Indian J. pure appl. Math., 26(7) : 705-724, July 1995

RUDIMENTARY NON-LINEAR EFFECTS NEAR CRACK EDGES IN DYNAMIC CRACK PROPAGATION

L. M. BROCK[1] AND H. G. GEORGIADIS[2]

[1]*Engineering Mechanics, University of Kentucky, Lexington, Kentucky* 40506, *U.S.A.*

[2]*Division of Applied Mechanics, Aristotle University of Thessaloniki, Thessaloniki GR* 54006, *Greece*

(*Received* 2 *June* 1994; *after revision* 13 *February* 1995; *accepted* 22 *February* 1995)

Some effects of localized, rudimentary non-linear behaviour in dynamic quasi-brittle fracture are considered in terms of the ongoing propagation of a 2D Mode I tension crack. As a convenient model, a vanishingly-thin zone of non-linear material response forms ahead of each crack edge. Two generic response laws are examined for these zones — a non-linear stress-strain relation and a restriction on stresses in quadratic form.

Both cases impose non-linear boundary conditions, but multiple solution sets can be constructed analytically and studied. One solution set suggests the existence of speeds, characteristic of the zones, that bound the rates at which crack/zone propagation can occur. Major attention, however, is focused upon solution sets associated with a uniform zone stress.

Results based on actual material parameters indicate that the two generic response laws lead to similar predictions for the tension load critical for crack/zone growth. Both critical loads are generally less than those predicted for zone yielding in the manner of the well-known Dugdale model, especially at higher crack speeds. These results are, moreover, largely independent of a specific fracture criterion.

1. INTRODUCTION

Combined numerical/analytical studies of dynamic 3-point bend impact tests[4, 11] imply that large rotations may occur very near crack edges in otherwise linear materials. Similarly, crack bridging ligaments can lead to non-linear relations near the crack edge[18]. Also, Becker and Gross[3] have shown that modifying the standard Dugdale[12] zone to embrace a von Mises yield law[14] produces a crack model that is more sensitive to mixed-Mode loading. Finally, Knowles and Sternberg[15] have shown that interesting crack region behaviour is revealed by accounting for finite deformations.

This article considers, therefore, some effects of very rudimentary, localized non-linear behaviuor on dynamic, quasi-brittle fracture by introducing cohesive traction zones[2] at the crack edges. Two generic non-linear response laws are treated for the zones. The first requires a non-linear relation between stress and strain, the second, a quadratic relation between stresses.

For simplicity, and to keep the results somewhat independent of a particular fracture criterion, ongoing 2D rapid crack propagation is treated. Specifically, a crack and its zones expand symmetrically under uniform tension in an unbounded, linearly elastic, isotropic solid. Moreover, crack and zone propagation rates are constant and subcritical. This allows two limit cases: the Broberg[6] brittle crack and the Dugdale[12] cracks of Brock[7] and Embley and Sih[13].

The analysis begins in the next section with a formal problem statement that, by slight variations, can be made to accomodate either the non-linear stress-strain or stress quadratic case. Both of these cases impose a non-linear boundary condition. Nevertheless, solution forms can be constructed analytically by standard linear dynamic problem methods that, for each case, reduce the problem to an inhomogeneous non-linear integral equation. Multiple solution candidates to the equations can be obtained analytically, giving rise to the question of uniqueness. In each case, however, one candidate involves a uniform stress field and, in this first-step analysis, is the one studied in more detail.

2. Problem Formulation

For $s \leq 0$, where s = (time) × (dilatational wave speed), consider an unbounded elastic half-plane in equilibrium under a uniform tension $\sigma_0 > 0$ applied in the y-direction, where (x, y) are the Cartesian coordinates defining the plane. For $s > 0$ a crack grows from $(x, y) = 0$ in both directions along the x-axis with a constant, subcritical speed. Simultaneously, vanishingly thin cohesive zones grow ahead of the crack edges. The zone tractions satisfy either a non-linear stress-strain law or a quadratic condition on stresses.

This process is illustrated schematically in Fig. 1, where the circles of radius $(s, s/m)$ represent the fronts of, respectively, the associated cylindrical dilatational and rotational waves. The dimensionless constant m is the ratio of dilatational to rotational speed and in order that Poisson's ratio remain positive, it must exceed the value $\sqrt{2}$. The dimensionless constants (c_0, c) are the crack and zone speeds non-dimensionalized with respect to the dilatational wave speed. The restriction to subcritical speeds requires that

$$c_0 < c < c_R < 1/m \qquad \ldots (2.1)$$

where c_R is the non-dimensionalized Rayleigh wave speed.

This problem is symmetric with respect to the x-axis (fracture path), so that attention can be focused on the half-plane $y \geq 0$. Here

$$u = u_0 \ (s \leq 0) \qquad \ldots (2.2)$$

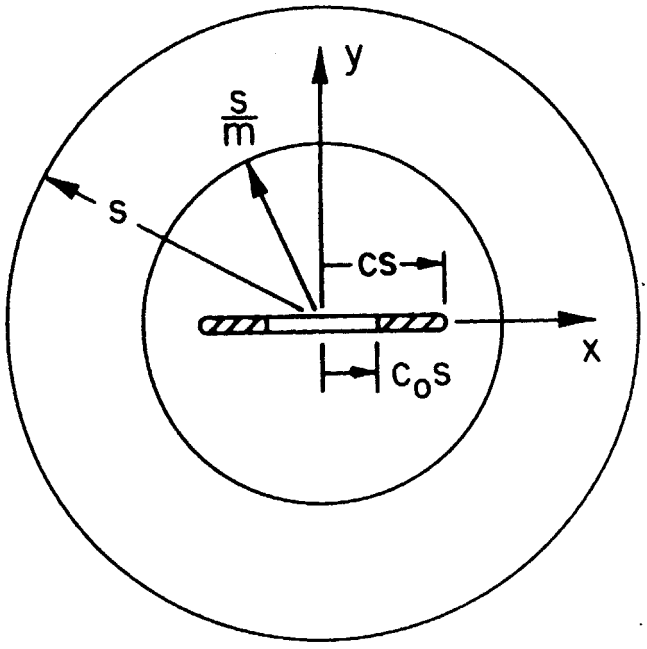

FIG. 1. Crack/zone growth process and associated wave pattern.

where $\boldsymbol{u}(x, y, s)$ is the displacement vector with (x, y)-components (u_x, u_y), while for $s > 0$

$$\nabla^2 \boldsymbol{u} + (m^2 - 1)\,\nabla(\nabla \cdot \boldsymbol{u}) - m^2 \boldsymbol{u}_{,ss} = 0 \qquad \text{... (2.3a)}$$

$$\frac{1}{\mu}\sigma_{xy} = u_{x,y} + u_{y,x},\ \frac{1}{\mu}(\sigma_x, \sigma_y) = (m^2 - 2)(\varepsilon_y, \varepsilon_x) + m^2(\varepsilon_x, \varepsilon_y) \qquad \text{... (2.3b)}$$

$$\varepsilon_x = u_{x,x},\ \varepsilon_y = u_{y,y}. \qquad \text{... (2.3c)}$$

Here, $\boldsymbol{u}_0$ is the solution field due to σ_0, $(\sigma_x, \sigma_y, \sigma_{xy})$ are the tractions and $(\varepsilon_x, \varepsilon_y)$, the normal strains. The operators (∇^2, ∇) are the 2D Laplacian and gradient, $(\ )_{,i}$ denotes i-differentiation, and the constant μ is the shear modulus. Along $y = 0$ the conditions

$$\sigma_{xy} = 0;\ u_y = 0\ (|x| > cs),\ \ \sigma_y = 0\ (|x| < c_0 s) \qquad \text{... (2.4a, b)}$$

hold for $s > 0$. For the non-linear stress-strain case the relation

$$E\varepsilon_y = \sigma_y \left[1 + k\left(\frac{\sigma_y}{\sigma}\right)^{p-1} \right] \quad (p > 1) \qquad \text{... (2.5a)}$$

holds for $y = 0$, $c_0 s < |x| < cs$, $s > 0$, while for the stress quadratic case we have, instead,

$$\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y = \sigma^2. \qquad \text{... (2.5b)}$$

In (2.5) the reference stresses (E, σ) and the dimensionless constants (k, p) characterize the zones. Equation (2.5a) is similar in form to a Ramberg-Osgood[16] law for simple tension, while (2.5b) resembles a von Mises multiaxial yield law[14]. Thus, (2.5a) might be used to represent crack bridging by ligaments in a fibrous material while Becker and Gross[3] have, in fact, used a rule like (2.5b) to represent von Mises-governed crack edges in equilibrium. Finally, we expect $\boldsymbol{u}$ to be continuous everywhere in $y > 0$, and that

$$\boldsymbol{u} = \boldsymbol{u}_0, \ \sqrt{(x^2 + y^2)} \to \infty \qquad \text{... (2.6)}$$

This problem contains the output parameters (c, c_0). Predicting them, as well as determining whether or not a given σ_0 is sufficient to cause fracture, requires that, in general three additional conditions be imposed upon the solution. For purposes of illustrating the effects of the two zone cases, however, a complete condition set is not necessary. Therefore, we impose, as one condition, the necessary requirement that stresses be finite at the zone leading edges $y = 0$, $x = cs^+$. For the second condition, we assume that outright material separation at the crack edges (zone trailing edges) implies that

$$\varepsilon_y = \varepsilon_c > 0 \quad (y = 0, \ |x| = c_0 s^+) \qquad \text{... (2.7)}$$

where ε_c, the strain value that occurs at the crack edges, proves useful as a parameter.

The non-linearity of (2.5a, b) may not guarantee a unique solution to the mathematical problem itself. This possibility will be addressed directly in terms of specific solution candidates. Construction of these begins in the next section.

3. Solution Scheme

Consider the superposition

$$\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{u}_F$$

where $\boldsymbol{u}_F$ also satisfies (2.3) and (2.4a), but (2.2) and (2.6) are replaced by

$$\boldsymbol{u} = 0 \ (y > 0, \ s \leq 0), \quad \boldsymbol{u} = 0 \quad (\sqrt{(x^2 + y^2)} \to \infty, \ s > 0) \qquad \text{... (3.2a, b)}$$

respectively, (2.4b) is replaced by

$$\sigma_y = -\sigma_0 \quad (y = 0, \ |x| < c_0 s) \qquad \text{... (3.3)}$$

and (2.5a, b) are both replaced by the single condition

$$\sigma_y = \mu F\left(\frac{x}{s}\right) - \sigma_0 \quad (y = 0,\ c_0 s < |x| < cs). \qquad \ldots (3.4)$$

Here $F(x/s) = F(-x/s)$ and is continuous and bounded for $c_0 < |x/s| < c$, but is otherwise largely arbitrary. In addition, the stresses due to u_F are bounded at $y = 0$, $x = cs^+$, while at $y = 0$, $x = c_0 s^+$, we have

$$\varepsilon_y = \varepsilon_c - (\varepsilon_y)_0\,,\ (\varepsilon_y)_0 = \sigma_0/\mu m^2 \qquad \ldots (3.5)$$

where $(\varepsilon_y)_0$ is the normal strain due to the tension field u_0.

Clearly, (3.1) will satisfy all the conditions on the two original problems except (2.5a) or (2.5b). Those can, it is hoped, be satisfied by choosing the correct function F. Its even-ness in x stems from the symmetry of the original problem about the origin. Its dependence on x/s follows from the observation that the original problems exhibit no characteristic length, and that the only non-homogeneous conditions on u_F are on its gradients. Therefore, all gradients of u_F should be homogeneous of degree zero in (x, y, s) (Brock[8]).

Because the solution u_0 is merely that for simple tension, the key task at hand is construction of u_F. This is accomplished in the next section in terms of the solution to a yet simpler problem.

4. Construction of F-Solution

Consider the same half-plane $y > 0$, with a continuous displacement field u_f satisfying (2.3) and (3.2) but, instead of (2.4), the unmixed conditions

$$\sigma_{xy} = 0;\ u_y = 0\ (|x| > cs),\ u_y = \int_{x/s}^{c} f(v)\left(s - \frac{x}{v}\right) dv \quad (|x| < cs) \quad \ldots (4.1a, b)$$

for all $y = 0$, $s > 0$. Here $f(v)$ is an arbitrary function that, for continuity of u_f at $y = 0$, $|x| = cs$ must satisfy the auxiliary conditions

$$\int_{-c}^{c} f(v)\left(1, \frac{1}{v}\right) dv = 0. \qquad \ldots (4.2)$$

The form of u_y for $|x| < cs$ and the lack of a characteristic length in the formulation indicate that u_f will, like u_F, be homogeneous of degree zero in (x, y, s).

This problem, with the unmixed conditions along $y = 0$, is readily solved by transform methods in the Appendix. In particular, we find that $(\sigma_y, \varepsilon_y)$ along $y = 0$, $x > 0$ for $s > 0$ are given by

$$\frac{\pi}{\mu} m^2 \sigma_y = \int_{-c}^{c} v f(v)\, dv \left(\int_{\xi}^{1} \frac{K^2}{a} - \int_{\xi}^{1/m} 4b \right) \frac{du}{u^4 (v - u)} \qquad \ldots (4.3a)$$

$$\pi m^2 \varepsilon_y = -\int_{-c}^{c} vf(v)\,dv \left(\int_{\xi}^{1} Ka + \int_{\xi}^{1/m} 2b \right) \frac{du}{u^4 (v-u)} \qquad \text{... (4.3b)}$$

where $\xi = x/s$ and

$$a = \sqrt{(1-u^2)},\ b = \sqrt{(1-m^2 u^2)}\ ,\ K = m^2 u^2 - 2. \qquad \text{... (4.4)}$$

For $y = 0$, $x < 0$ the u-integration limits are interchanged, and $(1, 1/m)$ replaced by their negatives.

Comparison of the conditions on (u_F, u_f) shows that the former can be obtained from the latter if the function f can be found so that (3.3)–(3.5) are satisfied, and that the resulting u_F exhibit finite stresses at $y = 0$, $| x | = cs$. Imposing (3.3)–(3.5) upon (4.3) gives, upon some rearrangement of terms,

$$-\int_{c_0}^{c} \frac{M}{u^4} I(u)\,du - \left(\int_{c}^{1} Ka + \int_{c}^{1/m} 2b \right) I(u) \frac{du}{u^4} = \pi \left(m^2 \varepsilon_c - \frac{\sigma_0}{\mu} \right) \qquad \text{... (4.5a)}$$

$$-\int_{\xi}^{c} \frac{R}{u^4 a} I(u)\,du + \left(\int_{c}^{1} \frac{K^2}{a} - \int_{c}^{1/m} 4b \right) I(u) \frac{du}{u^4} = -\frac{\pi}{\mu} m^2 \sigma_0 \qquad \text{... (4.5b)}$$

$$-\int_{\xi}^{c} \frac{R}{u^4 a} I(u)\,du + \left(\int_{c}^{1} \frac{K^2}{a} - \int_{c}^{1/m} 4b \right) I(u) \frac{du}{u^4} = \pi m^2 \left[F(\xi) - \frac{\sigma_0}{\mu} \right] \qquad \text{... (4.5c)}$$

for $(\xi = c_0^+,\ 0 < \xi < c_0,\ c_0 < \xi < c)$, respectively. Similar equations hold for $(\xi = -c_0^-, -c_0 < \xi < 0, -c < \xi < -c_0)$. In (4.5)

$$I(u) = \int_{-c}^{c} \frac{vf(v)}{v-u}\,dv \qquad \text{... (4.6)}$$

where the integration must be taken in the Cauchy principal value sense for the first terms in (4.5) because $0 < u < c$, while

$$M = 2b + Ka, \qquad R = 4ab - K^2. \qquad \text{... (4.7)}$$

Here R is the Rayleigh function with branch cuts

$$\text{Im } (u) = 0,\ 1/m < | \text{ Re } (u) | < 1,$$

and zeros at the origin and at $u = \pm c_R$.

The term M shares the same branch cuts, and vanishes both at the origin and in the range $| u | < 1/m$. However, the latter instance does not seem to be related to a wave speed, e.g. c_R.

The second terms on the left-hand sides of (4.5b, c) are constants, so that a necessary condition for the satisfaction of these equations would in view of (4.6) be

$$I(u) = 0 \quad (0 < u < c_0), \quad I(u) = \pi m^2 \frac{u^4 a}{R} F' \quad (c_0 < u < c) \qquad \ldots (4.8)$$

where $(\,)'$ denotes derivation with respect to the argument. It is easily shown that the same conditions can be obtained for $(-c_0 < u < 0, -c < u < -c_0)$, respectively. Therefore, (4.8) is a Cauchy singular integral equation for the function $vf(v)$ in the interval $|v| < c$. A solution to this equation that will preclude stress singularities at $y = 0$, $x = cs$, $(\xi = c)$ can be obtained as (Carrier *et al.*[10])

$$f(v) = \frac{-m^2 v}{\pi\sqrt{(c^2 - v^2)}} \left(\int_{-c}^{-c_0} + \int_{c_0}^{c} \right) F' tQ \frac{dt}{t - v} + \frac{f_0 v}{(c_0^2 - v^2)\sqrt{(c^2 - v^2)}} \quad (|v| < c) \qquad \ldots (4.9a)$$

$$Q(t) = \frac{t^2}{R}\sqrt{(c^2 - t^2)}\sqrt{(1 - t^2)} \qquad \ldots (4.9b)$$

where f_0 is an arbitrary constant and Cauchy principal value integration is understood.

Substitution of (4.9) into (4.6) gives for $|u| > c$ integrations that can be performed exactly by Cauchy residue theory :

$$I(u) = \frac{u^2}{\sqrt{(u^2 - c^2)}} \left[-\frac{\pi f_0}{c_0^2 - u^2} + m^2 \left(\int_{-c}^{-c_0} + \int_{c_0}^{c} \right) F' Q \frac{dt}{t - u} \right] = I_+. \qquad \ldots (4.10)$$

Here the evenness of $F(t)$ eliminates upon integration over t a u-independent contribution of the residue theory application.

The function $I(u)$ is, of course, governed by (4.8) when $|u| < c$. Therefore, substitution of (4.8) and (4.10) into (4.5a-c) gives, respectively,

$$-\int_{c_0}^{c} \frac{aM}{R} F' du - \frac{1}{\pi m^2} \left(\int_{c}^{1} Ka + \int_{c}^{1/m} 2b \right) I_+ \frac{du}{u^4} = \varepsilon_c - \frac{\sigma_0}{\mu m^2} \qquad \ldots (4.11a)$$

$$F(c_0) - F(c) + \frac{1}{\pi m^2} \left(\int_{c}^{1} \frac{K^2}{a} - \int_{c}^{1/m} 4b \right) I_+ \frac{du}{u^4} = -\frac{\sigma_0}{\mu} \qquad \ldots (4.11b)$$

$$-F(c) + \frac{1}{\pi m^2} \left(\int_{c}^{1} \frac{K^2}{a} - \int_{c}^{1/m} 4b \right) I_+ \frac{du}{u^4} = -\frac{\sigma_0}{\mu}. \qquad \ldots (4.11c)$$

Identical formulas arise when results for $x < 0$ are used. If the function F were known, then (4.11) would provide three equations for the unknown parameters (f_0, c, c_0). Solving those equations can be made easier by transforming the u-integrations involving the integrand $I_+(u)$ onto paths along the $\mathrm{Im}(u)$-axis. This entails dropping the principal value interpretation of I_+ and accounting for poles at $u = c_0$ and $u = t$ $(c_0 < t < c)$, but is not difficult. The more computationally efficient result is then

$$-\int_{c_0}^{c} \frac{aM}{R} F'\, du - \frac{f_0}{m^2}\left[\frac{\pi M(c_0)}{2c_0^3\sqrt{(c^2-c_0^2)}} + I_m(c_0)\right] - \int_{c_0}^{c} I_M(t)\, F'Q\, dt$$

$$= \varepsilon_c - \frac{\sigma_0}{\mu m^2} \qquad \ldots (4.12a)$$

$$F(c_0) - F(c) - \frac{f_0}{m^2} I_R(c_0) - \int_{c_0}^{c} I_R(t)\, F'Q\, dt = -\frac{\sigma_0}{\mu} \qquad \ldots (4.12b)$$

$$-F(c) - \frac{f_0}{m^2}\left[\frac{\pi}{2c_0 Q(c_0)} + I_R(c_0)\right] - \int_{c_0}^{c} I_R(t)\, F'Q\, dt = \frac{-\sigma_0}{\mu} \qquad \ldots (4.12c)$$

where

$$[I_M(t), I_R(t)] = \frac{2}{\pi}\int_0^{\infty} \frac{dv}{v^2(v^2+t^2)\sqrt{(v^2+c^2)}}\left[M(iv),\ \frac{R(iv)}{\sqrt{(1+v^2)}}\right]. \qquad \ldots (4.13)$$

It should be noted that the residues due to the poles at $u = t$ $(c_0 < t < c)$ do not appear, the resulting t-integrations having vanished because $F(t)$ is an odd function.

Any one equation of the set (4.12) can be chosen to eliminate the unknown constant f_0. However, merely taking the difference of (4.12b) and (4.12c) gives the compact result

$$\frac{-\pi f_0}{2m^2 c_0 Q(c_0)} = F(c_0). \qquad \ldots (4.14)$$

Now, any pair of the set (4.12) act as formulas for (c, c_0).

Equation (4.14) is, in effect, a jump condition for the normal traction at the crack edge/material separation points ($y = 0$, $x = c_0 s$), between the F-field in the zones and the zero-stress state on the crack surfaces.

The construction of $\boldsymbol{u}_F$ is now complete. In the next section, we attempt to obtain a particular function F so that (3.1) represents the solution to the original non-linear stress-strain zone problem. The stress quadratic zone problem is handled in section 6.

5. Original Problem : Non-Linear Stress-Strain

Clearly, the field (3.1) will satisfy the original problem if a stress function $F(x/s)$ can be found such that (2.5a) is satisfied. The $\boldsymbol{u}_0$-contributions in (3.1) to $(\varepsilon_y, \sigma_y)$ in (2.5a) are easily shown to be

$$\varepsilon_y = \frac{\sigma_0}{\mu m^2}, \quad \sigma_y = \sigma_0. \qquad \text{... (5.1a, b)}$$

From (4.3b, c), (4.9) and (4.11), the corresponding u_F-contributions for $c_0 < \xi < c$ follow immediately as

$$\varepsilon_y = -\int_{\xi}^{c} \frac{aM}{R} F'\,du - \frac{1}{\pi m^2}\left(\int_{c}^{1} Ka + \int_{c}^{1/m} 2b\right) I_+ \frac{du}{u^4} \qquad \text{... (5.2a)}$$

$$\frac{1}{\mu}\sigma_y = F(\xi) - F(c) + \frac{1}{\pi m^2}\left(\int_{c}^{1} \frac{K^2}{a} - \int_{c}^{1/m} 4b\right) I_+ \frac{du}{u^4}. \qquad \text{... (5.2b)}$$

One can write

$$\int_{\xi}^{c} \frac{aM}{R} F'\,du = \int_{c_0}^{c} \frac{aM}{R} F'\,du - \int_{c_0}^{\xi} \frac{aM}{R} F'\,du \qquad \text{... (5.3)}$$

so that, in view of (4.12a, b), (5.2a, b) become, respectively,

$$\varepsilon_y = \int_{c_0}^{\xi} \frac{aM}{R} F'\,du + \varepsilon_c - \frac{\sigma_0}{\mu m^2} \qquad \text{... (5.4a)}$$

$$\sigma_y = \mu F(\xi) - \sigma_0. \qquad \text{... (5.4b)}$$

Substitution of (5.1) and (5.4) into (2.5a) in light of (3.1) and the relation

$$e = E/\mu \qquad \text{... (5.5)}$$

gives, upon division by μ,

$$e\left(\varepsilon_c + \int_{c_0}^{\xi} a\frac{M}{R} F'\,du\right) = F + k\left(\frac{\mu}{\sigma}\right)^{p-1} F^p \quad (c_0 < \xi < c,\ p > 1). \qquad \text{... (5.6)}$$

Because $F(\xi) = F(-\xi)$, (5.6) is the governing, inhomogeneous non-linear integral equation for $F(\xi)$, and yields the relation

$$e\varepsilon_c = F(c_0) + k\left(\frac{\mu}{\sigma}\right)^{p-1} F(c_0)^p \qquad \text{... (5.7)}$$

for the values of F at the crack edges ($y = 0$, $|x| = c_0 s^+$).

Differentiation of (5.6) produces

$$F'\left[eX - 1 - kp\left(\frac{\mu}{\sigma}F\right)^{p-1}\right] = 0, \quad X = a\frac{M}{R} \qquad \text{... (5.8a, b)}$$

which defines two candidate sets for the solution of (5.6). The first— and simplest— is F = constant ($F' = 0$). The second is the set of solutions to the equation

$$kp\left(\frac{\mu}{\sigma}F\right)^{p-1} = eX - 1. \qquad \ldots (5.9)$$

As noted at the outset, attention will focus on the first candidate — uniform stress. First, however, an examination of (5.9) is of value : If we assume that $eX > 1$ ($c_0 < \xi < c$), then (5.9) gives the positive, real solution

$$F = \frac{\sigma}{\mu}\left(\frac{eX-1}{kp}\right)^{\delta}, \quad \delta = \frac{1}{p-1} \qquad \ldots (5.10)$$

for $c_0 < \xi < c$. Differentiation of (5.10) and substitution into (5.7) gives an integration that is readily performed analytically. Simple algebra in view of (5.7) then shows that (5.6) is formally satisfied. Therefore, so long as (5.7) holds, (5.10) represents a solution to (5.6).

If, however, $eX < 1$ ($c_0 < \xi < c$), then (5.10) yields real values only for even values of p, and those values are negative. This, in turn, requires that $\varepsilon_c < 0$ in (5.7). At this point, therefore, it is worthwhile to examine the function $eX - 1$. For convenience, we write

$$eX - 1 = -\frac{R_e}{R}, \quad R_e = R - eMa. \qquad \ldots (5.11)$$

A study for all $m > \sqrt{2}$ shows that $R_e(\xi)$ has the same branch cuts as R, but real zeros in the cut ξ-plane at the origin and as follows:

$$e < \frac{m^2}{m^2-1} : R_e(\pm c_e) = 0 \quad (c_R < c_e < 1/m) \qquad \ldots (5.12a)$$

$$m^2 < e < 2(m^2-1) : R_e(\pm c_e) = 0 \quad (c_e > 1) \qquad \ldots (5.12b)$$

$$e > 2(m^2-1) : R_e(\pm c_e^+),\ R_e(\pm c_e^-) = 0 \quad (c_e^- < c_R < 1/m,\ c_e^+ > 1) \ldots (5.12c)$$

It can also be shown that $R \geq 0$ for $|\xi| < c_R$, while

$$e < \frac{m^2}{m^2-1} : R_e \geq 0\ (|\xi| < c_e),\ R_e \leq 0\ (c_e < |\xi| < c_R) \qquad \ldots (5.13a)$$

$$\frac{m^2}{m^2-1} < e < 2(m^2-1) : R_e \geq 0 \quad (|\xi| < 1/m) \qquad \ldots (5.13b)$$

$$e > 2(m^2-1) : R_e \leq 0 \quad (|\xi| < c_e^-),\ R_e \geq 0\ (c_e^- < |\xi| < c_R). \qquad \ldots (5.13c)$$

Conditions (5.12a) and (5.13a, b) show that $eX > 1$ for $c_R < |\xi|$, while (5.12b) and (5.14c) show that $eX > 1$ for $0 < |\xi| < c_e^- < c_R$. However, if the stress parameter E in (2.5a) is taken to be the same as Young's modulus in the region outside the strip zone, then we must have

$$e = \frac{3m^2 - 4}{m^2 - 1} \qquad \ldots (5.14)$$

so that for all $m > \sqrt{2}$,

$$\frac{m^2}{m^2 - 1} < e < m^2 < 2(m^2 - 1). \qquad \ldots (5.15)$$

Thus, the zone defined by (5.10) requires that its material be stiffer for small ($|\sigma_y| << \sigma$) stresses than the surrounding material in order for the crack/zone to grow at subcritical speeds. Moreover, the actual allowable speeds must then be less than the Rayleigh wave value noted in brittle fracture (e.g. Broberg[6]).

At this point, we return to the simple uniform stress case :

For $F = F(c_0)$ = constant, (5.6) reduces to (5.7), and (4.12) and (4.14) produce the simple formulas

$$\frac{-\pi f_0}{2m^2 c_0 Q(c_0)} = F \qquad \ldots (5.16a)$$

$$-\frac{f_0}{m^2} I_R(c_0) = \frac{\sigma_0}{\mu} \qquad \ldots (5.16b)$$

$$-\frac{f_0}{m^2}\left[\frac{\pi M(c_0)}{2m^2 c_0^3 \sqrt{(c^2 - c_0^2)}} + I_M(c_0)\right] = \varepsilon_c - \frac{\sigma_0}{\mu m^2} \qquad \ldots (5.16c)$$

for (f_0, c, c_0). Solving (5.16) for $(\sigma_0/\mu, \varepsilon_c)$ in terms of F yields

$$\frac{\sigma_0}{\mu} = -\frac{2}{\pi} c_0 Q(c_0) I_R(c_0) F \qquad \ldots (5.17a)$$

$$\varepsilon_c = X_0 F, \quad X_0 = \frac{M(c_0)}{R(c_0)}\sqrt{(1 - c_0^2)} + \frac{2}{\pi} c_0 Q(c_0)\left[I_M(c_0) - \frac{I_R(c_0)}{m^2}\right] \qquad \ldots (5.17b)$$

which, upon substitution in (5.17), gives

$$eX_0 F = F + k\left(\frac{\mu}{\sigma}\right)^{p-1} F^p. \qquad \ldots (5.18)$$

Discarding the trivial solution, (5.18) can be solved to yield

$$F = \frac{\sigma}{\mu}\left(\frac{eX_0 - 1}{k}\right)^{\delta}. \qquad \ldots (5.19)$$

From (5.11) and (5.17b) it follows that

$$eX_0 - 1 = -\frac{R_e(c_0)}{R(c_0)} + \frac{2}{\pi} c_0 Q(c_0)\left[I_M(c_0) - \frac{I_R(c_0)}{m^2}\right]. \qquad \ldots (5.20)$$

The first term in (5.20) has been discussed above, and a study of (4.13) indicates that the second term is positive for $0 < c_0 < c < c_R$, $m > \sqrt{2}$. Therefore, for $c_0 < \xi < c$, $eX_0 - 1 > eX - 1$. However, the same study indicates that $eX_0 < 1$ when the term E in (2.5a) is forced to satisfy (5.14) — and, thus, (5.15) — white $eX_0 > 1$ for $0 < c_0 < c < c_e^-$ when $e \geq 2(m^2 - 1)$. This zero c_e^- exceeds its $eX - 1$ counterpart and, indeed, one can have $c_R < c_e^- < 1/m$. Therefore, while the zone model defined by (5.19) also requires a larger stiffness in the zone for small stresses, it allows larger crack/zone speeds than the (5.10)-based model.

Specifically, of course, these speeds and the value of σ_0 required for the process will be those values that allow $eX_0 > 1$, and satisfy (5.17), (5.19) and the third condition which, as noted at the outset, has been left unspecified. As also noted earlier, the major goal of this article is to determine the effect of non-linear zone response on dynamic fracture, and a simple means of doing this in the absence of a complete fracture criterion is a direct comparison with a standard model of the loads σ_0 and crack edge strains ε_c associated with a given (c, c_0). The standard Dugdale[12] model is convenient for purposes of comparison because its solution can be obtained from (3.1) merely by replacing μF with $\sigma = \sigma_Y$ (c.f. Atkinson[1], Brock[7] Embley and Sih[13]), where σ_Y is the yield stress. It is then easily shown that the ratios of the present $(\varepsilon_c, \sigma_0)$ to the standard values are

$$\frac{(\varepsilon_c)_{NL}}{(\varepsilon_c)_s} = \frac{(\sigma_0)_{NL}}{(\sigma_0)_s} = \left(\frac{eX_0 - 1}{k}\right)^{\delta}. \qquad \ldots (5.21)$$

In Fig. 2, the ratio (5.21) is plotted versus c_0 for various sub-critical values of $c > c_0$, when $e = 2(m^2 - 1)$. Two materials are considered (Boyer and Gall[5]) a 2024-T4 aluminum with the properties

$$\mu = 27.44\ Gpa,\ m = 1.98,\ k = 1.98,\ p = 5.37 \qquad \ldots (5.22a)$$

and a 4130-HT steel defined by

$$\mu = 79.29\ GPa,\ m = 1.87,\ k = 1.07,\ p = 7.94. \qquad \ldots (5.22b)$$

The (k, p)-parameters in (5.24) are obtained by setting $e = 2(m^2 - 1)$ and then fitting (2.5a) to the simple tension data

$$\sigma = \sigma_Y = 0.379\ GPa,\ \varepsilon_Y = 0.007;\ \sigma = 0.483\ GPa,\ \varepsilon = 0.02 \quad \ldots (5.23a)$$

for 2024-T4 aluminum, and the corresponding data

$$\sigma = \sigma_Y = 1.344\ GPa,\ \varepsilon_Y = 0.007;\ \sigma = 1.62\ GPa,\ \varepsilon = 0.02 \ldots (5.23b)$$

for 4130-HT steel. The subscript Y denotes that the 0.2% yield point data has been taken to define the onset of permanent set. The zero c_R of R can be obtained exactly in terms of the parameter m from the formula[9]

$$c_R = \sqrt{[2(m^2-1)]}\,\frac{G(0)}{m^2},\ lnG(0) = -\frac{1}{\pi}\int_1^m \frac{dt}{t}\,\tan^{-1}\frac{4t^2\sqrt{(t^2-1)}\,\sqrt{(m^2-t^2)}}{(m^2-2t^2)^2} \quad \ldots (5.24)$$

as

$$c_R = 0.469,\ 0.496 \quad \ldots (5.25a, b)$$

for the aluminum and steel, respectively.

Figure 2 shows that the ratio (5.21) is generally near unity — except at high crack/zone rates ($c,\ c_0 \to c_R$). Thus, the non-linear model predicts roughly the same critical load σ_0, and the same strain ε_c at the crack edge, for a given set of crack/zone growth parameters ($c,\ c_0$). The similarity is interesting in view of the data presented in Fig. 3. There, some typical ε_c calculations from the present analysis are plotted versus c_0 for various values of $c > c_0$ for the same two materials. The plots denoted by "NL" show that ε_c possible for $0 < c_0 < c < c_R$ is less than ε_Y. That is, the model involves crack edge strains that are below values normally deemed to be critical for yield. For the standard Dugdale[12] zone, this behaviour is simply a consequence of a rigid-perfectly plastic model. For the present model, however, it implies that the zone has not yielded.

At this point we consider the stress quadratic zone.

6. Original Problem : Stress Quadratic

The relations (5.1) and (5.4) again hold, but (2.5b) requires a knowledge of σ_x for $y = 0,\ c_0 < |\xi| < c$. From (2.3b,c) it follows that

$$(m^2-2)\,\sigma_x = m^2\,\sigma_y - 4\,(m^2-1)\,\mu\varepsilon_y. \quad \ldots (6.1)$$

Therefore, from (5.1), (5.4) and (3.1), (2.5b) can be rewritten as the inhomogeneous, non-linear integral equation

$$F^2 + \left(\varepsilon_0 - \int_{c_0}^{\xi} XF'\,du\right)^2 - \left(\varepsilon_0 - \int_{c_0}^{\xi} XF'\,du\right)F = \left(\frac{\sigma}{\mu}\right)^2 \quad (c_0 < \xi < c) \quad \ldots (6.2)$$

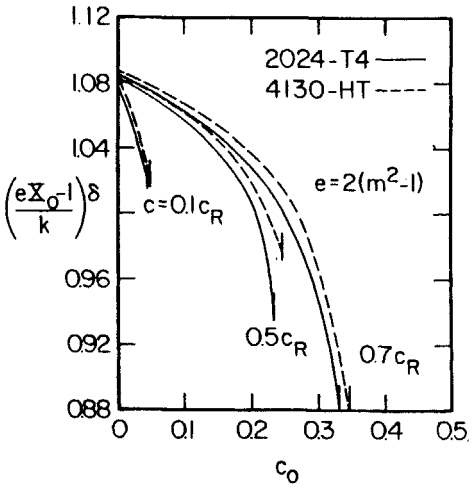

FIG. 2. Ratio of non-linear stress-strain to standard parameters versus non-dimensionalized crack speed.

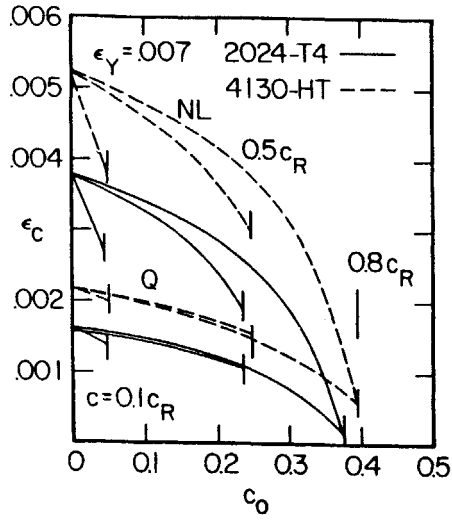

FIG. 3. Crack edge strain versus non-dimensionalized crack speed.

where

$$(m^2-2)\varepsilon_0 = m^2 F(c_0) - 4(m^2-1)\,\varepsilon_c, \quad X = \frac{R_x}{R}, \qquad R_x = 4ab + KN, \quad N = m^2u^2 + 2a^2. \qquad \text{... (6.3)}$$

For $\xi = c_0^{\downarrow}$, (6.2) gives the important relation

$$F^2(c_0) + \varepsilon_0^2 - \varepsilon_0 F(c_0) = \left(\frac{\sigma}{\mu}\right)^2. \qquad \text{... (6.4)}$$

The function R_x is the σ_x-counterpart to the Rayleigh function R associated with σ_y, and has the same branch cuts and zero at the origin. However, R_x has no real zeros elsewhere in the cut u-plane and, in fact, $R_x < 0$, $0 < |u| < 1/m$. Differentiation of (6.2) produces

$$F'\left[(2+X)F - \left(\varepsilon_0 - \int_{c_0}^{\xi} XF'\,du\right)(1+2X)\right] = 0 \qquad \text{... (6.5)}$$

which defines two candidate sets for the solution of (6.2). The more complicated set follows as the solutions to the equation

$$\varepsilon_0 - \int_{c_0}^{\xi} XF'\,du - \Omega F = 0, \quad \Omega = \frac{2+X}{1+2X}. \qquad \text{... (6.6a, b)}$$

Substitution of (6.6) into (6.2) leads to the simple result

$$F = \frac{\sigma}{\mu} \frac{1}{\sqrt{(1 + \Omega^2 - \Omega)}} \quad (c_0 < \xi < c). \qquad \ldots (6.7)$$

Here the (+) sign has been taken in a square-root operation, because it is expected that the zone will be in tension. Differentiation of (6.7) and use of (6.6b) shows that (6.6a) is indeed satisfied so long as (6.4) holds. Study of (R_x, R) indicates that

$$-1 < \Omega < 0.5 \quad (0 < |u| < c_R) \qquad \ldots (6.8)$$

so that (6.7) will give real solutions. Therefore, the difficulties immediately noted for the non-uniform solution set to (5.7) studied earlier do not arise here.

However, in keeping with the aim of the present article, we abandon for the moment further study of (6.7) and pursue the uniform stress case $F' = 0$. Then, (6.2) reduces to (6.4) and (5.17) again hold. Substitution of (5.17) into (6.4) allows the solution

$$F = \frac{\sigma}{\mu} \frac{1}{\sqrt{(1 + \Omega_0^2 - \Omega_0)}}, \quad \Omega_0 = X(c_0) + \frac{2}{\pi} c_0 \Omega(c_0) I_x(c_0) \qquad \ldots (6.9a, b)$$

where $I_x(t)$ follows from $I_R(t)$ upon replacement of R(iv) by R_x(iv). Study of Ω_0 for $(c, c_0) < c_R$ indicates that (6.9a) will produce real values, so that (6.9), (5.17) and (3.1) can be taken as a valid solution to (6.2).

Again, a simple means of determining the effects of the non-linear zone response is to compare directly the $(\sigma_0, \varepsilon_c)$ for the present solution with the standard Dugdale[12] model, i.e. $(\sigma_y = \sigma)$. In view of (6.4), (6.9) and results by Brock[7] and Embley and Sih[13], it is easily shown that the ratios of the present $(\varepsilon_c, \sigma_0)$ to the standard values are

$$\frac{(\varepsilon_c)_Q}{(\varepsilon_c)_s} = \frac{(\sigma_0)_Q}{(\sigma_0)_s} = \frac{1}{\sqrt{(1 + \Omega_0^2 - \Omega_0)}}. \qquad \ldots (6.10)$$

This parameter is graphed in Fig. 4 vs. c_0 for subcritical values of $c > c_0$ for subcritical values of $c > c_0$ in the case of, respectively, 2024-T4 aluminum and 4130-HT steel (Boyer and Gall[5]). For the former, we take, cf. (5.22) and (5.23)

$$\mu = 27.44\ GPa,\ m = 1.98,\ \sigma = \sigma_Y = 0.379\ GPa\ (\varepsilon_Y = 0.007) \quad \ldots (6.11a)$$

while, for the latter,

$$\mu = 79.29\ GPa,\ m = 1.87,\ \sigma = \sigma_Y = 1.344\ GPa\ (\varepsilon_Y = 0.007). \quad \ldots (6.11b)$$

Figure 4 shows, upon comparison with Fig. 2, a ratio that is even smaller. Some possible values of ε_c can be found in Fig. 3; there, the data denoted by "Q" is less than ε_Y, indicating again a consequence of a rigid-perfectly plastic model. In Fig. 5, we plot versus c_0 for various subcritical values of $c > c_0$ the ratios σ_0/σ obtained from (5.17a) and (6.9a) for 2024-T4 aluminum and 4130-HT steel. Figure 5 shows that this ratio increases with zone speed (c) and varies inversely with the net growth rate $(c - c_0)$ of the zone. However, the ratio is well below unity, especially as $(c, c_0 \to c_R)$. This merely indicates that a crack/zone growth process model needs a tensile load σ_0 far less than that required for permanent set in an uncracked material.

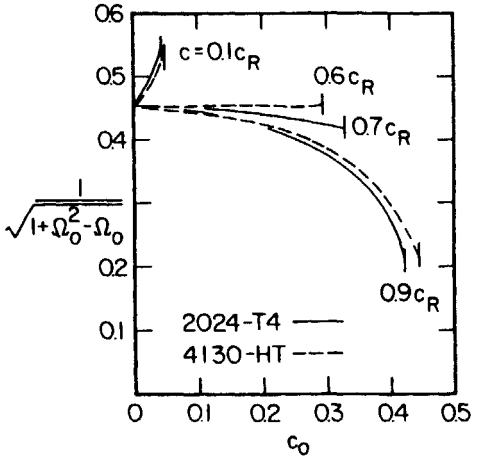

FIG. 4. Ratio of stress quadratic to standard parameters versus non-dimensionalized crack speed.

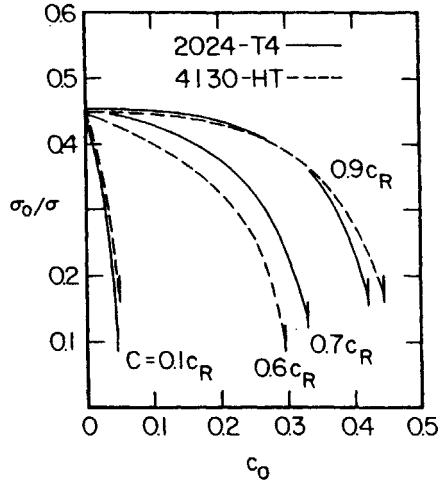

FIG. 5. Possible critical tension loads versus non-dimensionalized crack speed.

It should also be noted that the quasi-static analogy when there is no applied σ_x for this case reduces to the standard, e.g. $\mu F = \sigma = \sigma_Y$, model, because it can be shown[17], that, for the stresses generated by removal of the applied tractions, $\sigma_x = \sigma_y$ everywhere along $y = 0$. Whether by use of (6.1) or by derivations similar to those carried out in the Appendix, it can be shown for the dynamic problem that the σ_x-analogy to (4.3a,b) is

$$\frac{\pi}{\mu} m^2 \sigma_x = \int_{-c}^{c} vf(v)\, dv \left(\int_{\xi}^{1} \frac{KN}{a} + \int_{\xi}^{1/m} 4b \right) \frac{du}{u^4 (v-u)}, \quad \xi = \frac{x}{s}. \qquad \ldots (6.12)$$

Therefore, $\sigma_x \neq \sigma_y$ for $y = 0$, $s > 0$, and results obtained here imply that the multiaxiality of a yield criterion in the zone might affect rapid fracture more than slow crack growth, and that rapid fracture may proceed at lower tension levels.

7. Other Considerations

One criticism of the Dugdale[12] zone model is that it allows a logarithmic singularity at the crack edges — here $y = 0$, $| x | = c_0 s$ (Brock[7]). The presence of the pole-type term in the singular integral equation solution (4.9a) indicates the same problem here. The jump condition (4.14) shows that the singularity disappears only when the zone function F is continuous (vanishes) at the crack edge. This cannot occur for the uniform F-fields, but such continuity would seem possible for the other, non-uniform F-candidates also obtained earlier.

However, a check of (5.7) and (5.10) for the case (2.5a) shows that setting $F(c_0) = 0$ requires

$$\varepsilon_c = eX(c_0) - 1 = 0. \qquad \text{... (7.1a, b)}$$

The first condition implies that there can be no normal strain at the crack edge. The second condition implies that $c_0 = c_e^-$, and yet the previous study of $eX - 1$ showed that a positive, real F results from (5.10) only when $c_0 < c < c_e^-$.

For the case (2.5b), setting $F(c_0) = 0$ in view of (6.4) and (6.7) means that

$$\varepsilon_c = \frac{1}{4}\left(\frac{m^2-2}{m^2-1}\right)\frac{\sigma}{\mu}, \quad \Omega(c_0) \to \infty. \qquad \text{... (7.2a, b)}$$

Equation (7.2a) requires a specific normal strain at the crack edge, but (6.8) precludes (7.2b) for subcritical crack/zone growth rates.

Therefore, while the non-uniform stress solution candidates obtained here bear further study, they both also exhibit logarithmic singularities at the crack edge.

8. Brief Summary

This article examined the effects of rudimentary local non-linear behaviour in dynamic, quasi-brittle fracture by imposing non-linear responses in cohesive zones near dynamic 2D cracks growing under tension. Two cases were considered, and both imposed non- linear boundary conditions upon the otherwise linear elastodynamic mixed initial/boundary value problem.

An analytical approach based on standard solutions to purely linear problems reduced the actual problem for each case to a single, inhomogeneous, non-linear integral equation. The equation could, in each case, be solved exactly but, not surprisingly, for multiple solution candidates. Although all were examined, attention was here focused on those that involved a uniform stress.

Studies of these solutions for a 2024-T4 aluminum and a 4130-HT steel showed that, for a given set of crack and zone growth rates, both cases predicted roughly similar critical tension loads, although the non-linear stress strain case values were lower, especially at high crack/zone growth rates. Both cases predicted that the crack/zone growth process occurs for tensions smaller than that required for

permanent set in an uncracked material in simple tension. This effect is less noticeable for high zone speeds, but is enhanced when the instantaneous zone size is small in comparison to the instantaneous crack size. It was also noted that the stress quadratic case can collapse to a simple tension law for the analogous quasi-static problem, suggesting that rapid (dynamic) fracture is more sensitive to multiaxiality and proceeds at lower tension loads.

Both cases also exhibited the logarithmic singularity at the crack edge that also occurs for the standard rigid-perfectly plastic simple tension strip zone model. The other solution candidates, with their more robust behaviour, seemed to have potential for avoiding this. However, studies shows that this was not so — for subcritical crack/zone growth rates, at least. Both solution sets for the non-linear stress-strain case showed that zone-related characteristic speeds, analogous to the Rayleigh value, may exist, and, indeed, may be supersonic. The sub-sonic speeds apparently serve as bounds for the rates of crack/zone propagation. These speeds and the other solution candidates will be studied in future work.

The constant growth rates and lack of a characteristic length in the problems is a restriction on their application. On the other hand, the results obtained here do not depend on a complete set of crack/zone growth criteria. Besides this generality, moreover, the present results show that dynamic fracture is sensitive to localized non-linear effects. In particular, such effects seem to lower the load deemed to be critical for crack/zone growth.

ACKNOWLEDGEMENT

This research was carried out as part of the University of Kentucky NSF/EPSCoR project "Inverse problems and Quantitative Non-Destructive Evaluation." The support of NSF Grant MEA 9121700 is also acknowledged.

REFERENCES

1. C. Atkinson, *Arkiv for Fysik* **35** (1968), 469-76.
2. G. I. Barenblatt, in : *Advances in Applied Mechanics,* Vol. 17, eds. H. L. Dryden and T. V. Karman, 1962.
3. W. Becker and D. Gross, *Int. J. Fracture* **37** (1988)163-70.
4. W. Bohme, in: *Impact Loading and Dynamic Behaviour of Materials,* eds. C.Y. Chiem *et al.*, DGM, Oberursel, (1988).
5. H. E. Boyer and T. L. Gall, *Metals Handbook,* American Society for Metals, 1985.
6. K. B. Broberg, *Arkiv for Fysik* **18** (1960), 159-92
7. L. M. Brock, "Dynamic problems of crack propagation and extension of interface flaws," Ph.D. Dissertation, Northwestern University, Evanston, IL, 1972.
8. L. M. Brock, *Intern. J. Eng. Sci.* **15** (1977), 527-36.
9. L. M. Brock, *Intern. J. Solids & Structures* **29** (1992), 2217-34.
10. G. F. Carrier, M. Krook and C. E. Pearson, *Functions of a Complex Variable,* McGraw-Hill Book Co., Inc., New York, 1966.

11. V. G. DeGeorgi and L. M. Brock, *ASCE J. Engng Mech.* **116** (1990), 1276-89.
12. D. S. Dugdale, *J. Mech. & Phys. Solids* **8** (1960), 100-104.
13. G. T. Embley and G. C. Sih, *Engineering Fracture Mechanics* **4** (1972) 431-42.
14. R. Hill *The Mathematical Theory of Plasticity*, Clarendon Press, Oxford, 1950.
15. J. K. Knowles and E. Sternberg, *J. Elasticity* **10** (1980) 81-110.
16. W. Ramberg and W. R. Osgood, *NACA Technical Note No.* 902 (1943)
17. I. N. Sneddon and M. Lowengrub, *Crack Problems in the Classical Theory of Elasticity*, Wiley, New York, 1969.
18. J. R. Willis, Asymptotic analysis of crack bridging by ductile fibres, *XVIIth International Congress of Theoretical and Applied Mechanics, Haifa*, 1962.
19. I. N. Sneddon, *The Use of Integral transforms*, McGraw- Hill, Book Co., Inc., New York, 1972.
20. A. T. deHoop, *Appl. Scient. Res.* **B8** (1960), 349-60.

APPENDIX

Consider the unilateral and bilateral transforms[19]

$$F = \int_0^\infty f(s)\, e^{-\omega s}\, ds,\quad F^* = \int_{-\infty}^{\infty} F(x)\, e^{-\omega q x}\, dx \qquad \text{... (A.1a, b)}$$

where ω is real and positive, but q is, in general, complex. Application of (A.1a, b) to (2.3) and (3.2) gives the following set of double transforms for $y > 0$:

$$U_x^* = G_\alpha + G_\beta,\ U_y^* = -\frac{\alpha}{q} G_\alpha + \frac{q}{\beta} G_\beta,\ G_\alpha = Ce^{-\omega\alpha y},\ G_\beta = De^{-\omega\beta y} \qquad \text{... (A.2a)}$$

$$\frac{1}{\mu\omega}\Sigma_{xy}^* = -2\alpha G_\alpha - \frac{1}{\beta}(\beta^2 - q^2)G_\beta \qquad \text{... (A.2b)}$$

$$\frac{1}{\mu\omega}\Sigma_x^* = \frac{1}{q}(m^2 - 2\alpha^2)G_\alpha + 2qG_\beta \qquad \text{... (A.2c)}$$

$$\frac{1}{\mu\omega}\Sigma_y^* = \frac{1}{q}(\beta^2 - q^2)\ G_\alpha - 2qG_\beta. \qquad \text{... (A.2d)}$$

Here

$$\alpha = \sqrt{(1-q^2)}\ ,\quad \beta = \sqrt{(m^2 - q^2)} \qquad \text{... (A.3)}$$

where $\mathrm{Re}(\alpha, \beta) \geq 0$ in the complex q-plane cut along $\mathrm{Im}\,(q) = 0$, $|\ \mathrm{Re}\ (q)\ | > (1,\ 1/m)$, respectively. The coefficients $(C,\ D)$ are arbitrary functions of $(\omega,\ q)$. They are determined by imposing for $y = 0$ the conditions

$$\Sigma_{xy}^* = 0,\ U_y^* = \frac{1}{\omega^3}\int_{-c}^{c} \frac{f(v)}{q + v^{-1}}\, dv \qquad \text{... (A.4)}$$

that arise from application of (A.1a, b) to (4.1) in view of (3.2) and (4.2). Substitution (A.2) into (A.4) gives equations for (C, D), whereupon (A.2d) gives the following double transform for $y = 0$:

$$\frac{\omega}{\mu}\Sigma_y^* = \frac{-1}{m^2}\int_{-c}^{c} \frac{f(v)}{q+v^{-1}}\,dv\left[4\beta - \frac{1}{\alpha}(\beta^2-q^2)^2\right]. \qquad \text{... (A.5)}$$

The inverse of (A.1b) is[19]

$$F(x) = \frac{\omega}{2\pi i}\int_{\Gamma} F^* e^{\omega q x}\,dq \qquad \text{... (A.6)}$$

where, for this case, the integration path Γ can be taken along the entire Im (q)-axis. Substitution of (A.5) into (A.6), interchange of the v- and q-integrations, and the use of Cauchy theory to transform the latter onto paths around the branch cuts of (α, β) gives for $x > 0$

$$\frac{\pi}{\mu}m^2\Sigma_y = \int_{-c}^{c} f(v)\,dv\left[-\int_{m}^{\infty} 4z^2B + \int_{1}^{\infty}\frac{1}{A}(B^2-z^2)^2\right]\frac{e^{-\omega z x}}{z-v^{-1}}\,dz \qquad \text{... (A.7)}$$

where

$$A = \sqrt{(z^2-1)},\quad B = \sqrt{(z^2-m^2)}. \qquad \text{... (A.8)}$$

It is noted that the singularity at $z = 1/v$ lies on the branch cuts because $|\,v\,| < c$, $c < 1/m < 1$. Therefore, there are no residues in (A.7), but the z-integrations must be taken in the Cauchy principal value sense.

The inverse of exp ($-\omega z x$) is $H(s - zx)$, where H is the Heaviside function. Therefore, the inversion of (A.7) gives, cf. (deHoop[20]) :

$$\frac{\pi}{\mu}m^2\sigma_y = \int_{-c}^{c} vf(v)\,dv\left(\int_{\xi}^{1}\frac{K^2}{a} - \int_{\xi}^{1/m} 4b\right)\frac{du}{u^4(v-u)},\quad \xi = \frac{x}{s} \qquad \text{... (A.9)}$$

where the integration variable change $z = 1/u$ has been introduced, and

$$a = \sqrt{(1-u^2)},\quad b = \sqrt{(1-m^2u^2)},\quad K = m^2u^2 - 2. \qquad \text{... (A.10)}$$

The u-integrations must be taken in the Cauchy principal value sense when $0 < \xi < c$, and it is understood that integrations vanish when the lower limit equals or exceeds the upper limit. In this light, the form of (A.9) shows that the first and second terms are, respectively, dilatational and rotational waves traveling behind the wavefronts ($x = s$, s/m). Similarly, one can obtain for $y = 0$, $x > 0$:

$$\pi m^2\varepsilon_y = -\int_{-c}^{c} vf(v)\,dv\left(\int_{\xi}^{1} Ka + \int_{\xi}^{1/m} 2b\right)\frac{du}{u^4(v-u)}. \qquad \text{... (A.11)}$$

For $y = 0$, $x < 0$, (A.8) and (A.10) are modified by interchanging the upper and lower u-integration limits, and by replacing (1, $1/m$) by their negatives.