmatrix}$$

$$= \left(\sum_{t=5}^{\infty} q_t e^{-t}, \sum_{t=0}^{4} q_t e^{-2t} \right)$$

for every $q \in L^2(\mu)$, is a Lagrange multiplier of (1_p). Moreover

$$S_p(dg_{x_p})_p(q) = S_p(-q) = -\left(\sum_{t=5}^{\infty} q_t e^{-t}, \sum_{t=0}^{4} q_t e^{-2t} \right), \qquad \text{... (14)}$$

for every $p \in P_0$ and $q \in P$. Thus (12) and (14) coincide as Remark 13 states.

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