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WEAKER FORMS OF IRRESOLUTE FUNCTIONS

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In this paper, we introduce and investigate θ-irresolute functions which lie strictly between irresolute functions and quasi-irresolute functions introduced by Crossely and Hildebrand[4] and Di Maio and Noiri[6] respectively.

1. INTRODUCTION AND PRELIMINARIES

Crossely and Hildebrand[4] introduced the notion of irresolute functions which is stronger than that of semi-continuity, but is independent of that of continuity. Di Maio and Noiri[6] introduced and investigated the notion of quasi-irresolute functions. The purpose of this paper is to introduce a new class of functions called θ-irresolute and show that it lies strictly between above-mentioned two classes.

Throughout the present paper, spaces always mean topological spaces and $f : X \to Y$ denotes a single valued function of a space X into a space Y. Let X be a space and A a subset of X. We denote the closure of A and the interior of A by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A is said to be semiopen[8], denoted by $A \in SO(X)$, if there exists an open set U such that $U \subset A \subset \mathrm{Cl}(U)$, or equivalently if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$. The complement of a semiopen set is said to be semiclosed[3]. The intersection of all semiclosed sets containing A is called the semi-closure[3] of A and is denoted by $s\text{-Cl}(A)$. The semi-interior[3] of A, denoted by $s\text{-Int}(A)$, is defined by the union of all semiopen sets contained in A. The semi-θ-closure[5] of A and the semi-θ-interior[5] of A are respectively defined as follows :

$$s\text{-Cl}_\theta(A) = \{x \in X \mid A \cap s\text{-Cl}(U) \neq \phi \text{ for each } U \in SO(X) \text{ containing } x\},$$

$$s\text{-Int}_\theta(A) = \{x \in X \mid s\text{-Cl}(U) \subset A \text{ for each } U \in SO(X) \text{ containing } x\}.$$

2. θ-IRRESOLUTE FUNCTIONS

Definition 2.1 — A function $f : X \to Y$ is said to be irresolute[4] if $f^{-1}(V) \in SO(X)$ for each $V \in SO(Y)$.

Definition 2.2 — A function $f : X \to Y$ is said to be quasi-irresolute[6] if for each $x \in X$ and each $V \in SO(Y)$ containing $f(x)$, there exists $U \in SO(X)$ containing x such that $f(U) \subset s\text{-Cl}(V)$.

Definition 2.3 — A function $f : X \to Y$ is said to be θ-irresolute if for each $x \in X$ and each $V \in SO(Y)$ containing $f(x)$, there exists $U \in SO(X)$ containing x such that $f(s\text{-Cl}(U)) \subset s\text{-Cl}(V)$.

Theorem 2.1 — Every irresolute function is θ-irresolute.

PROOF : Let $x \in X$ and $V \in SO(Y)$ containing $f(x)$. Since f is irresolute, there exists $U \in SO(X)$ containing x such that $f(U) \subset V \subset s\text{-Cl}(V)$. Hence $U \subset f^{-1}(s\text{-Cl}(V))$. Since f is irresolute and $s\text{-Cl}(V)$ is semiclosed in Y, $f^{-1}(s\text{-Cl}(V))$ is semiclosed. Hence $s\text{-Cl}(U) \subset s\text{-Cl}(f^{-1}(s\text{-Cl}(V))) = f^{-1}(s\text{-Cl}(V))$. That is, $f(s\text{-Cl}(U)) \subset s\text{-Cl}(V)$. Thus f is θ-irresolute.

The converse of Theorem 2.1 is not necessarily true as can be seen from the following example.

Example 2.1 — Let $X = \{a, b, c, d\}$, $\tau_1 = \{X, \phi, \{a, b, c\}\}$ and $\tau_2 = \{X, \phi, \{a, b\}\}$. Let $f : (X, \tau_1) \to (X, \tau_2)$ be the identity function. Then f is θ-irresolute but not irresolute.

Every θ-irresolute function is quasi-irresolute. But the converse is not true as can be seen from the following example.

Example 2.2 — Let $X = \{a, b, c, d\}$, $\tau_1 = \{X, \phi, \{a\}, \{a, d\}, \{b, c\}, \{a, b, c\}\}$ and $\tau_2 = \{X, \phi, \{d\}, \{b, c\}, \{b, c, d\}\}$. Define $f : (X, \tau_1) \to (X, \tau_2)$ by $f(a) = d$, $f(b) = c$, $f(c) = b$ and $f(d) = a$. Then f is quasi-irresolute but not θ-irresolute.

Theorem 2.2 — For a function $f : X \to Y$, the following are equivalent :

(a) f is θ-irresolute.

(b) For each subset B of Y, $s\text{-Cl}_\theta(f^{-1}(B)) \subset f^{-1}(s\text{-Cl}_\theta(B))$.

(c) For each $B \in SO(Y)$, $s\text{-Cl}_\theta(f^{-1}(B)) \subset f^{-1}(s\text{-Cl}_\theta(B))$.

(d) For each $B \in SO(Y)$, $f^{-1}(B) \subset s\text{-Int}_\theta(f^{-1}(s\text{-Cl}(B)))$.

(e) For each subset A of X, $f(s\text{-Cl}_\theta(A)) \subset s\text{-Cl}_\theta(f(A))$.

(f) For each subset B of Y, $f^{-1}(s\text{-Int}_\theta(B)) \subset s\text{-Int}_\theta(f^{-1}(B))$.

(g) For each semi-θ-closed set B of Y, $f^{-1}(B)$ is semi-θ-closed.

(h) For each semi-θ-open set B of Y, $f^{-1}(B)$ is semi-θ-open.

PROOF : (a) ⇒ (b) : Let B be any subset of Y and $x \notin f^{-1}(s\text{-Cl}_\theta(B))$. Then $f(x) \notin s\text{-Cl}_\theta(B)$ and there exists $V \in SO(Y)$ containing $f(x)$ such that $s\text{-Cl}(V) \cap B = \phi$. By (a), there exists $U \in SO(X)$ containing x such that $f(s\text{-Cl}(U)) \subset s\text{-Cl}(V)$. Then we obtain $s\text{-Cl}(U) \cap f^{-1}(B) = \phi$ and hence $x \notin s\text{-Cl}_\theta(f^{-1}(B))$. Thus $s\text{-Cl}_\theta(f^{-1}(B)) \subset f^{-1}(s\text{-Cl}_\theta(B))$.

(b) ⇒ (c) : Since $s\text{-Cl}(B) = s\text{-Cl}_\theta(B)$ for each $B \in SO(Y)$, this is obvious.

(c) ⇒ (d) : Let $B \in SO(Y)$. Since $Y - s\text{-Cl}(B) \in SO(Y)$,

$$X - s\text{-Int}_\theta(f^{-1}(s\text{-Cl}(B))) = s\text{-Cl}_\theta(X - f^{-1}(s\text{-Cl}(B)))$$

$$= s\text{-Cl}_\theta(f^{-1}(Y - s\text{-Cl}(B))) \subset f^{-1}(s\text{-Cl}(Y - s\text{-Cl}(B)))$$

$$= f^{-1}(Y - s\text{-Int}(s\text{-Cl}(B))) \subset X - f^{-1}(B).$$

Hence $f^{-1}(B) \subset s\text{-Int}_\theta(f^{-1}(s\text{-Cl}_\theta(B)))$.

(d) $\Rightarrow$ (a) : Let $x \in X$ and $V \in SO(Y)$ containing $f(x)$. Then by (d), $x \in f^{-1}(V) \subset s\text{-Int}_\theta(f^{-1}(s\text{-Cl}(V)))$. There exists $U \in SO(X)$ containing x such that $s\text{-Cl}(U) \subset f^{-1}(s\text{-Cl}(V))$ which implies $f(s\text{-Cl}(U)) \subset s\text{-Cl}(V)$. Hence f is θ-irresolute.

(b) $\Leftrightarrow$ (e) : Straightforward.

(d) $\Leftrightarrow$ (f) : Straightforward.

(b) $\Rightarrow$ (g) : Let B be any semi-θ-closed set of Y. By (b), $s\text{-Cl}_\theta(f^{-1}(B)) \subset f^{-1}(s\text{-Cl}_\theta(B)) = f^{-1}(B)$ and hence $f^{-1}(B)$ is semi-θ-closed.

(g) $\Rightarrow$ (b) : Let B be any subset of Y. Then $s\text{-Cl}_\theta(B)$ is semi-θ-closed in Y and by (g), $f^{-1}(s\text{-Cl}_\theta(B))$ is semi-θ-closed. Hence $s\text{-Cl}_\theta(f^{-1}(B)) \subset s\text{-Cl}_\theta(f^{-1}(s\text{-Cl}_\theta(B))) = f^{-1}(s\text{-Cl}_\theta(B))$.

(g) $\Leftrightarrow$ (h) : Obvious.

A set N of a space X is said to be semi-neighbourhood[2] (briefly, semi-nbd) if there exists $U \in SO(X)$ such that $x \in U \subset N$. Bohn and Lee[2] showed that $A \in SO(X)$ if and only if A is the semi-nbd of each of its points.

Definition 2.4 — A set A of a space X is said to be a semi-θ-nbd of $x \in X$ if there exists a $U \in SO(X)$ containing x such that $s\text{-Cl}(U) \subset A$.

Theorem 2.3 — Let X be a space and let $A \subset X$. Then A is a semi-θ-open if and only if A is the semi θ-nbd of each of its points.

PROOF : It is obvious.

Theorem 2.4 — A function $f : X \to Y$ is θ-irresolute if and only if for each $x \in X$ and each semi-θ-nbd N of $f(x)$, $f^{-1}(N)$ is semi-θ-nbd of x.

PROOF : For $x \in X$ let N be a semi-θ-nbd of $f(x)$. Then there exists $V \in SO(Y)$ containing $f(x)$ such that $s\text{-Cl}(V) \subset N$. By θ-irresolute of f, there exists $U \in SO(X)$ containing x such that $f(s\text{-Cl}(U)) \subset s\text{-Cl}(V) \subset N$. This implies that $s\text{-Cl}(U) \subset f^{-1}(s\text{-Cl}(U))) \subset f^{-1}(N)$ and consequently $f^{-1}(N)$ is semi-θ-nbd of x.

Conversely, Let $x \in X$ and $V \in SO(Y)$ containing $f(x)$. Then $s\text{-Cl}(V)$ is semi-θ-nbd of $f(x)$. By hypothesis, $f^{-1}(s\text{-Cl}(V))$ is semi-θ-nbd of x. Hence there exists $U \in SO(X)$ containing x such that $s\text{-Cl}(U) \subset f^{-1}(s\text{-Cl}(V))$ and thus $f(s\text{-Cl}(U)) \subset s\text{-Cl}(V)$, which shows that f is θ-irresolute.

Theorem 2.5 — Let $f : X \to Y$ be a function and $g : X \to X \times Y$ the graph function, given by $g(x) = (x, f(x))$ for each $x \in X$. If g is θ-irresolute, then f is θ-irresolute.

PROOF : Let $x \in X$ and $V \in SO(Y)$ containing $f(x)$. Then $X \times V \in SO(X \times Y)$ containing $g(x)$ and hence there exists $U \in SO(X)$ containing x such that $g(s\text{-Cl}(U)) \subset s\text{-Cl}(X \times V)$. Then by Lemma 4 of Noiri[12], we have $s\text{-Cl}(X \times V) \subset X \times s\text{-Cl}(V)$. Since g is graph of f, $f(s\text{-Cl}(U)) \subset s\text{-Cl}(V)$. This shows that f is θ-irresolute.

A subset A of a space X is said to be preopen[10] if $A \subset \text{Int}(\text{Cl}(A))$. The family of all preopen sets in a space X is denoted by $PO(X)$.

Lemma 2.1[14] — Let A and X_0 be subsets of a space X.

(a) If $A \in SO(X)$ and $X_0 \in PO(X)$, then $A \cap X_0 \in SO(X_0)$.

(b) If $A \subset X_0 \in PO(X)$, then $X_0 \cap s\text{-Cl}(A) = s\text{-Cl}_{X_0}(A)$, where $s\text{-Cl}_{X_0}(A)$ denotes the semi-closure of A in the subspace X_0 of X.

Theorem 2.6 — If $f : X \to Y$ is a θ-irresolute function and $X_0 \in PO(X)$, then the restriction $f|_{X_0} : X_0 \to Y$ is θ-irresolute.

PROOF : Let $x \in X_0$ and $V \in SO(Y)$ containing $f(x)$. There exists $U \in SO(X)$ containing x such that $f(s\text{-Cl}(U)) \subset s\text{-Cl}(V)$. Since $X_0 \in PO(X)$, by Lemma 2.1, we have $x \in U \cap X_0 \in SO(X_0)$ and $s\text{-Cl}_{X_0}(U \cap X_0) = X_0 \cap s\text{-Cl}(U \cap X_0) \subset X_0 \cap s\text{-Cl}(U)$. Hence we have

$$(f|_{X_0})(s\text{-Cl}(U \cap X_0)) \subset (f|_{X_0})(X_0 \cap s\text{-Cl}(U))$$

$$= f(X_0 \cap s\text{-Cl}(U)) \subset s\text{-Cl}(V).$$

This shows that $f|_{X_0}$ is θ-irresolute.

Theorem 2.7 — If $f : X \to Y$ is quasi-irresolute, then the following are true :

(a) For each subset A of X, $f(s\text{-Cl}(A)) \subset s\text{-Cl}_\theta(f(A))$.

(b) For each subset B of Y, $s\text{-Cl}(s\text{-Int}(s\text{-Cl}(f^{-1}(B)))) \subset f^{-1}(s\text{-Cl}_\theta(B))$.

PROOF : (a) Let $x \in s\text{-Cl}(A)$ and $V \in SO(Y)$ containing $f(x)$. Since f is quasi-irresolute, there exists $U \in SO(X)$ containing x such that $f(U) \subset s\text{-Cl}(V)$. Now, we have

$$x \in s\text{-Cl}(A) \Rightarrow U \cap \neq \phi \Rightarrow f(U) \cap f(A) \neq \phi \Rightarrow s\text{-Cl}(V) \cap f(A) \neq \phi$$

$$\Rightarrow f(x) \in s\text{-Cl}_\theta(f(A)) \Rightarrow x \in f^{-1}(s\text{-Cl}_\theta(f(A))).$$

Hence $s\text{-Cl}(A) \subset f^{-1}(s\text{-Cl}_\theta(f(A)))$ and thus $f(s\text{-Cl}(A)) \subset s\text{-Cl}_\theta(f(A))$.

(b) Let $x \in s\text{–Cl}(s\text{-Int}(s\text{-Cl}(f^{-1}(B))))$ and $V \in SO(Y)$ containing $f(x)$. By hypothesis, there exists $U \in SO(X)$ containing x such that $f(U) \subset s\text{-Cl}(V)$. Now we have

$$x \in s\text{-Cl}(s\text{-Int}(s\text{-Cl}(f^{-1}(B)))) \subset s\text{-Cl}(f^{-1}(B)) \Rightarrow U \cap f^{-1}(B) \neq \phi$$

$$\Rightarrow s\text{-Cl}(V) \cap B \neq \phi \Rightarrow f(x) \in s\text{-Cl}_\theta(B) \Rightarrow x \in f^{-1}(s\text{-Cl}_\theta(B)).$$

Hence $s\text{-Cl}(s\text{-Int}(s\text{-Cl}(B)))) \subset f^{-1}(s\text{-Cl}_\theta(B))$.

Theorem 2.8 — A function $f : X \to Y$ is quasi-irresolute if and only if for each $x \in X$ and each semi-θ-nbd N of $f(x)$, $f^{-1}(N)$ is semi-nbd of x.

PROOF : Similar to the proof of Theorem 2.4.

Theorem 2.9 — Let $f : X \to Y$ and $g : Y \to Z$ be functions.

(a) If f is θ-irresolute and g is θ-irresolute, then $g \circ f$ is θ-irresolute.

(b) If f is quasi-irresolute and g is θ-irresolute, then $g \circ f$ is quasi-irresolute.

PROOF : (a) Let $x \in X$ and $W \in SO(Z)$ containing $g(f(x))$. Then there exists $V \in SO(Y)$ containing $f(x)$ such that $g(s\text{-Cl}(V)) \subset s\text{-Cl}(W)$. Since f is θ-irresolute, there exists $U \in SO(X)$ containing x such that $f(s\text{-Cl}(U)) \subset s\text{-Cl}(V)$. Hence $(g \circ f)(s\text{-Cl}(U)) \subset s\text{-Cl}(W)$ and thus $g \circ f$ is θ-irresolute.

(b) Similar to (a).

A function $f : X \to Y$ is said to be pre-semi-open[4] if $f(U) \in SO(Y)$ for each $U \in SO(X)$.

Theorem 2.10 — Let $f : X \to Y$ and $g : Y \to Z$ be functions. If $g \circ f : X \to Z$ is θ-irresolute and f is pre-semi-open bijection, then g is θ-irresolute.

PROOF : Let $y \in Y$. Then $y = f(x)$ for some $x \in X$. Let $W \in SO(Z)$ containing $g(f(x))$. Since $g \circ f$ is θ-irresolute, there exists $U \in SO(X)$ containing x such that $(g \circ f)(s\text{-Cl}(U)) \subset s\text{-Cl}(W)$ which implies $f(s\text{-Cl}(U)) \subset g^{-1}(s\text{-Cl}(W))$. Since f is pre-semi-open and injection, $s\text{-Cl}(f(U)) \subset f(s\text{-Cl}(U))$. For, let $t \notin f(s\text{-Cl}(U))$. Then $f^{-1}(t) \notin s\text{-Cl}(U)$. Therefore, there exists $V \in SO(X)$ contaning $f^{-1}(t)$ such that $V \cap U = \phi$. Since f is injection, $f(V) \cap f(U) = \phi$. Since f is pre-semi-open, $f(V) \in SO(V)$. Hence $t \notin s\text{-Cl}(f(U))$. Thus $f(U) \in SO(Y)$ containing $f(x)$ such that $s\text{-Cl}(f(U)) \subset f(s\text{-Cl}(U)) \subset g^{-1}(s\text{-Cl}(W))$. This shows that g is θ-irresolute.

A space X is said to be semi-Urysohn[9] if for every distinct points x and y in X, there exist U, $V \in SO(X)$ such that $x \in Y$, $y \in V$ and $s\text{-Cl}(U) \cap s\text{-Cl}(V) = \phi$.

Theorem 2.11 — If $f : X \to Y$ is θ-irresolute injection and Y is semi-Urysohn, then X is semi-Urysohn.

PROOF : Let x_1 and x_2 be any distinct two points of X. Then $f(x_1) \neq f(x_2)$. By semi-Urysohn property of Y, there exist $V_1, V_2 \in SO(Y)$ containing $f(x_1)$ and $f(x_2)$ respectively such that $s\text{-Cl}(V_1) \cap s\text{-Cl}(V_2) = \phi$. Since f is θ-irresolute, there exist $U_1, U_2 \in SO(X)$ containing x_1 and x_2 respectively such that $f(s\text{-Cl}(U_1)) \subset s\text{-Cl}(V_1)$ and $f(s\text{-Cl}(U_2)) \subset s\text{-Cl}(V_2)$. Hence $s\text{-Cl}(U_1) \cap s\text{-Cl}(U_2) = \phi$ and thus X is semi-Urysohn.

Theorem 2.12 — If $f : X \to Y$ is θ-irresolute and Y is semi-Urysohn, then the set $A = \{(x_1, x_2) \in X \times X \mid f(x_1) = f(x_2)\}$ is semi-θ-closed in $X \times X$.

PROOF : Let $(x_1, x_2) \in X \times X - A$. Then $f(x_1) \neq f(x_2)$. Since Y is semi-Urysohn, there exists $V_i \in SO(Y)$ containing $f(x_i)$ $(i = 1, 2)$ such that $s\text{-Cl}(V_1) \cap s\text{-Cl}(V_2) = \phi$. Since f is θ-irresolute, there exists $U_i \in SO(X)$ containing x_i such that $f(s\text{-Cl}(U_i)) \subset s\text{-Cl}(V_i)$, where $i = 1, 2$. Put $U = U_1 \times U_2$, then by Theorem 11 of Levine[8] $(x_1, x_2) \in U \in SO(X \times X)$ and $A \cap s\text{-Cl}(U) = \phi$. Hence $(x_1, x_2) \notin s\text{-Cl}_\theta(A)$ and thus A is semi-θ-closed in $X \times X$.

Theorem 2.13 — If $f : X \to Y$ is θ-irresolute and Y is semi-Urysohn, then the graph $G(f)$ is semi-θ-closed in $X \times Y$.

PROOF : Let $(x, y) \notin G(f)$. Then $f(x) \neq y$ and hence there exists V, $W \in SO(Y)$ such that $f(x) \in V$, $y \in W$ and $s\text{-Cl}(V) \cap s\text{-Cl}(W) = \phi$. Since f is θ-irresolute, there exists $U \in SO(X)$ containing x such that $f(s\text{-Cl}(U)) \subset s\text{-Cl}(V)$. Therefore, $s\text{-Cl}(W) \cap f(s\text{-Cl}(U)) = \phi$ and hence $s\text{-Cl}(U \times W) \cap G(f) = \phi$. This shows $(x, y) \notin s\text{-Cl}_\theta(G(f))$ and $G(f)$ is semi-θ-closed in $X \times Y$.

A space X is said to be s-closed[5] if for each semi-open cover $\{U_\alpha \mid \alpha \in \Lambda\}$ of X, there exists a finite subset Λ_0 of Λ such that $X = \bigcup\{s\text{-Cl}U_\alpha \mid \alpha \in \Lambda_0\}$.

Theorem 2.14 — If $f : X \to Y$ is θ-irresolute surjection and X is s-closed, then Y is s-closed.

PROOF : Let $\{V_\alpha \mid \alpha \in \Lambda\}$ be any semiopen cover of Y. For each $x \in X$, there exists an $\alpha(x) \in \Lambda$ such that $f(x) \in V_{\alpha(x)}$. Since f is θ-irresolute, there exists a $U_x \in SO(X)$ containing x such that $f(s\text{-Cl}(U_x)) \subset s\text{-Cl}(V_{\alpha(x)})$. Since $\{U_x \mid x \in X\}$ is semiopen cover of X, there exists a finite points, say, $x_1, x_2, \ldots, x_n$ of X such that $X = \bigcup\{s\text{-Cl}(U_x) \mid i = 1, 2, \ldots, n\}$. Since f is surjective,

$$Y = \bigcup_{i=1}^{n} f(s\text{-}Cl(U_{x_i})) \subset \bigcup_{i=1}^{n} s\text{-}Cl(V_{\alpha(x_i)}).$$

This shows that Y is s-closed.

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