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A COMPLEX INVERSION THEOREM FOR A MODIFIED *H*-TRANSFORMATION OF DISTRIBUTIONS

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In order to unify and extend the existing results on various distributional transformations, the authors introduce a distributional $\overline{H}$-function transform of functions and distributions with the help of the kernel $H(xt)$. The complex inversion theorem for this transformation is established in the conventional sense as well as in the distributional sense.

1. INTRODUCTION

In the present paper the authors define and study a distributional integral transformation associated with $\overline{H}$-function due to Inayat Hussain[4], which is more general than the H-function of Fox[3].

The importance of the results obtained in this paper lies in the fact that they serve as a key formula for several hitherto scattered results on various integral transforms of functions and distributions introduced and studied earlier by many workers notably by Zemanian[11,12], Charmichael and Pathak[2], Saxena[9], Joshi and Saxena[5], Malgonde and Saxena[7], and others.

The $\overline{H}$-function transform $F(x)$ of a complex valued function $f(t)$, $0 \le t \le \infty$, is defined by the equation :

$$F(x) = \int_0^\infty e^{-wxt}\, \overline{H}_{p,q}^{m,n}\left[xt \,\middle|\, \begin{matrix} (\alpha_j, A_j;\, a_j)_{1,n},\ (\alpha_j, A_j)_{n+1,p} \\ (\beta_j, B_j)_{1,m},\ (\beta_j, B_j;\, b_j)_{m+1,q} \end{matrix} \right] f(t)\, dt \quad \ldots (1.1)$$

where w is a positive number and $\overline{H}_{p,q}^{m,n}[z]$ is the generalization of the familiar H-function introduced by Inayat Hussain[4]. The $\overline{H}$-function defined by Inayat Hussain[4] is as follows :

$$\overline{H}_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (\alpha_j, A_j;\, a_j)_{1,n},\ (\alpha_j, A_j)_{n+1,p} \\ (\beta_j, B_j)_{1,m},\ (\beta_j, B_j;\, b_j)_{m+1,q} \end{matrix} \right] = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Theta(\eta)\, z^\eta\, d\eta \quad \ldots (1.2)$$

where

$$\Theta(\eta) = \frac{\prod_{j=1}^{m} \Gamma(\beta_j - B_j \eta) \prod_{j=1}^{n} \left\{ \Gamma(1 - \alpha_j + A_j \eta) \right\}^{a_j}}{\prod_{j=m+1}^{q} \left\{ \Gamma(1 - \beta_j + B_j \eta) \right\}^{b_j} \prod_{j=n+1}^{p} \Gamma(\alpha_j - A_j \eta)} \quad \ldots (1.3)$$

which contains fractional powers of some of the Γ-functions. Here α_j $(j = 1, \ldots, p)$ and β_j $(j = 1, \ldots, q)$ are complex parameters,

$$A_j > 0 \quad \text{for} \quad j = 1, \ldots, p,$$

$$B_j > 0 \quad \text{for} \quad j = 1, \ldots, q,$$

and the exponents a_j $(j = 1, \ldots, n)$ and b_j $(j = m + 1, \ldots, q)$ can take on non-integer values.

When the exponents a_j and b_j all take integer values, the $\overline{H}$-function reduces to the familiar H-function due to Fox[3]. A detailed account of the H-function is available from the monograph of Mathai and Saxena[8]. The sufficient conditions for absolute convergence of the contour integral (1.1), given by Buschman and Srivastava[1], are as follows :

$$\Omega = \sum_{j=1}^{m} |B_j| + \sum_{j=1}^{n} |a_j A_j| - \sum_{j=m+1}^{q} |b_j B_j| - \sum_{j=n+1}^{p} |A_j| > 0$$

and $$|\arg(z)| < \frac{1}{2}\pi\Omega.$$

2. The Testing Function Spaces $H_{a,b}$ $H(a, b)$ and Their Duals

Let I be an open set $0 < t < \infty$. $D(I)$ the space of all smooth functions on I having compact support.

For fixed real numbers a and $b > 0$, $H_{a,b}$ is the space of all complex valued functions $\phi(t)$ on $I(0, \infty)$ such that for each non-negative integer k

$$\delta_{a,b,k}(\phi) = \sup_{0<t<\infty} |e^{-bt} t^{a+k} D_t^k \phi(t)|. \quad \ldots (2.1)$$

The $\delta_{a,b,k}$ are seminorms on $H_{a,b}$. Moreover, $H_{a,b}$ is understood to possess the topology generated by the multinorms $\{\delta_{a,b,k}\}_{k=0}^{\infty}$. Thus $H_{a,b}$ is a countably multinormed space. Now we say that a sequence $\{\phi_\nu\}_{\nu=1}^{\infty}$ converges to ϕ in $H_{a,b}$ if for $k = 0, 1, 2, \ldots$, $\delta_{a,b,k}(\phi_\nu - \phi) \to 0$ as $\nu \to \infty$. Hence $H_{a,b}$ is complete countably multinormed space.

Now we define countable union space $H(a, b)$ as in Joshi and Saxena[6]. For this let b denote either a finite real number or $+\infty$. If we choose a monotonic increasing

sequence $\{b_\nu\}_{\nu=1}^{\infty}$ such that $b_\nu \to b^-$, then the countable union space $H(a, b)$ is defined by

$$H(a, b) = \bigcup_{\nu=1}^{\infty} H_{a, b_\nu}. \qquad \ldots (2.2)$$

Thus $H(a, b)$ is the space of all smooth functions $\phi(t)$ on I such that

$$\delta_{a, b, k}(\phi) = \sup_{0 < \infty < \infty} | e^{-bt} t^{a+k} D_t^k \phi(t) | . \qquad \ldots (2.3)$$

The spaces $H'_{a, b}$ and $H'(a, b)$ denote the conjugate (dual) of the space $H_{a, b}$ and $H(a, b)$ respectively.

3. The Distributional $\overline{H}$-Function Transform

Theorem 1 — The function

$$\overline{H}(xt) = e^{-wxt}\, \overline{H}_{p, q}^{m, n}\left[xt \,\middle|\, \begin{matrix} (\alpha_j, A_j; a_j)_{1, n}, (\alpha_j, A_j)_{n+1, p} \\ (\beta_j, B_j)_{1, m}, (\beta_j, B_j; b_j)_{m+1, q} \end{matrix} \right] \qquad \ldots (3.1)$$

is in $H_{a, b}$ for Re $(x) > -(b/w)$, $b > 0$ and $\mathrm{Re}(a + c) \geq 0$

where $c = \min_{1 \leq j \leq m} (\beta_j / B_j)$.

Proof : We have to show that

$$\sup_{0 < t < \infty} | e^{-bt} t^{a+k} D_t^k \{\overline{H}(xt)\} | \text{ is bounded,}$$

where $\overline{H}(xt)$ is given by (3.1).

Let

$$\sup_{0 < t < \infty} \left| e^{-bt} t^{a+k} \frac{d^k}{dt^k} \left\{ e^{-wxt}\, \overline{H}_{p, q}^{m, n}\left[xt \,\middle|\, \begin{matrix} (\alpha_j, A_j; a_j)_{1, n}, (\alpha_j, A_j)_{n+1, p} \\ (\beta_j, B_j)_{1, m}, (\beta_j, B_j; b_j)_{m+1, q} \end{matrix} \right] \right\} \right|$$

$$= \sup_{0 < t < \infty} \left| e^{-bt} t^{a+k} \sum_{r=0}^{k} \binom{k}{r} (-wx)^{k-r} e^{-wxt} \right.$$

$$\left. \times t^{-r}\, \overline{H}_{p+1, q+1}^{m, n+1}\left[xt \,\middle|\, \begin{matrix} (0, 1)(\alpha_j, A_j; a_j)_{1, n}, (\alpha_j, A_j)_{n+1, p} \\ (\beta_j, B_j)_{1, m}, (\beta_j, B_j; b_j)_{m+1, q}, (r, 1) \end{matrix} \right] \right| \qquad \ldots (3.2)$$

Now as $t \to 0$, (3.2) is bounded if minimum power of t is non-negative i.e. $\mathrm{Re}(a + c) \geq 0$ where $c = \min_{1 \leq j \leq m} (\beta_j / B_j)$. Also for large value of t, i.e. as $t \to \infty$, (3.2) will be finite if $\mathrm{Re}(x) > -(b/w)$.

Hence the theorem is proved.

We now define the distributional $\overline{H}$-function transform. If $f \in H'(a, b)$, then for $\mathrm{Re}(a + c) \geq 0$ and $\mathrm{Re}(x) > -(b/w)$, the distributional $\overline{H}$-function transform $F(x)$ is defined by

$$F(x) = \langle f(t), \overline{H}(xt) \rangle \qquad \ldots (3.3)$$

where $\overline{H}(xt)$ is defined in (3.1).

Here for each x, $0 < x < \infty$, the R.H.S. of (3.3) has a sense as the application of $f(t) \in H'(a, b)$ to $\overline{H}(xt) \in H(a, b)$. Now it can be readily seen that if $F(x)$ is defined by (3.3) in distributional sense, then

$$F^r(x) = \left\langle f(t), \frac{\partial^r}{\partial x^r} \{\overline{H}(xt)\} \right\rangle. \qquad \ldots (3.4)$$

4. Inversion Formula

In this section we establish an inversion formula for $\overline{H}$-function transform in conventional sense.

Theorem 2 — Let $y^{s-1} f(y) \in L(0, \infty)$ and let $f(y)$ be of bounded variation in the neighbourhood of the point $y = t$. Further let

$$F(x) = \int_0^\infty e^{-wxt}\, \overline{H}_{p,q}^{m,n}\left[xt \,\middle|\, \begin{matrix} (\alpha_j, A_j; a_j)_{1,n}, & (\alpha_j, A_j)_{n+1,p} \\ (\beta_j, B_j)_{1,m}, & (\beta_j, B_j; b_j)_{m+1,q} \end{matrix} \right] f(t)\, dt$$

then

$$\frac{1}{2}[f(t+) + f(t-)] = \lim_{\nu \to \infty} \frac{1}{2\pi i} \int_{\sigma - i\nu}^{\sigma + i\nu} [\chi(s)]^{-1} t^{-s} M(s)\, ds \qquad \ldots (4.1)$$

where

$$\chi(s) = \frac{1}{w^{1-s}}\, \overline{H}_{p+1,q}^{m,n+1}\left[\frac{1}{w} \,\middle|\, \begin{matrix} (s, 1)\,(\alpha_j, A_j; a_j)_{1,n}, & (\alpha_j, A_j)_{n+1,p} \\ (\beta_j, B_j)_{1,m}, & (\beta_j, B_j; b_j)_{m+1,q} \end{matrix} \right] \qquad \ldots (4.2)$$

and

$$M(s) = \int_0^\infty x^{-s} F(x)\, dx \qquad \ldots (4.3)$$

provided

$$\min_{1 \leq j \leq m} \mathrm{Re}\left(\frac{\beta_j}{B_j} - s + 1 \right) > 0$$

$$\Omega = \sum_{j=1}^{m} |B_j| + \sum_{j=1}^{n} |a_j A_j| - \sum_{j=m+1}^{q} |b_j B_j| - \sum_{j=n+1}^{p} |A_j| > 0$$

and $|\arg(x)| < \frac{1}{2}\pi\Omega.$

PROOF : We have

$$\int_0^\infty x^{s} F(x)\,dx$$

$$= \int_0^\infty x^{s} \int_0^\infty e^{-wxt}\, \overline{H}_{p,q}^{m,n}\left[xt \,\middle|\, \begin{matrix} (\alpha_j, A_j; a_j)_{1,n}, & (\alpha_j, A_j)_{n+1,p} \\ (\beta_j, B_j)_{1,m}, & (\beta_j, B_j; b_j)_{m+1,q} \end{matrix} \right] f(t)\,dt\,dx. \qquad \ldots (4.4)$$

Now we see that the t-integral is convergent when the $\overline{H}$-function transform of $|f(t)|$ exists. Also, x-integral is convergent for the conditions stated in the theorem. The resulting integral is absolutely convergent if $t^{s-1} f(t) \in L(0, \infty)$. Hence the interchange of the order of integration in (4.4) is justified by virtue of de la Vallée Poussin's well known result, and we have

$$\int_0^\infty x^{s} F(x)\,dx$$

$$= \int_0^\infty f(t)\,dt \int_0^\infty x^{s} e^{-wxt} \times \overline{H}_{p,q}^{m,n}\left[xt \,\middle|\, \begin{matrix} (\alpha_j, A_j; a_j)_{1,n}, & (\alpha_j, A_j)_{n+1,p} \\ (\beta_j, B_j)_{1,m}, & (\beta_j, B_j; b_j)_{m+1,q} \end{matrix} \right] dx$$

$$= \int_0^\infty f(t)\,dt \int_0^\infty x^{s} e^{-wxt} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Theta(\eta)\,(xt)^{\eta}\,d\eta\,dx \qquad \ldots (4.5)$$

where $\Theta(\eta)$ is given by (1.3).

Again interchanging the order of integration, which is permissible, (4.5) reduces to

$$\int_0^\infty x^{s} F(x)\,dx = \int_0^\infty f(t)\,dt \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Theta(\eta)\, t^{\eta}\,d\eta \times \int_0^\infty e^{-wxt} x^{s+\eta}\,dx. \qquad \ldots (4.6)$$

After evaluating the x-integral, we obtain (4.6) as

$$\int_0^\infty x^{s} F(x)\,dx = \int_0^\infty f(t)\,dt \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Theta(\eta)\, t^{\eta}\,d\eta \frac{\Gamma(\eta - s + 1)}{(wt)^{\eta - s + 1}}$$

$$= \int_0^\infty f(t)\,dt\, t^{s-1} \frac{1}{w^{1-s}} \cdot \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Theta(\eta)\,\Gamma(\eta - s + 1) \left(\frac{1}{w}\right)^{\eta} d\eta$$

$$= \int_0^\infty t^{s-1} f(t)\, dt \frac{1}{w^{1-s}}\ \overline{H}^{m,n+1}_{p+1,q}\left[\frac{1}{w}\,\middle|\,\begin{matrix}(s,1)\,(\alpha_j, A_j; a_j)_{1,n}\,,\ (\alpha_j, A_j)_{n+1,p}\\ (\beta_j, B_j)_{1,m},\ (\beta_j, B_j; b_j)_{m+1,q}\end{matrix}\right]$$

$$= \chi(s) \int_0^\infty t^{s-1} f(t)\, dt \qquad \ldots (4.7)$$

where

$$\chi(s) = \frac{1}{w^{1-s}}\ \overline{H}^{m,n+1}_{p+1,q}\left[\frac{1}{w}\,\middle|\,\begin{matrix}(s,1)\,(\alpha_j, A_j; a_j)_{1,n}\,,\ (\alpha_j, A_j)_{n+1,p}\\ (\beta_j, B_j)_{1,m},\ (\beta_j, B_j; b_j)_{m+1,q}\end{matrix}\right]$$

Applying the result (Titchmarch[10], p. 46) in (4.7) we obtain the required result.

5. Complex Inversion Theorem in Distributional Sense

Before establishing the main theorem we need the following three Lemmas.

Lemma 1 — If $f \in H'(a, b)$ and

$$\overline{H}(xu) = e^{-wxu}\ \overline{H}^{m,n}_{p,q}\left[xu\,\middle|\,\begin{matrix}(\alpha_j, A_j; a_j)_{1,n}\,,\ (\alpha_j, A_j)_{n+1,p}\\ (\beta_j, B_j)_{1,m},\ (\beta_j, B_j; b_j)_{m+1,q}\end{matrix}\right]$$

then

$$\int_0^\infty x^{-s} \langle f(u),\ \overline{H}(xu)\ \rangle\, dx = \langle f(u),\ \int_0^\infty x^{-s}\, \overline{H}(xu)\ dx\ \rangle. \qquad \ldots (5.1)$$

Lemma 2 — If $\phi \in D(I)$ and $\psi(s) = \int_0^\infty y^{-s}\, \phi(y)\, dy$ then for and fixed positive real number r and $s = \sigma + i\nu$,

$$\frac{1}{2\pi} \int_{-r}^{r} \langle f(u),\ u^{s-1} \rangle\, \psi(s)\, d\nu = \langle f(u),\ \frac{1}{2\pi} \int_{-r}^{r} u^{s-1}\, \psi(s)\, d\nu \rangle. \qquad \ldots (5.2)$$

Lemma 3 — Let (i) $\phi(t) \in D(I)$ and (ii) σ and r are real numbers such that $\sigma > 1$, then

$$\frac{1}{\pi} \int_0^\infty \phi(y) \left(\frac{u}{y}\right)^{\sigma} \frac{\sin\left(r \log \frac{u}{y}\right)}{u \log\left(\frac{u}{y}\right)}\, dy \qquad \ldots (5.3)$$

converges in $H(a, b)$ to $\phi(u)$ as $r \to \infty$. The proofs of the Lemmas 1, 2 and 3 can be developed as in Malgonde and Saxena[7].

Theorem 3 — If $f \in H'(a, b)$ and $\phi \in D(I)$ then

$$\lim_{r \to \infty} \left\langle \frac{1}{2\pi i} \int_{\sigma - ir}^{\sigma + ir} [\chi(s)]^{-1} y^{-s} M(s)\, ds,\ \phi(y) \right\rangle = \langle f, \phi \rangle \qquad \ldots (5.4)$$

where $\chi(s)$ and $M(s)$ are defined in Theorem 2.

The proof of (5.4) can be developed by the application of the above three lemmas and following the same procedure as adopted by Joshi and Saxena[5] (pp. 327-328).

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