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PERIODIC AND FIXED POINT THEOREMS IN *d*-COMPLETE TOPOLOGICAL SPACES

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Periodic point theorems and fixed point theorems are proved for some self maps of d-complete topological spaces which satisfy Caristi-type conditions. Also, some common fixed point theorems are proved in d-complete spaces and d-complete symmetric spaces.

Let (X, t) be a topological space and $d : X \times X \to [0, \infty)$ such that $d(x, y) = 0$ if and only if $x = y$. X is said to be d-complete if $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$ implies that the sequence $\{x_n\}_{n=1}^{\infty}$ is convergent in (X, t). Complete metric spaces and complete quasi-metric spaces are examples of d-complete topological spaces. The d-complete semi-metric spaces form an important class of examples of d-complete topological spaces.

Let X be an infinite set and t any T_1 non-discrete first countable topology for X. There exists a complete metric d for X such that $t \le t_d$ and the metric topology t_d is non-discrete. Now (X, t, d) is d-complete since $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$ implies that $\{x_n\}_{n=1}^{\infty}$ is Cauchy in t_d. Thus, $x_n \to x$ in t_d and therefore in the topology t. The construction of t_d is given by Hicks and Crisler[5].

Recently, Hicks[4] and Hicks and Rhoades[6, 7] proved several metric space fixed point theorems in d-complete topological spaces. We shall prove additional theorems in this setting.

Let $T : X \to X$ be a mapping. A point $x \in X$ is called a periodic point for T if and only if there exists a positive integer k such that $T^k x = x$. If $k = 1$, then x is called a fixed point for T. T is ω-continuous at x if $x_n \to x$ implies $Tx_n \to Tx$ as

$n \to \infty$. The set $O(x, \infty) = \{x, Tx, T^2x, ...\}$ is called the orbit of x. A real-valued function $G : X \to [0, \infty)$ is T-orbitally lower semi-continuous relative to x if and only if $\{x_n\}_{n=1}^{\infty}$ is a sequence in $O(x, \infty)$ and $\lim x_n = p$ implies $G(p) \leq \liminf G(x_n)$. A real-valued function $G : X \to [0, \infty)$ is said to be T-orbitally weak lower semi-continuous (w.l.s.c.) relative to x if and only if $\{x_n\}_{n=1}^{\infty}$ is a sequence in $O(x, \infty)$ and $\lim x_n = p$ implies $G(p) \leq \limsup G(x_n)$.

Ćirić[1] proposed the following as a periodic point theorem for a complete quasi-metric space using the concept of T-orbital weak lower semi-continuity. Recall that in a quasi-metric space, $d(x, y) = d(y, x)$ may not hold.

Ćirić's claim is the following. Suppose $T : X \to X$, $n : X \to N$ and $\Phi : X \to [0, \infty)$, where X is a complete quasi-metric space. If for some $x_0 \in X$ there exists a subsequence $S = \{x_n\}_{n=0}^{\infty}$ in $O(x_0, \infty)$ such that $T^{n(x_n)} x_n \in S$ and $d(y, T^{n(y)} y) \leq \Phi(y) - \Phi(T^{n(y)} y)$ holds for each $y \in S$, then we have

(a) $\lim x_n = p$ exists,

(b) $T^{n(p)} p = p$ if and only if $G(x) = d(x, T^{n(x)} x)$ is T-orbitally w.l.s.c. relative to x_0,

(c) $d(x_0, x_n) \leq \Phi(x_0)$, and

(d) if $y \to d(z, y)$ is T-orbitally w.l.s.c. relative to x_0 for $z \in S$, then $d(x_n, p) \leq \Phi(x_n)$ and $d(x_0, p) \leq \Phi(x_0)$.

The following example shows that Ćirić's claim does not hold even in a complete metric space. However, if we require that $S = \{x_n\}_{n=0}^{\infty}$ be defined by $x_{n+1} = T^{n(x_n)} x_n$ for $n = 0, 1, 2, ...$, then the theorem holds.

Example 1 — Suppose $X = [0, 1]$ with the usual metric, $Tx = 1 - x$ for $x \in X$, $n(x) = 2$ for all $x \in X$, $x_0 = \frac{1}{4}$, $S = \{T^n x_0\}_{n=0}^{\infty}$, and $\Phi(x) = 1$ for all $x \in X$. This satisfies the statement of Ćirić's claim but does not satisfy the conclusion. In particular, $\lim x_n = \lim T^n x_0$ does not exist.

If we assume X is a d-complete topological space, then we obtain Theorem 1. Note that we do not obtain parts (c) and (d) of Ćirić's claim.

Theorem 1 — Suppose $T : X \to X$, $n : X \to N$ and $\Phi : X \to [0, \infty)$, where (X, t) is a d-complete topological space. If for some $x_0 \in X$, the sequence $S = \{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = T^{n(x_n)} x_n$ satisfies

(A) $d(y, T^{n(y)} y) \leq \Phi(y) - \Phi(T^{n(y)} y)$ for each $y \in S$,

then we have

(a) $\lim x_n = p$ exists, and

(b) $T^{n(p)}p = p$ if and only if $G(x) = d(x, T^{n(x)}x)$ is T-orbitally w.l.s.c. relative to x_0.

PROOF : Since $x_{n+1} = T^{n(x_n)}x_n$ for $n = 0, 1, ...,$ we have

$$d(x_n, x_{n+1}) = d(x_n, T^{n(x_n)}x_n) \leq \Phi(x_n) - \Phi(x_{n+1}).$$

For $m \geq 0$,

$$\sigma_m = \sum_{n=0}^{m} d(x_n, x_{n+1}) \leq \sum_{n=0}^{m} [\Phi(x_n) - \Phi(x_{n+1})] = \Phi(x_0) - \Phi(x_{m+1}) \leq \Phi(x_0).$$

The sequence $\{\sigma_m\}_{m=0}^{\infty}$ of partial sums of the infinite series $\sum_{n=0}^{\infty} d(x_n, x_{n+1})$ is a nondecreasing sequence bounded above by $\Phi(x_0)$ and therefore converges. Since X is d-complete, we have $\lim x_n = p$ for some $p \in X$.

For (b), if $T^{n(p)}p = p$, then $G(p) = d(p, T^{n(p)}p) = 0 \leq \lim\sup d(x_n, T^{n(x_n)}x_n) = \lim\sup G(x_n)$. Suppose that $G(p) \leq \lim\sup G(x_n)$. Now $G(x_n) = d(x_n, x_{n+1}) \to 0$ as $n \to \infty$ since $\sum_{n=0}^{\infty} d(x_n, x_{n+1}) < \infty$. Therefore, $G(p) = d(p, T^{n(p)}p) = 0$. Hence p is a periodic point of T and (b) holds.

Corollary 1 (Hicks[4], Theorem 1) — Let X be a d-complete topological space. Suppose $T : X \to X$ and $\Phi : X \to [0, \infty)$. Suppose there exists an x_0 such that $d(y, Ty) \leq \Phi(y) - \Phi(Ty)$ for all $y \in O(x_0, \infty)$.

Then we have :

(1) $\lim T^n x = p$ exists.

(2) $Tp = p$ if and only if $G(x) = d(x, Tx)$ is T-orbitally lower semi-continuous at x_0.

PROOF : Let $S = x_0, Tx_0, T^2 x_0, ...$. Define $n : X \to N$ to be $n(x) = 1$ for every $x \in X$. Then apply Theorem 1.

Corollary 2 — Suppose $T : X \to X$ and $n : X \to N$, where (X, t) is a d-complete topological space. If for some $x_0 \in X$, the sequence $S = \{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = T^{n(x_n)}x_n$ satisfies $d(z, w) \leq kd(y, z)$ for each $y \in S$, where $k \in (0, 1)$, $z = T^{n(y)}y$ and $w = T^{n(z)}z$, then

(a) $\lim x_n = p$ exists, and

(b) $T^{n(p)}p = p$ if and only if $G(x) = d(x, T^{n(x)}x)$ is T-orbitally w.l.s.c. relative to x_0.

PROOF : Set $\Phi(y) = \frac{1}{1-k} d(y, T^{n(y)}y)$ for $y \in S$. Now $d(z, w) \leq kd(y, z)$ implies $d(y, z) - d(z, w) \geq (1-k)\, d(y, z)$, or $\frac{1}{1-k}[d(y, z) - d(z, w)] \geq d(y, z)$. Now

$$\frac{1}{1-k} d(y, z) = \frac{1}{1-k} d(y, T^{n(y)} y) = \Phi(y)$$

and

$$\frac{1}{1-k} d(z, w) = \frac{1}{1-k} d(T^{n(y)} y,\ T^{n(T^{n(y)} y)} (T^{n(y)} y)) = \Phi\,(T^{n(y)} y).$$

Thus $\quad d(y, T^{n(y)} y) \leq \Phi(y) - \Phi(T^{n(y)} y)$ for each $y \in S$. Apply Theorem 1.

Note that Corollary 2 is a generalization of Corollary 1 of Hicks[4]. Theorems 2 and 3 are extensions of fixed point theorems for quasi-metric spaces given in Ćirić[1] to d-complete topological spaces.

Theorem 2 — Suppose $T : X \to X$, $n : X \to N$ and $\Phi : X \to [0, \infty)$, where (X, t) is a d-complete topological space. If for some $x_0 \in X$, the sequence $S = \{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = T^{n(x_n)} x_n$ satisfies

(A) $d(y, T^{n(y)} y) \leq \Phi(y) - \Phi(T^{n(y)} y)$ for each $y \in S$

and for all $y \in \mathrm{Cl}[O(x_0, \infty)]$

(1) $y \neq Ty$ implies $\Phi(T^m y) < \Phi(y)$ for some positive integer $m = m(y)$,

then T has a fixed point.

The proof of Theorem 2 is identical to the proof of Theorem 3 of Ćirić[1].

Theorem 3 — Suppose $S, T : X \to X$ and $\Phi : X \to [0, \infty)$, where (X, t) is a d-complete topological space. If there is $x_0 \in X$ such that $d(y, Ty) + d(Ty, STy) \leq \Phi(y) - \Phi(STy)$ for all $y \in O_{ST}(x_0, \infty) := \{x_0, Tx_0, STx_0, T(ST)x_0, ..., (ST)^n x_0, T(ST)^n x_0, \ldots\}$, then we have

(a′) $\lim (ST)^n x_0 = \lim T(ST)^n x_0 = p$ exists, and

(b′) $Tp = p = Sp$ if and only if $G_1(x) = d(x, Tx)$ and $G_2(x) = d(x, Sx)$ are (S, T)-orbitally w.l.s.c. relative to x_0.

PROOF : Consider the sequence $\{z_n\}_{n=0}^{\infty}$ defined by $z_{2k} = (ST)^k x_0$ and $z_{2k+1} = T(ST)^k x_0$ ($k = 0, 1, 2, \ldots$). For $m = 2k + 1$,

$$\sigma_m = \sum_{n=0}^{2k+1} d(z_n, z_{n+1})$$

$$= [d(x_0, Tx_0) + d(Tx_0, STx_0)] + [d(STx_0, T(ST)x_0) + d(T(ST)x_0, (ST)^2 x_0)] + \ldots$$

$$+ [d((ST)^k x_0, T(ST)^k x_0) + d(T(ST)^k x_0, (ST)^{k+1} x_0)]$$

$$\leq [\Phi(x_0) - \Phi(STx_0)] + [\Phi(STx_0) - \Phi((ST)^2 x_0)] + \ldots$$

$$+ [\Phi((ST)^k x_0) - \Phi((ST)^{k+1} x_0)]$$

$$= \Phi(x_0) - \Phi((ST)^{k+1} x_0) \leq \Phi(x_0).$$

For $m = 2k$,

$$\sigma_m = \sum_{n=0}^{2k} d(z_n, z_{n+1}) \leq \left(\sum_{n=0}^{2k} d(z_n, z_{n+1}) \right) + d(z_{2k+1}, z_{2k+2}) = \sigma_{2k+1} \leq \Phi(x_0).$$

The sequence $\{\sigma_m\}_{m=0}^{\infty}$ of partial sums of the infinite series $\sum_{n=0}^{\infty} d(z_n, z_{n+1})$ is a nondecreasing sequence bounded above by $\Phi(x_0)$ and therefore converges. Since X is d-complete, there exists $p \in X$ such that $z_n \to p$ as $n \to \infty$. Hence

$$\lim (ST)^n x_0 = \lim T(ST)^n x_0 = p.$$

Suppose that G_1 is (S, T)-orbitally w.l.s.c. relative to x_0. $G_1(z_{2k}) = d(z_{2k}, Tz_{2k}) = d(z_{2k}, z_{2k+1}) \to 0$ as $k \to \infty$ since $\sum_{n=0}^{\infty} d(z_n, z_{n+1}) < \infty$. Now $\{z_{2k}\}_{k=0}^{\infty}$ is a sequence in $O_{ST}(x_0, \infty)$ such that $z_{2k} \to p$ as $k \to \infty$. Thus, $G_1(p) \leq \limsup G_1(z_{2k}) = 0$. So $d(p, Tp) = G_1(p) = 0$ implies $p = Tp$. Also, $d(p, Tp) = 0$ implies $G_1(x) = d(x, Tx) \geq 0 = G_1(p)$ for all $x \in X$. So G_1 is w.l.s.c. relative to x_0. Similarly, $p = Sp$ implies G_2 is w.l.s.c. relative to x_0.

Suppose that G_2 is (S, T)-orbitally w.l.s.c. relative to x_0. Now $\{z_{2k+1}\}_{k=0}^{\infty}$ is a sequence in $O_{ST}(x_0, \infty)$ such that $z_{2k+1} \to p$ as $k \to \infty$. So $G_2(p) \leq \limsup G_2(z_{2k+1})$. However, $G_2(z_{2k+1}) = d(z_{2k+1}, Sz_{2k+1}) = d(T(ST)^k x_0, (ST)^{k+1} x_0) = d(z_{2k+1}, z_{2k+2}) \to 0$ as $k \to \infty$ since $\sum_{n=0}^{\infty} d(z_n, z_{n+1}) < \infty$. Thus, $G_2(p) \leq 0$ implies $d(p, Sp) = G_2(p) = 0$. So $p = Sp$.

Theorems 4 and 5 are common fixed point theorems for mappings in d-complete topological spaces.

Theorem 4 — Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of self mappings of a d-complete topological space (X, t) satisfying $d(T_i x, T_j y) \leq \Phi(\max\{d(x, y), d(x, T_i x), d(y, T_j y), d(y, T_i x)\})$ for all $x, y \in X$, where d is continuous, $\Phi : [0, \infty) \to [0, \infty)$, $\Phi(0) = 0$, Φ is nondecreasing, Φ is upper semicontinuous, and $\Phi(t) < t$ for all $t > 0$. Then the sequence $\{T_n\}_{n=1}^{\infty}$ has a unique common fixed point in X if and only if there exists $x_0 \in X$ such that

$$\sum_{n=0}^{\infty} \Phi^n (d(x_0, T_1 x_0)) < \infty \qquad [\Phi^2(t) = \Phi(\Phi(t))].$$

PROOF : If there exists $p \in X$ such that $p = T_n p$ for $n = 1, 2, 3, \ldots$, then

$$\sum_{n=0}^{\infty} \Phi^n (d(p, T_1 p)) = \sum_{n=0}^{\infty} \Phi^n (0) = 0.$$

Suppose there exists $x_0 \in X$ such that $\sum_{n=0}^{\infty} \Phi^n (d(x_0, T_1 x_0)) < \infty$. Define the sequence $\{x_n\}_{n=1}^{\infty}$ by $x_n = T_n x_{n-1}$, $n = 1, 2, 3, 4, \ldots$. Now

$$\begin{aligned} d(x_1, x_2) = d(T_1 x_0, T_2 x_1) &\leq \Phi(\max \{d(x_0, x_1), d(x_0, T_1 x_0), d(x_1, T_2 x_1), d(x_1, T_1 x_0)\}) \\ &= \Phi(\max \{d(x_0, x_1), d(x_0, x_1), d(x_1, x_2), d(x_1, x_1)\}) \\ &= \Phi(\max \{d(x_0, x_1), d(x_1, x_2)\}). \end{aligned}$$

If $d(x_1, x_2) = 0$, then $d(x_1, x_2) = 0 \leq \Phi (d(x_0, x_1))$. If $d(x_1, x_2) > 0$, then we also get $d(x_1, x_2) \leq \Phi(d(x_0, x_1))$. Suppose $\max \{d(x_0, x_1), d(x_1, x_2)\} = d(x_1, x_2)$. Then $d(x_1, x_2) \leq \Phi(d(x_1, x_2)) < d(x_1, x_2)$, a contradiction. Thus,

$$\max \{d(x_0, x_1), d(x_1, x_2)\} = d(x_0, x_1) \text{ and } d(x_1, x_2) \leq \Phi(d(x_0, x_1)).$$

Also,

$$\begin{aligned} d(x_2, x_3) &= d(T_2 x_1, T_3 x_2) \\ &\leq \Phi(\max\{d(x_1, x_2), d(x_1, T_2 x_1), d(x_2, T_3 x_2), d(x_2, T_2 x_1)\}) \\ &= \Phi(\max\{d(x_1, x_2), d(x_1, x_2), d(x_2, x_3), d(x_2, x_2)\}) \\ &= \Phi(\max\{d(x_1, x_2), d(x_2, x_3)\}) \\ &= \Phi(d(x_1, x_2)) \\ &\leq \Phi(\Phi(d(x_0, x_1)) = \Phi^2(d(x_0, x_1)) \end{aligned}$$

since Φ is nondecreasing. Continuing in this manner, we get $d(x_n, x_{n+1}) \leq \Phi^n d(x_0, x_1)$. Since $\sum_{n=0}^{\infty} \Phi^n (d(x_0, x_1)) = \sum_{n=0}^{\infty} \Phi^n (d(x_0, T_1 x_0)) < \infty$, then $\sum_{n=0}^{\infty} d(x_n, x_{n+1}) < \infty$. Since X is d-complete, there exists $x \in X$ such that $x_n \to x$ as $n \to \infty$.

For fixed $n \in N$,

$$\begin{aligned} d(x_{m+1}, T_n x) &= d(T_{m+1} x_m, T_n x) \\ &\leq \Phi(\max\{d(x_m, x), d(x_m, T_{m+1} x_m), d(x, T_n x), d(x, T_{m+1} x_m)\}) \\ &= \Phi(\max\{d(x_m, x), d(x_m, x_{m+1}), d(x, T_n x), d(x, x_{m+1})\}). \end{aligned}$$

Since d is continuous and Φ is upper semicontinuous, letting $m \to \infty$ gives $d(x, T_n x) \leq \Phi(\max\{d(x, x), d(x, T_n x)\}) = \Phi(d(x, T_n x))$. If $d(x, T_n x) > 0$, then $\Phi(d(x, T_n x)) < d(x, T_n x)$, a contradiction. Thus, $x = T_n x, n = 1, 2, 3, \ldots$.

Suppose $y = T_n y$ for each n with $y \neq x$. Then

$$d(x, y) = d(T_n x, T_n y) \leq \Phi(\max\{d(x, y), d(x\ T_n x), d(y, T_n y), d(y, T_n x)\})$$

$$= \Phi(\max\{d(x, y), 0, d(y, x)\}) = \Phi(d(y, x))$$

since $d(x, y) > 0$ implies that $\Phi(d(x, y)) < d(x, y)$. Similarly, $d(y, x) \leq \Phi(d(x, y))$. Thus, $d(x, y) \leq \Phi(d(y, x)) < d(y, x) \leq \Phi(d(x, y)) < d(x, y)$, a contradiction. Thus, the common fixed point is unique.

Corollary 3 — Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of self mappings of a d-complete topological space (X, t) satisfying $d(T_i x, T_j y) \leq h \max \{d(x; y), d(x, T_i x), d(y, T_j y), d(y, T_i x)\}$ for all $x, y \in X$, where $0 \leq h < 1$ and d is continuous. Then the sequence $\{T_n\}_{n=1}^{\infty}$ has a unique common fixed point in X.

PROOF : Let $\Phi(t) = ht$ for $t \geq 0$. Note that $\Phi(0) = 0, \Phi$ is continuous and nondecreasing, and $\Phi(t) = ht < t$ for all $t > 0$ since $0 \leq h < 1$. Let $x_0 \in X$. Then $\sum_{n=1}^{\infty} \Phi^n(d(x_0, T_1 x_0)) = \sum_{n=1}^{\infty} h^n d(x_0, T_1 x_0) < \infty$ since $0 \leq h < 1$. Applying Theorem 4, we get a unique common fixed point for $\{T_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=0}^{\infty}$ defined by $x_n = T_n x_{n-1}$, $n = 1, 2, 3, \ldots$, converges to this fixed point.

Theorem 5 is based on work by Kang and Kim[3]. A symmetric on a set X is a real-valued function d on $X \times X$ such that the following two properties hold :

(1) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$,

(2) $d(x, y) = d(y, x)$.

Let A and B be mappings from a d-complete topological space X into itself. Jungck[2] defined compatibility. A and B are said to be compatible on X if $\lim d(ABx_n, BAx_n) = 0$ when $\{x_n\}_{n=1}^{\infty}$ is a sequence in X such that $\lim Ax_n = \lim Bx_n = q$ for some point q in X. Hicks and Saliga[8] gave the following alternative definition for A to be compatible with B. Given a map A, a map B is compatible with A if, for any sequence $\{x_n\}_{n=1}^{\infty}$ such that $\lim A x_n = \lim B x_n = t$ it follows that $\lim A(Bx_n) = Bt$. Using this alternative definition, if A and B are continuous on a metric space, then A is compatible with B is equivalent to B is compatible with A. In this case, we say that A and B are compatible. Also, if A and B are continuous on a metric space, then the two definitions of compatibility agree. When two maps defined on a d-complete topological space are said to be compatible, it is assumed that both definitions are being used.

Theorem 5 — Let A, B, S and T be mappings from a d-complete topological space (X, t) into itself satisfying the following conditions :

(a) $A(X) \subset T(X)$ and $B(X) \subset S(X)$,

(b) $d(Ax, By) \leq \Phi(\max\{d(Ax, Sx), d(By, Ty), d(Sx, Ty)\})$ for all $x, y \in X$, where $\Phi : [0, \infty) \to [0, \infty), \Phi(0) = 0, \Phi$ is nondecreasing, Φ is upper semicontinuous, and $\Phi(t) < t$ for all $t > 0$,

(c) one of A, B, S, or T is ω-continuous,

(d) pairs A, S and B, T are compatible on X,

(e) d is a continuous symmetric.

Then A, B, S and T have a unique common fixed point in X if and only if there exist $x_0, x_1 \in X$ such that $Ax_0 = Tx_1$ and $\sum_{n=0}^{\infty} \Phi^n(d(Ax_0, Bx_1)) < \infty$.

PROOF : If there exists $q \in X$ such that q is a common fixed point of A, B, S, and T, then $Aq = Tq$ and $\sum_{n=0}^{\infty} \Phi^n(d(Aq, Bq)) = \sum_{n=0}^{\infty} \Phi^n(0) = 0$.

Suppose there exist $x_0, x_1 \in X$ such that $Ax_0 = Tx_1$ and $\sum_{n=0}^{\infty} \Phi^n(d(Ax_0, Bx_1)) < \infty$. Choose $x_2 \in X$ such that $Sx_2 = Bx_1$. Choose $x_3 \in X$ such that $Tx_3 = Ax_2$. Continuing in this manner we can define a sequence $\{y_n\}_{n=0}^{\infty}$ in X such that $y_{2n+1} = Tx_{2n+1} = Ax_{2n}$ and $y_{2n} = Sx_{2n} = Bx_{2n-1}$, $n = 1, 2, 3, \ldots$. Now

$$
\begin{aligned}
d(y_{2n}, y_{2n+1}) &= d(y_{2n+1}, y_{2n}) \\
&= d(Ax_{2n}, Bx_{2n-1}) \\
&\leq \Phi(\max\{d(Ax_{2n}, Sx_{2n}), d(Bx_{2n-1}, Tx_{2n-1}), d(Sx_{2n}, Tx_{2n-1})\}) \\
&= \Phi(\max\{d(y_{2n+1}, y_{2n})\ d(y_{2n}, y_{2n-1})\}).
\end{aligned}
$$

If $d(y_{2n}, y_{2n+1}) = 0$, then $\max\{d(y_{2n+1}, y_{2n}), d(y_{2n}, y_{2n-1})\} = d(y_{2n}, y_{2n-1})$ so that

$$d(y_{2n}, y_{2n+1}) \leq \Phi(d(y_{2n}, y_{2n-1})).$$

If $d(y_{2n}, y_{2n+1}) > 0$, then $\Phi(d(y_{2n}, y_{2n+1})) < d(y_{2n}, y_{2n+1})$. Suppose $\max\{d(y_{2n+1}, y_{2n}), d(y_{2n}, y_{2n-1})\} = d(y_{2n+1}, y_{2n})$. Then $d(y_{2n+1}, y_{2n}) \leq \Phi(d(y_{2n+1}, y_{2n})) < d(y_{2n+1}, y_{2n})$, a contradiction. Thus, $\max\{d(y_{2n+1}, y_{2n}), d(y_{2n}, y_{2n-1})\} = d(y_{2n}, y_{2n-1})$ and $d(y_{2n}, y_{2n+1}) \leq \Phi(d(y_{2n}, y_{2n-1}))$. Similarly

$$
\begin{aligned}
d(y_{2n+1}, y_{2n+2}) &= d(Ax_{2n}, Bx_{2n+1}) \\
&\leq \Phi(\max\{d(Ax_{2n}, Sx_{2n}), d(Bx_{2n+1}, Tx_{2x+1}), d(Sx_{2n}, Tx_{2n+1})\}) \\
&= \Phi(d(y_{2n+1}, y_{2n})).
\end{aligned}
$$

Then $d(y_n, y_{n+1}) \leq \Phi(d(y_n, y_{n-1}))$, $n = 2, 3, 4, \ldots$. It follows that

$$d(y_n, y_{n+1}) \leq \Phi^{n-1} d(y_2, y_1), \quad n = 1, 2, 3, \ldots,$$

since Φ is nondecreasing. Thus,

$$\sum_{n=1}^{\infty} d(y_n, y_{n+1}) \leq \sum_{n=1}^{\infty} \Phi^{n-1}(d(y_2, y_1)) = \sum_{n=0}^{\infty} \Phi^n(d(Bx_1, Ax_0)) < \infty$$

implies that there exists $p \in X$ such that $y_n \to p$ as $n \to \infty$. Since $\{Tx_{2n+1}\}_{n=1}^{\infty}$, $\{Ax_{2n}\}_{n=1}^{\infty}$, $\{Sx_{2n}\}_{n=1}^{\infty}$, and $\{Bx_{2n-1}\}_{n=1}^{\infty}$ are subsequences of $\{y_n\}_{n=1}^{\infty}$, they each converge to p.

Suppose S is ω-continuous. Then $S^2x_{2n} \to Sp$ as $n \to \infty$. Since S and A are compatible, $ASx_{2n} \to Sp$ as $n \to \infty$. Now $d(ASx_{2n}, Bx_{2n-1}) \leq \Phi(\max\{d(ASx_{2n}, S^2x_{2n}), d(Bx_{2n-1}, Tx_{2n-1}), d(S^2x_{2n}, Tx_{2n-1})\})$. Since S is ω-continuous, d is continuous, and Φ is upper semicontinuous, letting $n \to \infty$ gives $d(Sp, p) \leq \Phi(\max\{d(Sp, Sp), d(p, p), d(Sp, p)\}) = \Phi(d(Sp, p))$. If $d(Sp, p) > 0$, then $\Phi(d(Sp, p)) < d(Sp, p)$, a contradiction. Thus $Sp = p$ since $d(Sp, p) = 0$. Also, $d(Ap, Bx_{2n-1}) \leq \Phi(\max\{d(Ap, Sp)\ d(Bx_{2n-1}, Tx_{2n-1}), d(Sp, Tx_{2n-1})\})$. Letting $n \to \infty$, we get $d(Ap, p) \leq \Phi(\max\{d(Ap, p), d(p, p), d(Sp, p)\}) = \Phi(d(Ap, p))$. If $d(Ap, p) > 0$, then $\Phi(d(Ap, p)) < d(Ap, p)$, a contradiction. Thus $Ap = p$.

Since $A(X) \subset T(X)$, $p \in T(X)$ and there exists $u \in X$ such that $p = Ap = Tu$. Now $d(p, Bu) = d(Ap, Bu) \leq \Phi$ (max $\{d(Ap, Sp), d(Bu, Tu), d(Sp, Tu)\}) = \Phi(d(Bu, p))$ which implies that $Bu = p$. Since B and T are compatible on X and $Tu = Bu = p$, $d(TBu, BTu) = 0$ and hence $Tp = TBu = BTu = Bp$. Now $d(p, Tp) = d(Ap, Bp) \leq \Phi$ (max $\{d(Ap, Sp), d(Bp, Tp), d(Sp, Tp)\}) = \Phi(d(p, Tp))$, or $Tp = p$. The proof for T ω-continuous is similar.

Next suppose that A is ω-continuous. Since A and S are compatible on X, A^2x_{2n} and SAx_{2n} converge to Ap as $n \to \infty$. Now,

$$d(A^2x_{2n}, Bx_{2n-1}) \leq \Phi(\max\{d(A^2x_{2n}, SAx_{2n}), d(Bx_{2n-1}, Tx_{2n-1}), d(SAx_{2n}, Tx_{2n-1})\}).$$

Letting $n \to \infty$ gives $d(Ap, p) \leq \Phi(\max\{d(Ap, Ap), d(p, p), d(Ap, p)\}) = \Phi(d(Ap, p))$. Thus $Ap = p$. Hence, there exists $v \in X$ such that $p = Ap = Tv$. Now

$$d(A^2x_{2n}, Bv) \leq \Phi(\max\{d(A^2x_{2n}, SAx_{2n}), d(Bv, Tv), d(SAx_{2n}, Tv)\}).$$

By letting $n \to \infty$, we obtain $d(Ap, Bv) \leq \Phi(\max\{d(Ap, Ap), d(Bv, p), d(Ap, p)\}) = \Phi(d(Bv, p))$. This implies $Bv = p$. Since B and T are compatible on X and $Tv = Bv = p$,

$$d(TBv, BTv) = 0 \text{ and } Tp = TBv = BTv = Bp.$$

Now $d(Ax_{2n}, Bp) \leq \Phi(\max\{d(Ax_{2n}, Sx_{2n}), d(Bp, Tp), d(Sx_{2n}, Tp)\})$. Letting $n \to \infty$ yields $d(p, Bp) \leq \Phi(\max\{d(p, p), d(Bp, Tp), d(p, Tp)\}) = \Phi(d(p, Tp)) = \Phi(d(p, Bp))$ so $p = Bp$.

Since $B(X) \subset S(X)$, there exists $w \in X$ such that $p = Bp = Sw$. Now

$$d(Aw, p) = d(Aw, Bp) \leq \Phi(\max\{d(Aw, Sw), d(Bp, Tp), d(Sw, Tp)\})$$
$$= \Phi(d(Aw, p))$$

implies $Aw = p$. Since A and S are compatible on X and $Aw = Sw = p$, $d(ASw, SAw) = 0$, and hence $Ap = ASw = SAw = Sp$. Therefore, p is a common fixed point of A, B, S and T. The proof for B ω-continuous is similar.

Suppose that p and z, $p \neq z$, are common fixed points of A, B, S and T. Then

$$d(p, z) = d(Ap, Bz) \leq \Phi(\max\{d(Ap, Sp), d(Bz, Tz), d(Sp, Tz)\})$$
$$= \Phi(\max\{d(p, p), d(z, z), d(p, z)\})$$
$$= \Phi(d(p, z)),$$

a contradiction since $d(p, z) > 0$ if $p \neq z$. So the common fixed point of A, B, S and T is unique.

Corollary 3 — Let A, B, S and T be mappings from a d-complete topological space (X, t) into itself satisfying the following conditions :

(a) $A(X) \subset T(X)$ and $B(X) \subset S(X)$,

(b) $d(Ax, By) \leq h \max\{d(Ax, Sx), d(By, Ty), d(Sx, Ty)\}$ for all $x, y \in X$ where $0 \leq h < 1$,

(c) one of A, B, S, or T is ω-continuous,

(d) pairs A, S and B, T are compatible on X,

(e) d is a continuous symmetric.

Then A, B, S and T have a unique common fixed point in X.

The proof is similar to the proof of Corollary 4.

Note : If we change conditions (c) and (d) of Theorem 5 to S and T are ω-continuous and pairs A, S and B, T satisfy Hicks' and Saliga's definition of compatibility on X, we obtain the same result.

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