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PSEUDOLINEAR FRACTIONAL MINMAX PROGRAMMING

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Necessary and sufficient optimality conditions are derived for a general fractional minmax programming problem and duality results are established under conditions of pseudolinearity and its generalization.

1. INTRODUCTION

Weir[5] considered the following general minmax problem :

$$\text{Minimize } f(x) = \sup_{y \in Y} \phi(x, y)$$

$$\text{subject to } g(x) \leq 0,$$

where $\phi(.,.) : R^n \times R^m \to R$ is a differentiable pseudoconvex function of x and $g(.) : R^n \to R^m$ is continuously differentiable quasiconvex function of x. Y is a specified compact subset of R^m.

Chew and Choo[1] introduced the pseudolinear functions and derived first and second order characterization for them. Recently, Kaul *et al.*[4] established duality results for multi-objective fractional programming problems under conditions of pseudolinearity and its generalization.

The aim of this paper is to establish a sufficient optimality condition for a fractional minmax programming problem involving a pseudolinear objective function and then to establish duality results. In addition, we will establish the Kuhn-Tucker necessary optimality conditions for our primal problem under conditions of pseudolinearity.

2. PRELIMINARIES

The following conventions for equalities and inequalities for vectors $x, y \in R^n$ are used :

$$x = y \iff x_i = y_i \quad \text{for all} \quad i = 1, 2, ..., n$$

$$x \geqq y \iff x_i \geqq y_i \quad \text{for all } i = 1, 2, ..., n$$

$$x \geq y \iff x \geqq y \quad \text{but } x \neq y$$

$$x > y \iff x_i > y_i \quad \text{for all } i = 1, 2, ..., n.$$

We consider the following fractional minmax programming problem

$$\text{(P) Minimize } \frac{f(x)}{g(x)} = \sup_{y \in Y} \left(\frac{\phi(x, y)}{\psi(x, y)} \right)$$

subject to $h(x) \leqq 0$,

where ϕ (.,.) : $S \times R^m \to R$, ψ (.,.) : $S \times R^m \to R$, $h : S \to R^m$ are differentiable functions and S is an open convex subset of R^n. Moreover, $\psi(x, y) > 0$ for every $y \in Y$, $x \in S$, Y is a specified compact subset of R^m. Let the feasible region of the problem (P) be denoted by $X = \{x : x \in S, h(x) \leqq 0\}$.

The following definition from Chew and Choo[1] will be needed in the sequel :

Definition 2.1 — A real differentiable function f defined on an open set X of R^n is pseudolinear on a subset S of X if it is both pseudoconvex and pseudoconcave on S.

The following equivalence was also established when S is a convex subset of R^n.

(i) f is pseudolinear on S.

(ii) There exists a real function p defined on $S \times S$ such that $p(x, y) > 0$ and

$$f(Y) = f(x) + p(x, y) \nabla f(x)^t (y - x)$$

for every x and y is S. Here $p(x, y)$ is called the proportional function of f.

The following Lemma and example are taken from Kaul *et al.*[4] :

Lemma 2.1 — If f and g are two pseudolinear functions defined on an open convex subset S of R^n with the same proportional function $p(x, y)$ and $g(x) > 0$ for every $x \in S$, then f/g is also pseudolinear on S with respect to new proportional function

$$\hat{p}(x, y) = \frac{g(x)\, p(x, y)}{g(y)} .$$

However, f/g is not necessarily pseudolinear with respect to the same proportional function p.

For f, g defined on (0, 1) by $f(x) = (7x + 3) / (2x + 5)$ and $g(x) = (9x + 4)/(2x + 5)$ are pseudolinear with respect to the same proportional function $p(x, y) = (2x + 5)/(2y + 5)$. But the $\frac{f(x)}{g(x)} = \frac{7x+3}{9x+4}$ is not a pseudolinear function with respect to the same p.

3. Necessary and Sufficient Optimality Conditions

For the fractional minmax problem

$$\text{Minimize } \frac{f(x)}{g(x)} = \sup_{y \in Y} \frac{\phi(x, y)}{\psi(x, y)}$$

$$\text{subject to } h(x) \leqq 0.$$

Mishra and Mukherjee[3] established the following necessary Fritz John optimality conditions : if x^* minimizes (P), these exists a positive integer α^*, scalars $\lambda_i^* \geq 0$, $i = 1, 2, ..., \alpha^*$ scalars $\mu_i^* \geqq 0$, $i = 1, 2, ..., m$, vectors

$$y^i \in Y(x^*) = \left\{ y \in Y : \frac{\phi(x^*, y)}{\psi(x^*, y)} = \sup_{z \in Y} \frac{\phi(x^*, z)}{(x^*, z)} \right\}, \ i = 1, 2, ..., \alpha^*$$

such that

$$\sum_{i=1}^{\alpha^*} \lambda_i^* \nabla_x \left(\frac{\phi(x^*, y^i)}{\psi(x^*, y^i)} \right) + \sum_{i=1}^{m} \mu_i^* \nabla_x h_i(x^*) = 0$$

$$\mu_i^* h_i(x^*) = 0, \ i = 1, 2, ..., m$$

$$(\lambda_i^*, \ i = 1, 2, ..., \alpha^*; \ \mu_i^*, \ i = 1, 2, ..., m) \neq 0.$$

In addition, if the vectors $\nabla_x h_i(x^*)$, $i \in \{i : h_i(x^*) = 0\}$ are linearly independent then at least one of the scalars λ_i^*, $i = 1, 2, ..., \alpha^*$ is non-zero. This condition is well known regularity assumption. Other regularity type conditions are also possible as we will now show.

Assume that h_i, $i = 1, 2, ..., m$ are pseudolinear at x^* and there exists a feasible x' for (P) with $h_i(x') < 0$, $i = 1, 2, ..., m$. If $\lambda_i^* = 0$, $i = 1, 2, ..., \alpha^*$, then (μ_i^*, $i = 1, 2, ..., m$) $\neq 0$. Then also, by assumption $\mu_i^* h_i(x^*) = 0$, $i = 1, 2, ..., m$ and $\sum_{i=1}^{m} \mu_i^* \nabla_x h_i(x^*) = 0$. As well, for $i \notin I(x^*) = \{i : h_i(x^*) = 0\}$, $\mu_i^* = 0$. For $i \in I(x^*)$, $h_i(x') - h_i(x^*) < 0$ and by pseudolinearity, $\mu_i^* p(x^*, x') \times \nabla_x h_i(x^*)^t (x' - x^*) < 0$. For $i \notin I(x^*)$, $p(x^*, x') \nabla_x h_i(x^*)^t (x' - x^*) = 0$. Noting that ($\mu_i^*$, i = 1, 2, ..., m) $\neq 0$ and hence $I(x^*) \neq \phi$, it follows that

$$\sum_{i=1}^{m} \mu_i^* p(x^*, x') \nabla_x h_i(x^*)^t (x' - x^*) < 0.$$

Contradicting $\sum_{i=1}^{m} \nabla_x \mu_i^* h_i(x^*) = 0.$

Hence, $(\lambda_i^*, i = 1, 2, ..., \alpha^*) \neq 0$. The assumption that h is pseudolinear at x^* and that there exists a feasible x' for which $h_i(x') < 0$, $i = 1, ..., m$ is analogue of Slater's weak constraint qualification[2]. More generally, the condition

$$\sum_{i=1}^{m} \nabla_x \mu_i^* h_i(x^*) = 0 \text{ and } \sum_{i=1}^{m} \mu_i^* h_i(x^*) = 0$$

$$\Rightarrow \mu^* = (\mu_i^*, i = 1, 2, ..., m) = 0$$

will imply $\lambda^* = (\lambda_i^*, i = 1, 2, ..., \alpha^*) \neq 0$, for if $\lambda^* = 0$, then $\mu^* = 0$ will contradict the Fritz John conditions. Note that this condition holds if the constraints $h_i(x^*) \leq 0$ which are active at x^* have linearly independent gradients.

In view of the above discussion, the problem (P) will be said to satisfy a constraint qualification at x^* if

(a) Slater's weak constraint qualification holds namely h is pseudolinear at x^*, and there is a feasible x' for (P) with $h_i(x') < 0$, $i = 1, 2, ..., m$; or

(b) $\sum_{i=1}^{m} \nabla_x \mu_i^* h_i(x^*) = 0$ and $\sum_{i=1}^{m} \mu_i^* h_i(x^*) = 0$

$$\Rightarrow \mu^* = (\mu_i^*, i = 1, 2, ..., m) = 0.$$

The above considerations lead to the following Kuhn-Tucker theorem.

Theorem 3.1 — Let x^* be an optimal solution for (P) at which a constraint qualification is satisfied. Then there exists a positive integer α^*, $1 \leq \alpha^* \leq n + 1$, scalar $\lambda_i^* \geq 0$, $i = 1, 2, ..., \alpha^*$ not all zero scalars $\mu_i^* \geq 0$, $i = 1, 2, ..., m$, vectors $y^i \in Y(x^*) = \left\{ y \in Y : \frac{\phi(x^*, y)}{\psi(x^*, y)} = \sup_{z \in Y} \frac{\phi(x^*, z)}{\psi(x^*, z)} \right\}$, $i = 1, 2, ..., \alpha^*$ such that

$$\sum_{i=1}^{\alpha^*} \lambda_i^* \nabla_x \left(\frac{\phi(x^*, y^i)}{\psi(x^*, y^i)} \right) + \sum_{i=1}^{m} \mu_i^* \nabla_x h_i(x^*) = 0$$

$$\mu_i^* h_i(x^*) = 0, \; i = 1, 2, ..., m.$$

Now we will establish the sufficiency of the conditions described in Theorem 3.1.

Theorem 3.2 — Let there be a positive integer α^*, $1 \leq \alpha^* \leq n + 1$ scalar $\lambda_i^* \geq 0$, $i = 1, 2, ..., \alpha^*$ not all zero, scalar $\mu_i^* \geq 0$, $i = 1, 2, ..., m$ vectors $y^i \in Y(x^*)$

$$= \left\{ y \in Y : \frac{\phi(x^*, y)}{\psi(x^*, y)} > \sup_{z \in Y} \frac{\phi(x^*, z)}{\psi(x^*, z)} \right\}, \; i = 1, 2, ..., \alpha^*$$

such that

$$\sum_{i=1}^{\alpha*} \lambda_i^* \nabla_x \left(\frac{\phi(x^*, y^i)}{\psi(x^*, y^i)} \right) + \sum_{i=1}^{m} \mu_i^* \nabla_x h_i(x^*) = 0$$

$$\mu_i^* h_i(x^*) = 0, \ i = 1, 2, ..., m.$$

If $\phi(., y)$ and $\psi(., y)$ are pseudolinear with respect to the same proportional function p for every $y \in Y$ and $h(.)$ is pseudolinear with respect to q, then x^* is a minmax solution to (P).

PROOF : Suppose that the conditions of the theorem are satisfied but x^* is not a minmax solution. Then there exists a feasible x^0 such that

$$\sup_{y \in Y} \frac{\phi(x^0, y)}{\psi(x^0, y)} < \sup_{y \in Y} \frac{\phi(x^*, y)}{\psi(x^*, y)} .$$

Now

$$\sup_{y \in Y} \frac{\phi(x^*, y)}{\psi(x^*, y)} = \frac{\phi(x^*, y^i)}{\psi(x^*, y^i)} , \ i = 1, 2, ..., \alpha^*,$$

and

$$\sup_{y \in Y} \frac{\phi(x^0, y^i)}{\psi(x^0, y^i)} \leqq \sup_{y \in Y} \frac{\phi(x^*, y^i)}{\psi(x^*, y^i)} , \ i = 1, 2, ..., \alpha^*.$$

Therefore

$$\frac{\phi(x^0, y^i)}{\psi(x^0, y^i)} < \frac{\phi(x^*, y^i)}{\psi(x^*, y^i)} , \ i = 1, 2, ..., \alpha^*.$$

Pseudolinearity of $\frac{\phi(., y^i)}{\psi(., y^i)}$ then implies

$$\tilde{p}_i(x^*, x^0) \nabla_x \left(\frac{\phi(x^*, y^i)}{\psi(x^*, y^i)} \right)^t (x^0 - x^*) < 0, \ i = 1, 2, ..., \alpha^*,$$

$$\left(\text{by Lemma 2.1, } \ \tilde{p}_i(x^*, x^0) = p_i(x^*, x^0) \frac{\phi(x^0, y^i)}{\psi(x^*, y^i)} \right).$$

Hence

$$\tilde{p}_i(x^*, x^0) \lambda_i \nabla_x \left(\frac{\phi(x^*, y^i)}{\psi(x^*, y^i)} \right)^t (x^0 - x^*) \leqq 0, \ i = 1, 2, ..., \alpha^*$$

with at least one strict inequality since $\lambda^* = (\lambda_i^*, \ i = 1, 2, ..., \alpha^*) \neq 0$. Since $\tilde{p}_i(x^*, x^0) > 0$ for all $i = 1, 2, ..., \alpha^*$, it follows that

$$\sum_{i=1}^{\alpha^*} \lambda_i^* \nabla_x \left(\frac{\phi(x^*, y^i)}{\psi(x^*, y^i)} \right)^t (x^0 - x^*) < 0. \qquad \text{... (1)}$$

Since x^0 is feasible for (P), we have

$$\mu_i^* h_i(x^0) - \mu_i^* h_i(x^*) \le 0, \; i = 1, 2, ..., m,$$

$$\mu_i^* h_i(x^0) - \mu_i^* h_i(x^*) < 0, \text{ for some } j.$$

Using pseudolinearity of $h_i(.)$, we get

$$q(x^*, x^0) \, \mu_i^* \nabla_x h_i(x^*)^t (x^0 - x^*) \le 0, \; i = 1, 2, ..., m.$$

Again as $q(x^0, x^*) > 0$, it follows that

$$\sum_{i=1}^{m} \mu_i^* \nabla_x h_i(x^*)^t (x^0 - x^*) \leqq 0. \qquad \text{... (2)}$$

Adding (1) and (2) we arrive to a contradiction to an equality constraint of the Theorem 3.2. □

4. Duality

For the minmax problem (P) we consider the following dual similar to that of Weir[5] :

(D) Maximize t

subject to

$$\lambda_i \left[\frac{\phi(u, w_i)}{\psi(u, w_i)} - t \right] \geqq 0, \; i = 1, 2, ..., \alpha, \; 1 \leqq \alpha \leqq n + 1,$$

$$\sum_{i=1}^{m} \mu_i h_i(u) \geqq 0, \quad (\alpha, \lambda, \omega) \in \psi, \quad (u, \mu) \in \Theta(\alpha, \lambda, \omega)$$

where ψ denotes the triplets $(\alpha, \lambda, \omega)$, where α ranges over the integers $1 \leqq \alpha \leqq n + 1$, $\lambda = (\lambda_i, \; i = 1, 2, ..., \alpha)$, $\lambda_i \geqq 0$, $i = 1, 2, ..., \alpha$, $\lambda \neq 0$, $\omega = (\omega_i, \; i = 1, 2, ..., \alpha)$ with $\omega_i \in Y$ for all $i = 1, 2, ..., \alpha$ and

$$\Theta(\alpha, \lambda, \omega) = \left\{ (u, \mu) \in R^n \times R^m : \sum_{i=1}^{\alpha} \lambda_i \nabla_x \left(\frac{\phi(u, w^i)}{\psi(u, w^i)} \right) + \sum_{i=1}^{m} \mu_i \nabla_x h_i(u) = 0, \quad \mu \geqq 0 \right\}.$$

Theorem 4.1 (Weak Duality) — Let $\phi(.,\ y)$ and $\psi(.,\ Y)$ be pseudolinear with respect to the proportional function p and let $\sum_{i=1}^{m} \mu_i h_i(.)$ be pseudolinear with respect to the proportional function q for all feasible x for (P) and for all feasible $(\alpha, \lambda, \omega) \in \psi$, $(u, \mu) \in \Theta(\alpha, \lambda, \omega)$ for (D). Then inf(P) $\geq$ sup(D).

PROOF : Suppose there exists x^0 feasible for (P) and feasible $(\alpha, \lambda, \omega) \in \psi$, $(u, \mu) \in \Theta(\alpha, \lambda, \omega)$ for (D) such that $\sup_{y \in Y} \frac{\phi(x^0, y)}{\psi(x^0, y)} < t$. Then $\frac{\phi(x^0, y)}{\psi(x^0, y)} < t$ for all $y \in Y$, and hence

$$\lambda_i \left(\frac{\phi(x^0, y)}{\psi(x^0, y)} \right) \leq \lambda_i t \text{ for all } i = 1, 2, ..., \alpha,$$

with at least one strict inequality since $\lambda \neq 0$. From the dual constraints it then follows that $\lambda_i \left(\frac{\phi(x^0, y)}{\psi(x^0, y)} \right) \leq \lambda_i \left(\frac{\phi(u, w^i)}{\psi(u, w^i)} \right)$, for all $y \in Y$ and $i = 1, 2, ..., \alpha$ with at least one strict inequality. Therefore

$$\lambda_i \left(\frac{\phi(x^0, w^i)}{\psi(x^0, w^i)} \right) \leq \lambda_i \left(\frac{\phi(u, w^i)}{\psi(u, w^i)} \right),\ i = 1, 2, ..., \alpha$$

and since $\phi(.,\ y)$ and $\psi(.,\ y)$ are pseudolinear with respect to the proportional function p. We have

$$p_i(u, x^0)\, \lambda_i \nabla_x \left(\frac{\phi(u, w^i)}{\psi(u, w^i)} \right)^t (x^0 - u) \leq 0,\ i = 1, 2, ..., \alpha$$

with at least one strict inequality.
Since $p(x^0, u) > 0$, we get

$$\sum_{i=1}^{\alpha} \lambda_i \nabla_x \left(\frac{\phi(u, w^i)}{\psi(u, w^i)} \right)^t (x^0 - u) < 0. \qquad ...(3)$$

But feasibility of x^0 for (P) and $(\alpha, \lambda, \omega) \in \psi$ and $(u, \mu) \in \Theta(\alpha, \lambda, \omega)$ for (D) implies

$$\sum_{i=1}^{m} \mu_i h_i(x^0) - \sum_{i=1}^{m} \mu_i h_i(u) \leq 0,$$

and pseudolinearity of $\sum_{i=1}^{m} \mu_i h_i(.)$ implies

$$\sum_{i=1}^{m} q_i(u, x^0)\, \mu_i \nabla_x h_i(u)^t (x^0 - u) \leq 0.$$

Again as $q(u, x^0) > 0$, we get

$$\sum_{i=1}^{m} \mu_i \nabla_x h_i(u)^t (x^0 - u) \leq 0. \quad \text{... (4)}$$

Adding (3) and (4) we arrive to a contradiction to the equality constraint of the dual problem (D).

Theorem 4.2 (Strong Duality) — Let x^* be optimal for (P) and let a constraint qualification be satisfied. Then there exists $(\alpha^*, \lambda^*, y^*) \in \psi$ and $\mu^* \in R^m$, $\mu^* \geqq 0$ with $(x^*, \mu^*) \in \Theta(\alpha^*, \lambda^*, y^*)$ such that $(\alpha^*, \lambda^*, y^*)$ and (x^*, μ^*) are feasible for (D). If also $\phi(., y)^{\cdot}$ and $\psi(., y)$ are pseudolinear with respect to the same proportional function p and $\sum_{i=1}^{m} \mu_i h_i(.)$ is pseudolinear with respect to the proportional function q for all feasible x for (P) and for all feasible $(\alpha, \lambda, \omega) \in \psi$ $(u, \mu) \in \Theta(\alpha, \lambda, \omega)$ for (D), then $(\alpha^*, \lambda^*, y^*)$ and (x^*, μ^*) is an optimal solution for (D).

PROOF : Since x^* is an optimal solution for (P) and a constraint qualification is satisfied, then Theorem 3.1 guarantees the existence of a positive integer α^*, $1 \leqq \alpha^* \leqq n+1$, scalars $\lambda_i^* \geqq 0$, $i = 1, 2, ..., \alpha^*$ not all zero, scalars $\mu_i^* \geqq 0$, $i = 1, 2, ..., m$, vectors

$$y^i \in Y(x^*) = \left\{ y \in Y : \frac{\phi(x^*, y)}{\psi(x^*, y)} = \sup_{z \in Y} \frac{\phi(x^*, z)}{\psi(x^*, z)} \right\}, \quad i = 1, 2, ..., \alpha^*$$

such that

$$\sum_{i=1}^{\alpha^*} \lambda_i^* \nabla_x \left(\frac{\phi(x^*, y^i)}{\psi(x^*, y^i)} \right) + \sum_{i=1}^{m} \mu_i^* \nabla_x h_i(x^*) = 0$$

$$\mu_i^* h_i(x^*) = 0. \ i = 1, 2, ..., m.$$

Thus, denoting $y^* = (y^1, y^2, ..., y^{\alpha^*})$ and $\lambda^* = (\lambda_1^*, \lambda_2^*, ..., \lambda_{\alpha^*}^*)$, $(\alpha^*, \lambda^*, y^*) \in \psi$, $(x^*, \mu^*) \in \Theta(\alpha^*, \lambda^*, y^*)$ and $t = ((\phi(x^*, y^i))/(\psi(x^*, y^i)))$, $i = 1, 2, ..., \alpha^*$ are feasible for the dual and the values of the primal and dual are equal. The optimality of $(\alpha^*, \lambda^*, y^*) \in \psi$, $(x^*, \mu^*) \in \Theta(\alpha^*, \lambda^*, y^*)$ for (D) follows by weak duality.

5. GENERALIZATION

The result discussed in the preceding sections also hold for a more general class of functions given in Kaul *et al.*[4].

Definition 5.1 — A real differentiable function f defined on an open set S of R^n is said to be η-pseudolinear if there exist functions $p : S \times S \to R$ and $\eta : S \times S \to R^n$ such that $p(x, y) > 0$ and

$$f(y) = f(x) + p(x, y)\ (\nabla f(x))^t \eta\,(x,y)$$

for every $x,\ y \in S$.

The regularity conditions can be easily extended to η-pesudolinear functions. The sufficiency of Theorem 3.1 can be established with η-pseudolinear functions and the duality results are also true for the class of η-pseudolinear functions. We only state the results and the proofs are on similar lines.

Theorem 5.1 — Let there be a positive integer $\alpha^*, 1 \leqq \alpha^* \leqq n + 1$, scalars $\lambda_i^* \geqq 0,\ i = 1, 2, ..., \alpha^*$ not all zero, scalars $\mu_i^* \geqq 0,\ i = 1, 2, ..., m$, vectors y^i

$$y^i \in Y(x^*) = \left\{ y \in Y : \frac{\phi(x^*, y)}{\psi(x^*, y)} = \sup_{z \in Y} \frac{\phi(x^*, z)}{\psi(x^*, z)} \right\},\ i = 1, 2, ..., \alpha^*$$

such that

$$\sum_{i=1}^{\alpha^*} \lambda_i^* \nabla_x \left(\frac{\phi(x^*, y^i)}{\psi(x^*, y^i)} \right) + \sum_{i=1}^{m} \mu_i^* \nabla_x h_i(x^*) = 0$$

$$\mu_i^* h_i(x^*) = 0,\ i = 1, 2, ..., m.$$

If $\phi(., y)$ and $\psi(., y)$ are η-pseudolinear with respect to the same proportional function p, for every $y \in Y$ and $h(.)$ is η-pseudolinear with respect to the proportional function q, then x^* is a minmax solution to (p).

Theorem 5.2 (Weak Duality) — Let $\phi(., y)$ and $\psi(., y)$ be η-pseudolinear with respect to the same proportional function p and $\sum_{i=1}^{m} \mu_i h_i(.)$ be η-pseudolinear with respect to the proportional function q for all feasible x for (p) and for all feasible $(\alpha, \lambda, \omega) \in \psi\ (u, \mu) \in \Theta\,(\alpha, \lambda, \omega)$ for (D). Then $\inf(p) \geq \sup(D)$.

Theorem 5.3 (Strong Duality) — Let x^* be optimal for (p) and let a constraint qualification be satisfied. Then there exists $(\alpha^*, \lambda^*, y^*) \in \psi$ and $\mu^* \in R^m, \mu^* \geq 0$ with $(x^*, \mu^*) \in \Theta\,(\alpha^*, \lambda^*, y^*)$ such that $(\alpha^*, \lambda^*, y^*)$ and (x^*, μ^*) are feasible for (D). If also $\phi(., y)$ and $\psi(., y)$ are η-pseudolinear with respect to the same proportional function p and $\sum_{i=1}^{m} \mu_i h_i(.)$ is η-pseudolinear with respect to the proportional function q for all feasible x for (p) and for all feasible $(\alpha, \lambda, \omega) \in \psi$, $(u, \mu) \in \Theta\,(\alpha, \lambda, \omega)$ for (D) then $(\alpha^*, \lambda^*, y^*)$ and (x^*, μ^*) is an optimal solution for (D).

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