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ON A CLASS OF DUAL INTEGRAL EQUATIONS INVOLVING GENERALIZED ASSOCIATED LEGENDRE FUNCTIONS

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In this paper, we consider certain dual integral equations involving generalized associated Legendre function of first kind and trigonometric functions as kernels. Solutions of these are obtained by using properties of the generalized associated Legendre functions and the inversion formula for the generalized Mehler-Fok transform involving generalized associated Legendre functions of first kind.

1. INTRODUCTION

In the solution of certain mixed boundary value problems of mathematical physics, it is frequently advantageous to reduce the solution of the problem to the determination of an unknown function by means of dual integral equations. Dual integral equations involving Bessel functions or trigonometric functions were studied in the literature extensively (see Titchmarsh[12], Sneddon[10,11], Srivastava[9], Noble[5], Nasim[6], Mandal[3] and the references cited therein).

Babloian[1] first considered dual integral equations involving Legendre function $P_{-1/2+i\tau}(\cosh \alpha)$ as kernel. Later Rukhovets and Ufliand[8] considered dual integral equations involving associated Legendre function $P^{m}_{-1/2+i\tau}(\cosh \alpha)$ $(m = 0, 1, 2, ...)$ as kernel. Pathak[7] generalized the kernels of these dual integral equations to associated Legendre function $P^{\mu}_{-1/2+i\tau}(\cosh \alpha)$ where $|\operatorname{Re} \mu| < \frac{1}{2}$. Virchenko[13] considered certain hybrid dual integral equations in which one of the equations contains the generalized Legendre function of first kind $P^{\mu,\nu}_{-1/2+i\tau}(ch\alpha)$ as kernel and the second equation contains the trigonometric function $\cos \alpha\tau$ or $\sin \alpha\tau$ as kernel. Recently, Mandal[4] considered dual integral equations involving $P^{\mu}_{-1/2+i\tau}(\cosh \alpha)$, $\operatorname{Re} \mu < \frac{1}{2}$ as kernel.

In the present paper, we consider a class of dual integral equations involving generalized associated Legendre function $P^{\mu,\nu}_{-1/2+i\tau}(\cosh\alpha)$ as kernel where $|\text{Re}\,\mu| < 1/2$, $|\text{Re}\,\nu| < 1 - \text{Re}\,\mu$. These are more general dual integral equations than those of Babloian[1] and Pathak[7] and we solve them formally by using properties of generalized associated Legendre functions and the inverse generalized Mehler-Fok transform formula given by Braaksma and Meulenbeld[2]. Some integrals involving generalized associated Legendre functions are also obtained which are not available elsewhere. These are used in the present study.

2. Some Integrals Involving Generalized Associated Legendre Functions

The following integral representations[13] are basic tools for our present investigation :

$$P^{\mu,\nu}_{-1/2+i\tau}(\cosh\alpha) = \pi^{-1/2}\,2^{(\nu-\mu+1)/2}\left\{\Gamma\left(\frac{1}{2}-\mu\right)\right\}^{-1} \times sh^{\mu}\alpha\int_0^{\alpha}(ch\alpha - ch\varphi)^{-\mu-1/2} \times F\left(\frac{\nu-\mu}{2}, -\frac{\nu+\mu}{2}; \frac{1}{2}-\mu; \frac{ch\alpha - ch\varphi}{1+ch\alpha}\right)\cos\tau\varphi\, d\varphi \quad \dots (2.1)$$

where $|\text{Re}\,\mu| < \frac{1}{2}$, $|\text{Re}\,\nu| < 1 - \text{Re}\,\mu$ and

$$\cos\tau\varphi = \pi^{-1/2}\,2^{(\mu-\nu-1)/2}\,\Gamma\left(\frac{1}{2}-\mu\right)\cos\mu\pi\,\frac{d}{d\varphi} \times \left[(ch\varphi+1)^{(\nu-\mu)/2}\int_0^{\varphi}(ch\varphi - ch\alpha)^{\mu-1/2}(ch\alpha+1)^{(\mu-\nu)/2} \times F\left(\frac{\mu-\nu}{2}, \frac{1+\mu-\nu}{2}; \frac{1}{2}+\mu; \frac{ch\varphi - ch\alpha}{1+ch\,\varphi}\right) \times P^{\mu,\nu}_{-1/2+i\tau}(ch\alpha)\,sh^{1-\mu}\alpha\, d\alpha\right]. \quad \dots (2.2)$$

The generalized Mehler-Fok transform formulae involving generalized associated Legendre function of first kind is given by Braaksma and Meulenbeld[2] in the following form :

$$f^{*}(\tau) = \int_0^{\infty} f(\alpha)\,P^{\mu,\nu}_{-1/2+i\tau}(ch\alpha)\,sh\alpha\, d\alpha \quad \dots (2.3)$$

then, under certain conditions (see Braaksma and Meulenbeld[2]),

$$f(\alpha) = \pi^{-2} 2^{\mu-\nu-1} \int_0^\infty \tau\, sh\, 2\pi\tau\, \Gamma\left(\frac{1-\mu+\nu}{2}+i\tau\right)\Gamma\left(\frac{1-\mu+\nu}{2}-i\tau\right)$$

$$\times \Gamma\left(\frac{1-\mu-\nu}{2}+i\tau\right)\Gamma\left(\frac{1-\mu-\nu}{2}-i\tau\right)P^{\mu,\nu}_{-1/2+i\tau}(ch\alpha) f^*(\tau)\, d\tau. \quad \ldots (2.4)$$

Let $F_c(\tau)$ denote the Fourier cosine transform of $f(x)$. Then (2.1) can be written as

$$F_c\left[H(\alpha-\varphi)(ch\alpha - ch\varphi)^{-\mu-1/2} F\left(\frac{\nu-\mu}{2}, -\frac{\nu+\mu}{2};\right.\right.$$

$$\left.\left.\times \frac{1}{2}-\mu;\ \frac{ch\alpha - ch\varphi}{1+ch\alpha}\right)\right]$$

$$= 2^{(\mu-\nu)/2}\,\Gamma\left(\frac{1}{2}-\mu\right) sh^{-\mu}\alpha\, P^{\mu,\nu}_{-1/2+i\tau}(ch\,\alpha) \quad \ldots (2.5)$$

where $H(x)$ is the Heaviside unit function, so that by the inversion formula for the Fourier cosine transform, we obtian

$$H(\alpha-\varphi)(ch\alpha - ch\varphi)^{-\mu-1/2} F\left(\frac{\nu-\mu}{2}, -\frac{\nu+\mu}{2}; \frac{1}{2}-\mu;\ \frac{ch\alpha - ch\varphi}{1+ch\alpha}\right)$$

$$= \pi^{-1/2}\, 2^{(\mu-\nu+1)/2}\,\Gamma\left(\frac{1}{2}-\mu\right) sh^{-\mu}\alpha$$

$$\times \int_0^\infty P^{\mu,\nu}_{-1/2+i\tau}(ch\alpha)\cos\varphi\tau\, d\tau. \quad \ldots (2.6)$$

Equation (2.6) can be rewritten as

$$\int_0^\infty \pi^{-2} 2^{\mu-\nu-1}\tau\, sh\, 2\pi\tau\, \Gamma\left(\frac{1-\mu+\nu}{2}+i\tau\right)\Gamma\left(\frac{1-\mu+\nu}{2}-i\tau\right)$$

$$\times\ \Gamma\left(\frac{1-\mu-\nu}{2}+i\tau\right)\Gamma\left(\frac{1-\mu-\nu}{2}-i\tau\right) P^{\mu,\nu}_{-1/2+i\tau}(ch\alpha)\, g^*(\tau)\, d\tau$$

$$= \pi^{1/2}\, 2^{(\nu-\mu-1)/2}\left\{\Gamma\left(\frac{1}{2}-\mu\right)\right\}^{-1} sh^{\alpha}(ch\alpha - ch\varphi)^{-\mu-1/2}$$

$$\times F\left(\frac{\nu-\mu}{2}, -\frac{\nu+\mu}{2}; \frac{1}{2}-\mu;\ \frac{ch\alpha - ch\varphi}{1+ch\alpha}\right) H(\alpha-\varphi)$$

where
$$g^{*}(\tau)=\pi^{2}\,2^{\nu-\mu+1}\,\tau^{-1}\,sh^{-1}\,2\pi\tau\cos\varphi\tau$$
$$\times\left\{\Gamma\left(\frac{1-\mu+\nu}{2}+i\tau\right)\Gamma\left(\frac{1-\mu+\nu}{2}-i\tau\right)\right.$$
$$\left.\times\;\Gamma\left(\frac{1-\mu-\nu}{2}+i\tau\right)\Gamma\left(\frac{1-\mu-\nu}{2}-i\tau\right)\right\}^{-1}.$$

Now compairing the above equation with (2.4) and then by generalized Mehler-Fok transform formula (2.3), the above equation gives

$$\cos\varphi\tau=\pi^{-3/2}\,2^{(\mu-\nu-3)/2}\left\{\Gamma\left(\frac{1}{2}-\mu\right)\right\}^{-1}\tau\,sh\,2\pi\tau\,\Gamma\left(\frac{1-\mu+\nu}{2}+i\tau\right)$$
$$\times\Gamma\left(\frac{1-\mu+\nu}{2}-i\tau\right)\Gamma\left(\frac{1-\mu-\nu}{2}+i\tau\right)\Gamma\left(\frac{1-\mu-\nu}{2}-i\tau\right)$$
$$\times\int_{\varphi}^{\infty}(ch\alpha-ch\varphi)^{-\mu-1/2}\,F\left(\frac{\nu-\mu}{2},\,-\frac{\nu+\mu}{2};\,\frac{1}{2}-\mu;\,\frac{ch\alpha-ch\varphi}{1+ch\alpha}\right)$$
$$\times P^{\mu,\nu}_{-1/2+i\tau}(ch\alpha)\,sh^{1+\mu}\alpha\,d\alpha. \qquad \dots(2.7)$$

Differentiating both sides of (2.7) with respect to φ, we obtain

$$\sin\tau\varphi=\pi^{-3/2}\,2^{(\mu-\nu-3)/2}\cos\pi\mu\,sh\,2\pi\tau\;\Gamma\left(\frac{1}{2}+\mu\right)\Gamma\left(\frac{1-\mu+\nu}{2}+i\tau\right)$$
$$\times\Gamma\left(\frac{1-\mu+\nu}{2}-i\tau\right)\Gamma\left(\frac{1-\mu-\nu}{2}+i\tau\right)\Gamma\left(\frac{1-\mu-\nu}{2}-i\tau\right)\frac{d}{d\varphi}$$
$$\times\int_{\varphi}^{\infty}(ch\alpha-ch\varphi)^{-\mu-1/2}\times F\left(\frac{\nu-\mu}{2},-\frac{\nu+\mu}{2};\,\frac{1}{2}-\mu;\,\frac{ch\alpha-ch\varphi}{1+ch\alpha}\right)$$
$$\times P^{\mu,\nu}_{-1/2+i\tau}(ch\alpha)\,sh^{1+\mu}\alpha\,d\alpha. \qquad \dots(2.8)$$

Integrating both sides of (2.2) with respect to φ from 0 to φ, we get

$$\sin\tau\varphi=\pi^{-1/2}\,2^{(\mu-\nu-1)/2}\;\Gamma\left(\frac{1}{2}-\mu\right)\tau\cos\mu\pi\,(ch\varphi+1)^{(\nu-\mu)/2}$$
$$\times\int_{0}^{\varphi}(ch\varphi-ch\alpha)^{\mu-1/2}\,(ch\alpha+1)^{(\mu-\nu)/2}$$
$$\times F\left(\frac{\mu-\nu}{2},\,\frac{1+\mu-\nu}{2};\,\frac{1}{2}+\mu;\,\frac{ch\varphi-ch\alpha}{1+ch\varphi}\right)$$
$$\times P^{\mu,\nu}_{-1/2+i\tau}(ch\alpha)\,sh^{1-\mu}\dot{\alpha}\,d\alpha. \qquad \dots(2.9)$$

Equation (2.9) can be written as

$$\pi^{1/2}\, 2^{(\nu-\mu+1)/2}\left\{\Gamma\left(\frac{1}{2}-\mu\right)\right\}^{-1} \tau^{-1} \sec \mu\pi\, (ch\varphi+1)^{(\mu-\nu)/2} \sin \tau\varphi$$

$$= \int_0^\infty (ch\varphi - ch\alpha)^{\mu-1/2}\,(ch\alpha+1)^{(\mu-\nu)/2}$$

$$\times F\left(\frac{\mu-\nu}{2}, \frac{1+\mu-\nu}{2}; \frac{1}{2}+\mu; \frac{ch\varphi - ch\alpha}{1+ch\varphi}\right)$$

$$\times sh^{1-\mu}\alpha\, H(\varphi-\alpha)\, P^{\mu,\nu}_{-1/2+i\tau}(ch\alpha)\, d\alpha.$$

Now comparing the above equation with (2.3) and then by utilizing the inverse generalized Mehler-Fok transform formula (2.4), the above equation gives

$$(ch\varphi - ch\alpha)^{\mu-1/2}\,(ch\alpha+1)^{(\mu-\nu)/2}$$

$$\times F\left(\frac{\mu-\nu}{2}, \frac{1+\mu-\nu}{2}; \frac{1}{2}+\mu; \frac{ch\varphi - ch\alpha}{1+ch\varphi}\right) \times sh^{1-\mu}\alpha\, H(\varphi-\alpha)$$

$$= \pi^{-3/2}\, 2^{(\mu-\nu-1)/2}\left\{\Gamma\left(\frac{1}{2}-\mu\right)\right\}^{-1} \sec \mu\pi\, (ch\varphi+1)^{(\mu-\nu)/2}$$

$$\times \int_0^\infty sh\, 2\pi\tau\; \Gamma\left(\frac{1-\mu+\nu}{2}+i\tau\right) \Gamma\left(\frac{1-\mu+\nu}{2}-i\tau\right) \Gamma\left(\frac{1-\mu-\nu}{2}+i\tau\right)$$

$$\times \Gamma\left(\frac{1-\mu-\nu}{2}-i\tau\right) P^{\mu,\nu}_{-1/2+i\tau}\,(ch\alpha) \sin \varphi\tau\, d\tau. \qquad \dots (2.10)$$

By Fourier sine inversion, the above equation becomes

$$P^{\mu,\nu}_{-1/2+i\tau}\,(ch\alpha) = (2\pi)^{3/2}\, 2^{(\nu-\mu)/2}\left\{\Gamma\left(\frac{1}{2}+\mu\right)\right\}^{-1} \operatorname{cosech} 2\pi\tau\, sh^{-\mu}\alpha$$

$$\times \left\{\Gamma\left(\frac{1-\mu+\nu}{2}+i\tau\right)\Gamma\left(\frac{1-\mu+\nu}{2}-i\tau\right)\right.$$

$$\times \left.\Gamma\left(\frac{1-\mu-\nu}{2}+i\tau\right)\Gamma\left(\frac{1-\mu-\nu}{2}-i\tau\right)\right\}^{-1}$$

$$\times \int_\alpha^\infty (ch\varphi+1)^{(\nu-\mu)/2}\,(ch\varphi - ch\alpha)^{-1/2+\mu}\,(ch\alpha+1)^{(\mu-\nu)/2}$$

$$\times F\left(\frac{\mu-\nu}{2}, \frac{1+\mu-\nu}{2}; \frac{1}{2}+\mu; \frac{ch\varphi - ch\alpha}{1+ch\varphi}\right) \sin \tau\varphi\, d\varphi.$$

By using a relation

$$F(a,b;c;z) = (1-z)^{-a} F\left(a,b;c;\frac{z}{z-1}\right),$$

the above equation gives the following integral representation for $P^{\mu,\nu}_{-1/2+i\tau}(ch\alpha)$:

$$P^{\mu,\nu}_{-1/2+i\tau}(ch\alpha) = (2\pi)^{3/2}\, 2^{(\nu-\mu)/2}\left\{\Gamma\left(\frac{1}{2}+\mu\right)\right\}^{-1} \operatorname{cosech} 2\pi\tau\, sh^{-\mu}\alpha$$

$$\times\left\{\Gamma\left(\frac{1-\mu+\nu}{2}+i\tau\right)\Gamma\left(\frac{1-\mu+\nu}{2}-i\tau\right)\Gamma\left(\frac{1-\mu-\nu}{2}+i\tau\right)\Gamma\left(\frac{1-\mu-\nu}{2}-i\tau\right)\right\}^{-1}$$

$$\times\int_{\alpha}^{\infty}(ch\varphi-ch\alpha)^{-1/2+\mu} F\left(\frac{\mu-\nu}{2},\frac{\mu+\nu}{2};\frac{1}{2}+\mu;\frac{ch\alpha-ch\varphi}{1+ch\alpha}\right)\sin\tau\varphi\, d\varphi \quad \text{... (2.11)}$$

where $|\operatorname{Re}\mu| < \frac{1}{2}$, $|\operatorname{Re}\nu| < 1-\operatorname{Re}\mu$.

3. Dual Integral Equations With Generalized Associated Legendre Function Kernels

We consider the dual integral equations

$$\int_0^{\infty} f(\tau)\, P^{\mu,\nu}_{-1/2+i\tau}(ch\alpha)\, d\tau = g(\alpha),\quad 0\le\alpha\le\alpha_1, \quad \text{... (3.1)}$$

$$\int_0^{\infty} f(\tau)\,\Gamma\left(\frac{1-\mu+\nu}{2}+i\tau\right)\Gamma\left(\frac{1-\mu+\nu}{2}-i\tau\right)\Gamma\left(\frac{1-\mu-\nu}{2}+i\tau\right)$$

$$\times\,\Gamma\left(\frac{1-\mu-\nu}{2}-i\tau\right)\tau\, sh\, 2\pi\tau\, P^{\mu,\nu}_{-1/2+i\tau}(ch\alpha)\, d\tau = h(\alpha),\quad \alpha>\alpha_1. \quad \text{... (3.2)}$$

To obtain the solution of these dual integral equations, multiplying (3.1) by

$$\pi^{-1} 2^{(\mu-\nu)/2}\,\Gamma\left(\frac{1}{2}-\mu\right)\cos\mu\pi\, sh^{1-\mu}\alpha\,(ch\varphi-ch\alpha)^{\mu-1/2}(ch\varphi+1)^{(\nu-\mu)/2}$$

$$\times\,(ch\alpha+1)^{(\mu-\nu)/2} F\left(\frac{\mu-\nu}{2},\frac{1+\mu-\nu}{2};\frac{1}{2}+\mu;\frac{ch\varphi-ch\alpha}{1+ch\varphi}\right), \quad \text{... (3.3)}$$

and integrating with respect to α from 0 to φ, and then differentiating with respect to φ and interchanging the order of integration and using (2.2), we find

$$F_c\,[f(\tau)] = g_1(\varphi), \quad 0 \le \varphi \le \alpha_1, \qquad \text{... (3.4)}$$

where

$$g_1(\varphi) = \pi^{-1}\, 2^{(\mu-\nu)/2}\, \Gamma\left(\frac{1}{2}-\mu\right) \cos \mu\pi \frac{d}{d\varphi}$$

$$\times \Bigg[(ch\varphi + 1)^{(\nu-\mu)/2} \times \int_0^{\varphi} (ch\varphi - ch\alpha)^{\mu-1/2} (ch\alpha + 1)^{(\mu-\nu)/2}$$

$$\times F\left(\frac{\mu-\nu}{2}, \frac{1+\mu-\nu}{2}; \frac{1}{2}+\mu; \frac{ch\varphi - ch\alpha}{1+ch\varphi}\right) sh^{1-\mu}\alpha\, g(\alpha)\, d\alpha \Bigg]. \qquad \text{... (3.5)}$$

Similarly, multiplying (3.2) by

$$\pi^{-2}\, 2^{((\mu-\nu)/2)-1} \left\{ \Gamma\left(\frac{1}{2}-\mu\right)\right\}^{-1} (ch\alpha - ch\varphi)^{-\mu-1/2}$$

$$\times F\left(\frac{-\mu+\nu}{2}, -\frac{\nu+\mu}{2}; \frac{1}{2}-\mu; \frac{ch\alpha - ch\varphi}{1+ch\alpha}\right) sh^{1+\mu}\alpha, \qquad \text{... (3.6)}$$

integrating with respect to α from φ to ∞, and then interchanging the order of integration and using (2.7), we get

$$F_c\,[f(\tau)] = h_1(\varphi), \quad \alpha_1 < \varphi < \infty, \qquad \text{... (3.7)}$$

where

$$h_1(\varphi) = \pi^{-2}\, 2^{((\mu-\nu)/2)-1} \left\{ \Gamma\left(\frac{1}{2}-\mu\right)\right\}^{-1} \int_{\varphi}^{\infty} (ch\alpha - ch\varphi)^{-\mu-1/2}$$

$$\times F\left(\frac{\nu-\mu}{2}, -\frac{\nu+\mu}{2}; \frac{1}{2}-\mu; \frac{ch\alpha - ch\varphi}{1+ch\alpha}\right) sh^{1+\mu}\alpha\, h(\alpha)\, d\alpha. \qquad \text{... (3.8)}$$

Hence using the inversion formula for the Fourier cosine transform, (3.4) and (3.7) give

$$f(\tau) = \left(\frac{2}{\pi}\right)^{1/2} \left[\int_0^{\alpha_1} g_1(\varphi) \cos \varphi\tau\, d\varphi + \int_{\alpha_1}^{\infty} h_1(\varphi) \cos \varphi\tau\, d\varphi \right] \qquad \text{... (3.9)}$$

which gives the solution of the dual integral equations (3.1) and (3.2).

Next, we consider the dual integral equations

$$\int_0^\infty \tau f(\tau) P^{\mu,\nu}_{-1/2+i\tau}(ch\alpha)\, d\tau = m(\alpha), \quad 0 \le \alpha \le \alpha_1, \qquad \text{... (3.10)}$$

$$\int_0^\infty f(\tau)\,\Gamma\left(\frac{1-\mu+\nu}{2}+i\tau\right)\Gamma\left(\frac{1-\mu+\nu}{2}-i\tau\right)\Gamma\left(\frac{1-\mu-\nu}{2}+i\tau\right)$$

$$\times\,\Gamma\left(\frac{1-\mu-\nu}{2}-i\tau\right) sh\,2\pi\tau\, P^{\mu,\nu}_{-1/2+i\tau}(ch\alpha)\, d\tau = n(\alpha), \quad \alpha > \alpha_1.$$

... (3.11)

Multiplying (3.10) by the expression (3.3) and integrating with respect to α from 0 to φ, then interchanging the order of integration and using (2.9), we find

$$F_s[f(\tau)] = m_1(\varphi), \quad 0 \le \varphi < \alpha_1, \qquad \text{... (3.12)}$$

where $$m_1(\varphi) = \pi^{-1}\, 2^{(\mu-\nu)/2}\,\Gamma\left(\frac{1}{2}-\mu\right)\cos\mu\pi\,(ch\varphi+1)^{(\nu-\mu)/2}$$

$$\times\int_0^\varphi (ch\varphi - ch\alpha)^{\mu-1/2}(ch\alpha+1)^{(\mu-\nu)/2}$$

$$\times\, F\left(\frac{\mu-\nu}{2},\ \frac{1+\mu-\nu}{2};\ \frac{1}{2}+\mu;\ \frac{ch\varphi-ch\alpha}{1+ch\varphi}\right) sh^{1-\mu}\alpha\, m(\alpha)\, d\alpha.$$

... (3.13)

Similarly, multiplying (3.11) by the expression (3.6) and integrating with respect to α from φ to ∞, and then differentiating with respect to φ and interchanging the order of integration and using (2.8), we get

$$F_s[f(\tau)] = n_1(\varphi), \quad \alpha_1 < \varphi < \infty, \qquad \text{... (3.14)}$$

where $$n_1(\varphi) = -\pi^{-1}\, 2^{((\mu-\nu)/2)-1}\left\{\Gamma\left(\frac{1}{2}-\mu\right)\right\}^{-1}\frac{d}{d\varphi}\int_\varphi^\infty (ch\alpha - ch\varphi)^{-\mu-1/2}$$

$$\times\, F\left(\frac{\nu-\mu}{2},\ -\frac{\nu+\mu}{2};\ \frac{1}{2}-\mu;\ \frac{ch\alpha-ch\varphi}{1+ch\alpha}\right) sh^{1+\mu}\alpha\, n(\alpha)\, d\alpha.$$

... (3.15)

Hence, using the inversion formula for the Fourier sine transform, (3.12) and (3.14) give

$$f(\tau) = \left(\frac{2}{\pi}\right)^{1/2}\left[\int_0^{\alpha_1} m_1(\varphi)\sin\varphi\tau\, d\varphi + \int_{\alpha_1}^\infty n_1(\varphi)\sin\varphi\tau\, d\varphi\right], \qquad \text{... (3.16)}$$

which gives the solution of the dual integral equations (3.10) and (3.11).

Now we consider more general dual integral equations

$$\int_0^\infty f(\tau)\,[1+w(\tau)]\,P^{\mu,\nu}_{-1/2+i\tau}(ch\alpha)\,d\tau = g(\alpha)\,,\quad 0 \le \alpha \le \alpha_1, \qquad \ldots (3.17)$$

$$\int_0^\infty f(\tau)\,\Gamma\left(\frac{1-\mu+\nu}{2}+i\tau\right)\Gamma\left(\frac{1-\mu+\nu}{2}-i\tau\right)\Gamma\left(\frac{1-\mu-\nu}{2}+i\tau\right)$$
$$\times\,\Gamma\left(\frac{1-\mu-\nu}{2}-i\tau\right)\tau\, sh\, 2\pi\tau\, P^{\mu,\nu}_{-1/2+i\tau}(ch\alpha)\,d\tau = h(\alpha)\,,\quad \alpha_1 < \alpha < \infty, \qquad \ldots (3.18)$$

where $w(\tau)$ is a known weight function.

Equation (3.17) can be written as

$$\int_0^\infty f(\tau)\,P^{\mu,\nu}_{-1/2+i\tau}(ch\alpha)\,d\tau = g(\alpha) - \int_0^\infty f(\tau)\,w(\tau)\,P^{\mu,\nu}_{-1/2+i\tau}(ch\alpha)\,d\tau,\ 0 \le \alpha \le \alpha_1. \qquad \ldots (3.19)$$

Then by (3.9), the solution of (3.19), (3.18) is obtained as

$$f(\tau) = \left(\frac{2}{\pi}\right)^{1/2}\left[\int_0^{\alpha_1} G(\varphi)\cos\tau\varphi\,d\varphi + \int_{\alpha_1}^\infty h_1(\varphi)\cos\tau\varphi\,d\varphi\right] \qquad \ldots (3.20)$$

where $h_1(\varphi)$ is the same as defined by (3.8), but

$$G(\varphi) = \pi^{-1}\,2^{(\mu-\nu)/2}\,\Gamma\left(\frac{1}{2}-\mu\right)\cos\mu\pi\,\frac{d}{d\varphi}$$
$$\times\left[(ch\varphi+1)^{(\nu-\mu)/2}\int_0^\varphi (ch\varphi - ch\alpha)^{\mu-1/2}\,(ch\alpha+1)^{(\mu-\nu)/2}\right.$$
$$\times\,F\left(\frac{\mu-\nu}{2},\,\frac{1+\mu-\nu}{2};\,\frac{1}{2}+\mu;\,\frac{ch\varphi - ch\alpha}{1+ch\varphi}\right) sh^{1-\mu}\alpha$$
$$\left.\times\left\{g(\alpha) - \int_0^\infty f(\tau)\,w(\tau)\,P^{\mu,\nu}_{-1/2+i\tau}(ch\alpha)\,d\tau\right\}d\alpha\right],\quad 0 \le \varphi \le \alpha_1. \qquad \ldots (3.21)$$

Hence the solution of the pair of eqns. (3.17), (3.18) has been reduced to the solution of a Fredholm integral equation of second kind for the function $G(\varphi)$, which, following Pathak[7] can be written in the form

$$G(\varphi) + \int_0^{\alpha_1} K(\varphi, u)\,G(u)\,du = G^*(\varphi)\,,\quad 0 \le \varphi \le \alpha_1, \qquad \ldots (3.22)$$

where $K(\varphi, u) = (2\pi)^{-1/2}\ \{w_c(\varphi+u) - w_c(|\varphi - u|)\},\ w_c(\varphi) = F_c\,[w(\tau)]$,

$G^*(\varphi) = g_1(\varphi) - g_2(\varphi)$, $g_1(\varphi)$ is defined by (3.5),

$g_2(\varphi) = F_c\,[w(\tau)\,H_1(\tau)]$, with $$H_1(\tau) = \left(\frac{2}{\pi}\right)^{1/2} \int_{\alpha_1}^{\infty} h_1(\varphi) \cos \tau\varphi \, d\varphi.$$

Similarly, the solution of the general pair of equations

$$\int_0^\infty \tau f(\tau)\, P^{\mu,\nu}_{-1/2+i\tau}(ch\alpha)\, d\tau = m(\alpha), \quad 0 \le \alpha \le \alpha_1,$$

$$\int_0^\infty f(\tau)\,[1 + w(\tau)]\,\Gamma\left(\frac{1-\mu+\nu}{2} + i\tau\right)\Gamma\left(\frac{1-\mu+\nu}{2} - i\tau\right)$$

$$\times\,\Gamma\left(\frac{1-\mu-\nu}{2} + i\tau\right)\Gamma\left(\frac{1-\mu-\nu}{2} - i\tau\right)$$

$$\times\, sh\, 2\pi\tau\, P^{\mu,\nu}_{-1/2+i\tau}(ch\alpha)\, d\tau = n(\alpha), \quad \alpha_1 < \alpha < \infty$$

is given by

$$f(\tau) = \left(\frac{2}{\pi}\right)^{1/2} \left[\int_0^{\alpha_1} m_1(\varphi) \sin \tau\varphi \, d\varphi + \int_{\alpha_1}^{\infty} N(\varphi) \sin \tau\varphi \, d\varphi\right]$$

where $m_1(\varphi)$ is the same as defined by (3.13), but $N(\varphi)$ satisfies the Fredholm integral equation of second kind given by

$$N(\varphi) = N^*(\varphi) + \int_{\alpha_1}^{\infty} K_1(\varphi, u)\, N(u)\, du, \quad \alpha_1 < \varphi < \infty,$$

where $$K_1(\varphi, u) = (2\pi)^{-1/2} \{w_c(\varphi + u) - w_c(|\varphi - u|)\},$$

$N^*(\varphi) = n_1(\varphi) - n_2(\varphi)$, $n_1(\varphi)$ is defined by (3.15)

and $$n_2(\varphi) = \left(\frac{2}{\pi}\right) \int_0^{\alpha_1} m_1(y) \left\{\int_0^\infty w(\tau) \sin \tau\varphi \sin \tau y \, d\tau\right\} dy.$$

4. Dual Integral Equations with Trigonometric Function Kernels

First we consider the dual integral equations

$$\int_0^\infty f(\tau) \cos \tau\varphi \, d\tau = g(\varphi), \quad 0 \le \varphi \le \alpha_1, \qquad \dots (4.1)$$

$$\int_0^\infty f(\tau)\left\{\Gamma\left(\frac{1-\mu+\nu}{2}+i\tau\right)\Gamma\left(\frac{1-\mu+\nu}{2}-i\tau\right)\ \Gamma\left(\frac{1-\mu-\nu}{2}+i\tau\right)\right.$$

$$\left.\times\ \Gamma\left(\frac{1-\mu-\nu}{2}-i\tau\right)\right\}^{-1} csch\, 2\pi\tau \sin\tau\varphi\, d\tau = h(\varphi),\quad \alpha_1<\varphi<\infty. \qquad \text{... (4.2)}$$

To solve these pair of equations, we multiply (4.1) by

$$\pi^{-1/2}\, 2^{(\mu-\nu-1)/2}\left\{\Gamma\left(\frac{1}{2}-\mu\right)\right\}^{-1} sh^{\mu}\alpha\,(ch\alpha-ch\varphi)^{-\mu-1/2}$$

$$\times F\left(\frac{\nu-\mu}{2},\ -\frac{\nu+\mu}{2};\ \frac{1}{2}-\mu;\frac{ch\alpha-ch\varphi}{1+ch\alpha}\right) \qquad \text{... (4.3)}$$

and integrate it with respect to φ from 0 to α, then interchange the order of integration and then apply (2.1) to obtain

$$\int_0^\infty f(\tau)\, P^{\mu,\nu}_{-1/2+i\tau}(ch\alpha)\, d\tau = G_1(\alpha),\quad 0\le\alpha\le\alpha_1 \qquad \text{... (4.4)}$$

where $$G_1(\alpha) = \pi^{-1/2}\, 2^{(\mu-\nu-1)/2}\left\{\Gamma\left(\frac{1}{2}-\mu\right)\right\}^{-1} sh^{\mu}\alpha\int_0^\alpha (ch\alpha-ch\varphi)^{-\mu-1/2}$$

$$\times F\left(\frac{\nu-\mu}{2},\ -\frac{\nu+\mu}{2};\ \frac{1}{2}-\mu;\frac{ch\alpha-ch\varphi}{1+ch\alpha}\right) g(\varphi)\, d\varphi. \qquad \text{... (4.5)}$$

Also, if we multiply (4.2) by

$$(2\pi)^{3/2}\, 2^{(\nu-\mu)/2}\left\{\Gamma\left(\frac{1}{2}+\mu\right)\right\}^{-1} sh^{-\mu}\alpha\ (ch\varphi-ch\alpha)^{\mu-1/2}$$

$$\times F\left(\frac{\mu-\nu}{2},\ \frac{\mu+\nu}{2};\ \frac{1}{2}+\mu;\frac{ch\alpha-ch\varphi}{1+ch\alpha}\right), \qquad \text{... (4.6)}$$

integrate it with respect to φ from α to ∞, then interchange the order of integration and then apply (2.10) to get

$$\int_0^\infty f(\tau)\ P^{\mu,\nu}_{-1/2+i\tau}(ch\alpha)\, d\tau = H_1(\alpha),\ \alpha_1<\alpha<\infty, \qquad \text{... (4.7)}$$

where $$H_1(\alpha) = (2\pi)^{3/2}\, 2^{(\nu-\mu)/2}\left\{\Gamma\left(\frac{1}{2}+\mu\right)\right\}^{-1} sh^{-\mu}\alpha\int_\alpha^\infty (ch\varphi-ch\alpha)^{\mu-1/2}$$

$$\times F\left(\frac{\mu-\nu}{2},\ \frac{\mu+\nu}{2};\ \frac{1}{2}+\mu;\frac{ch\alpha-ch\varphi}{1+ch\alpha}\right) h(\varphi)\, d\varphi. \qquad \text{... (4.8)}$$

Then by generalized Mehler-Fok transform formulae (2.3), (2.4) we obtain the solution of the pair of equations (4.1), (4.2) from (4.4) and (4.7), given by

$$f(\tau) = \pi^{-2}\, 2^{\mu-\nu-1}\, \tau\, sh\, 2\pi\tau\, \Gamma\left(\frac{1-\mu+\nu}{2}+i\tau\right) \Gamma\left(\frac{1-\mu+\nu}{2}-i\tau\right)$$

$$\times \Gamma\left(\frac{1-\mu-\nu}{2}+i\tau\right)\Gamma\left(\frac{1-\mu-\nu}{2}-i\tau\right)\left[\int_0^{\alpha_1} G_1(\alpha)\, P^{\mu,\nu}_{-1/2+i\tau}(ch\alpha)\, sh\alpha\, d\alpha\right.$$

$$\left. + \int_{\alpha_1}^{\infty} H_1(\alpha)\, P^{\mu,\nu}_{-1/2+i\tau}(ch\alpha)\, sh\alpha\, d\alpha\right]. \qquad \dots (4.9)$$

Next, the solution of the dual integral equations

$$\int_0^{\infty} \tau^{-1} f(\tau) \sin\tau\varphi\, d\tau = m(\varphi), \quad 0 \le \varphi \le \alpha_1, \qquad \dots (4.10)$$

$$\int_0^{\infty} f(\tau)\left\{\Gamma\left(\frac{1-\mu+\nu}{2}+i\tau\right)\Gamma\left(\frac{1-\mu+\nu}{2}-i\tau\right)\ \Gamma\left(\frac{1-\mu-\nu}{2}+i\tau\right)\right.$$

$$\left.\times\ \Gamma\left(\frac{1-\mu-\nu}{2}-i\tau\right)\right\}^{-1} csch\, 2\pi\tau \sin\tau\varphi\, d\tau = h(\varphi), \quad \alpha_1 < \varphi < \infty,$$

$$\dots (4.11)$$

is given by (4.9) where we define

$$G_1(\alpha) = \pi^{-1/2}\, 2^{(\mu-\nu-1)/2}\left\{\Gamma\left(\frac{1}{2}-\mu\right)\right\}^{-1} sh^{\mu}\alpha \int_0^{\alpha} (ch\alpha - ch\varphi)^{-\mu-1/2}$$

$$\times F\left(\frac{\nu-\mu}{2}, -\frac{\nu+\mu}{2}; \frac{1}{2}-\mu; \frac{ch\alpha - ch\varphi}{1+ch\alpha}\right) m'(\varphi)\, d\varphi, \quad \dots (4.12)$$

instead of (4.5). The function $H_1(\alpha)$ is same as defined by (4.8).

The solution of the pair of equations

$$\int_0^{\infty} \tau f(\tau) \sin\tau\varphi\, d\tau = n(\varphi), \quad 0 \le \varphi \le \alpha_1, \qquad \dots (4.13)$$

$$\int_0^{\infty} f(\tau)\left\{\Gamma\left(\frac{1-\mu+\nu}{2}+i\tau\right)\Gamma\left(\frac{1-\mu+\nu}{2}-i\tau\right)\Gamma\left(\frac{1-\mu-\nu}{2}+i\tau\right)\right.$$

$$\left.\times\left\{\Gamma\left(\frac{1-\mu-\nu}{2}-i\tau\right)\right\}^{-1} csch\, 2\pi\tau \sin\tau\varphi\, d\tau = h(\varphi), \quad \alpha_1 < \varphi < \infty,\right.$$

$$\dots (4.14)$$

is obtained by integrating (4.13) with respect to φ from 0 to φ.

It then assumes the form (4.1) with $g(\varphi)$ now defined by

$$g(\varphi) = C - \int_0^{\varphi} n(u)\, du \qquad \text{... (4.15)}$$

where C is constant of integration defined by

$$C = \int_0^{\infty} f(\tau)\, d\tau. \qquad \text{... (4.16)}$$

The solution is then given in terms of C by (4.9), (4.5) and (4.8) with $g(\varphi)$ defined by (4.15).

The solution of the dual integral equations

$$\int_0^{\infty} f(\tau) \cos \tau\varphi \, d\tau = g(\varphi), \quad 0 \le \varphi \le \alpha_1, \qquad \text{... (4.17)}$$

$$\int_0^{\infty} \tau^{-1} f(\tau) \left\{ \Gamma\left(\frac{1-\mu+\nu}{2} + i\tau \right) \Gamma\left(\frac{1-\mu+\nu}{2} - i\tau \right) \Gamma\left(\frac{1-\mu-\nu}{2} + i\tau \right) \right.$$

$$\left. \times \left\{ \Gamma\left(\frac{1-\mu-\nu}{2} - i\tau \right) \right\} \right\}^{-1} csch\, 2\pi\tau \cos \tau\varphi \, d\tau = p(\varphi), \quad \alpha_1 < \varphi < \infty, \qquad \text{... (4.18)}$$

is given by (4.9), (4.5) and (4.8) with $h(\varphi)$ now defined by

$$h(\varphi) = -p'(\varphi). \qquad \text{... (4.19)}$$

Finally, the solution of the pair of equations

$$\int_0^{\infty} f(\tau) \cos \tau\varphi \, d\tau = g(\varphi), \quad 0 \le \varphi \le \alpha_1, \qquad \text{... (4.20)}$$

$$\int_0^{\infty} \tau f(\tau) \left\{ \Gamma\left(\frac{1-\mu+\nu}{2} + i\tau \right) \Gamma\left(\frac{1-\mu+\nu}{2} - i\tau \right) \Gamma\left(\frac{1-\mu-\nu}{2} + i\tau \right) \right.$$

$$\left. \times \; \Gamma\left(\frac{1-\mu-\nu}{2} - i\tau \right) \right\}^{-1} csch\, 2\pi\tau \cos \tau\varphi \, d\tau = q(\varphi), \quad \alpha_1 < \varphi < \infty, \qquad \text{... (4.21)}$$

is given by (4.9), (4.5) and (4.8) with $h(\varphi)$ now defined by

$$h(\varphi) = C_1 - \int_{\varphi}^{\infty} q(u)\, du. \qquad \text{... (4.22)}$$

The unknown constant C is determined in the same way as in (4.16).

It may be mentioned that all the results and the dual integral equations and their solutions considered by Pathak[7] and Babloian[1] can be deduced as special cases of the results obtained in this paper simply on setting $\mu = \nu$ and $\mu = \nu = 0$ respectively and using the facts that

$$P^{\mu,\mu}_{-1/2+i\tau}(x) = P^{\mu}_{-1/2+i\tau}(x) \text{ and } P^{0}_{-1/2+i\tau}(x) = P_{-1/2+i\tau}(x).$$

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