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FUZZY INTERIOR IDEALS IN SEMIGROUPS

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In this paper we investigate some properties of a fuzzy interior ideal of a semigroup, which is introduced by Kuroki[3]. We also consider fuzzy characteristic interior ideals.

The concept of fuzzy set, introduced by Zadeh[5], was applied to the elementary theory of groupoids and groups[4] and of semigroups[2, 3]. Kuroki[3] discussed fuzzy interior ideals of semigroups. In this paper we investigate some properties of fuzzy interior ideals, and discuss fuzzy characteristic interior ideals in semigroups.

Let S be a semigroup. By a subsemigroup of S we mean a nonempty subset A of S such that $A^2 \subseteq A$.

A fuzzy set δ in a semigroup S is a function from S into [0, 1]. For a fuzzy set δ in S and $t \in [0, 1]$, the set

$$\delta_t = \{x \in S : \delta(x) \geq t\}$$

is called a level subset of δ.

A fuzzy set δ in a semigroup S is called a fuzzy subsemigroup of S if $\delta(xy) \geq \min \{\delta(x), \delta(y)\}$ for all $x, y \in S$.

Lemma 1 Rosenfeld[4] — Let A be a nonempty subset of a semigroup S and δ_A the characteristic function of A. Then A is a subsemigroup of S if and only if δ_A is a fuzzy subsemigroup of S.

Definition 1 — A subsemigroup A of a semigroup S is called an interior ideal of S if $SAS \subseteq A$.

Definition 2 Kuroki[3] — A fuzzy subsemigroup δ of a semigroup S is called a fuzzy interior ideal of S if $\delta(xay) \geq \delta(a)$ for all x, a and y in S.

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The following theorem shows that the concept of a fuzzy interior ideal in a semigroup is an extension of an interior ideal.

Theorem 1 (Kuroki[3]) — Let A be a nonempty subset of a semigroup S. Then the following conditions are equivalent :

(i) A is an interior ideal of S.

(ii) The characteristic function δ_A of A is a fuzzy interior ideal of S.

Theorem 2 — If δ is a fuzzy interior ideal of a semigroup S, then the level subset δ_t of δ is an interior ideal of S for every $t \in [0, 1]$, when $\delta_t \neq \phi$.

PROOF : Assume that $\delta_t \neq \phi$. Let $a, b \in \delta_t$ for every $t \in [0, 1]$. Then $\delta(a) \geq t$ and $\delta(b) \geq t$, which imply that $\delta(ab) \geq \min \{\delta(a), \delta(b)\} \geq t$, so that $ab \in \delta_t$. Hence δ_t is a subsemigroup of S. Now let $x, y \in S$ and $a \in \delta_t$ for every $t \in [0, 1]$. Then $\delta(xay) \geq \delta(a) \geq t$, which implies that $xay \in \delta_t$. Consequently δ_t is an interior ideal of S. □

Definition 3 — Let δ be a fuzzy interior ideal of a semigroup S. The interior ideals $\delta_t, t \in [0, 1]$, are called level interior ideals of δ.

Theorem 3 — Let A be an interior ideal of a semigroup S. Then for every $t \in (0, 1)$, there exists a fuzzy interior ideal δ of S such that $\delta_t = A$.

PROOF : Let A be an interior ideal of S and let δ be a fuzzy set in S defined by

$$\delta(x) = \begin{cases} t, & \text{if } x \in A, \\ 0, & \text{if } x \notin A, \end{cases}$$

where t is a fixed number in $(0, 1)$. Let $x, y \in S$. If $x, y \in A$ then $xy \in A$. Hence $\delta(xy) = t = \min \{\delta(x), \delta(y)\}$. If $x, y \notin A$ then $\delta(x) = 0 = \delta(y)$, and so $\delta(xy) \geq \min \{\delta(x), \delta(y)\}$. If exactly one of x and y belongs to A then exactly one of $\delta(x)$ and $\delta(y)$ is equal to 0. So $\delta(xy) \geq \min \{\delta(x), \delta(y)\}$. Hence δ is a fuzzy subsemigroup of S. Now let x, a and y be any elements of S. If $a \in A$ then $xay \in A$. Thus $\delta(xay) = t = \delta(a)$. If $a \notin A$ then $\delta(a) = 0$, and so $\delta(xay) \geq 0 = \delta(a)$. Therefore δ is a fuzzy interior ideal of S, and clearly $\delta_t = A$. □

Theorem 4 — Suppose δ is a fuzzy interior ideal of a semigroup S. Then two level interior ideals $\delta_{t_1}, \delta_{t_2}$ (with $t_1 < t_2$) of δ are equal if and only if there is no $x \in S$ such that $t_1 \leq \delta(x) < t_2$.

PROOF : Assume that $\delta_{t_1} = \delta_{t_2}$ for $t_1 < t_2$ and that there exists $x \in S$ such that $t_1 \leq \delta(x) < t_2$. Then δ_{t_2} is a proper subset of δ_{t_1}. This is a contradiction.

Conversely suppose there is no $x \in S$ such that $t_1 \leq \delta(x) < t_2$. We have that $t_1 < t_2$ implies $\delta_{t_2} \subseteq \delta_{t_1}$. If $x \in \delta_{t_1}$ then $\delta(x) \geq t_1$. Since $\delta(x) \not< t_2$, we get $\delta(x) \geq t_2$ or $x \in \delta_{t_2}$. Thus $\delta_{t_1} = \delta_{t_2}$, completing the proof. □

Theorem 5 — Let δ be a fuzzy interior ideal of a semigroup S. If $\mathrm{Im}(\delta)$ is finite, then the family of interior ideals $\delta_t, t \in \mathrm{Im}(\delta)$, constitutes all the level interior ideals of δ.

PROOF : Suppose that $\mathrm{Im}(\delta)$ is finite. Since $\mathrm{Im}(\delta) \subseteq [0, 1]$, without loss of generality we may assume that $\mathrm{Im}(\delta) = \{t_1, t_2, ..., t_n\}$ where $t_1 < t_2 < .... < t_n$. Let $t \in [0, 1]$ and $t \notin \mathrm{Im}(\delta)$. If $t < t_1$ then $\delta_{t_1} \subseteq \delta_t$. Since $\delta_{t_1} = S$, we have that $\delta_t = S$ and $\delta_t = \delta_{t_1}$. If $t_i < t < t_{i+1}$ $(1 \le i \le n-1)$ then there is no $x \in S$ such that $t \le \delta(x) < t_{i+1}$. It follows from Theorem 4 that $\delta_t = \delta_{t_{i+1}}$. Hence the level interior ideal δ_t is in $\{\delta_{t_i} : i = 1, 2, ..., n\}$. The proof is complete. □

Lemma 2 — Let δ be a fuzzy interior ideal of a semigroup S. If s and t belong to $\mathrm{Im}(\delta)$ such that $\delta_s = \delta_t$, then $s = t$.

PROOF : Assume that $s \neq t$, say $s < t$. Then there is $x \in S$ such that $\delta(x) = s < t$, and so $x \in \delta_s$ and $x \notin \delta_t$. Thus $\delta_s \neq \delta_t$, a contradiction. This completes the proof. □

Theorem 6 — Let S be a semigroup and let γ and δ be two fuzzy interior ideals of S with identical family of level interior ideals. If $\mathrm{Im}(\gamma) = \{s_1, s_2, ..., s_n\}$ and $\mathrm{Im}(\delta) = \{t_1, t_2, ..., t_m\}$, where $s_1 > s_2 > ... > s_n$ and $t_1 > t_2 > ... > t_m$, then

(i) $m = n$;

(ii) $\gamma_{s_k} = \delta_{t_k}$, $k = 1, 2, ..., n$;

(iii) if $x \in S$ such that $\gamma(x) = s_k$ then $\delta(x) = t_k$ for $k = 1, 2, ..., n$.

PROOF : Using Theorem 5 we have that the only level interior ideals of γ and δ are γ_{s_k} and δ_{t_k}, respectively. Since γ and δ have the identical family of level interior ideals, it follows that $n = m$, so that (i) holds.

To prove (ii), using Theorem 5 again we get $\{\gamma_{s_1}, \gamma_{s_2}, ..., \gamma_{s_n}\} = \{\delta_{t_1}, \delta_{t_2}, ..., \delta_{t_n}\}$, and by Theorem 4 we have $\gamma_{s_1} \subset \gamma_{s_2} \subset ... \subset \gamma_{s_n} = S$ and $\delta_{t_1} \subset \delta_{t_2} \subset ... \subset \delta_{t_n} = S$. Hence $\gamma_{s_k} = \delta_{t_k}$ for $k = 1, 2, ..., n$ and (ii) holds. Now let $x \in S$ be such that $\gamma(x) = s_k$ and let $\delta(x) = t_j$. Noticing that $x \in \delta_{t_k}$, i.e., $\delta(x) \ge t_k$, we get $t_j \ge t_k$. Thus $\delta_{t_j} \subset \delta_{t_k}$. Since $x \in \delta_{t_j} = \gamma_{s_j}$, we obtain $s_k = \gamma(x) \ge s_j$. It follows that $\gamma_{s_k} \subset \gamma_{s_j}$. By (ii), $\delta_{t_k} = \gamma_{s_k} \subset \gamma_{s_j} = \delta_{t_j}$. Therefore $\delta_{t_k} = \delta_{t_j}$, and by Lemma 2 we have $t_k = t_j$. Hence $\delta(x) = t_k$. The proof is complete. □

Theorem 7 — Let γ and δ be two fuzzy interior ideals of a finite semigroup S such that the families of level interior ideals of γ and δ are identical. Then $\gamma = \delta$ if and only if $\mathrm{Im}(\gamma) = \mathrm{Im}(\delta)$.

PROOF : Necessity is obvious. Assume that $\mathrm{Im}(\gamma) = \mathrm{Im}(\delta) = \{t_1, ..., t_n\}$ where $t_1 > t_2 > ... > t_n$. Let $x_1, ..., x_n$ be distinct elements of S such that $\gamma(x_k) = t_k$ for $1 \le k \le n$. By Theorem 6(iii), $\delta(x_k) = t_k$ for $1 \le k \le n$. Since for any $x \in S$ there exists some t_k such that $\gamma(x) = t_k$, and so $x \in \gamma_{t_k} = \delta_{t_k}$. Hence $\delta(x) \ge t_k$, it follows that $\delta(x) \ge \gamma(x)$. By the same argument, we get $\gamma(x) \ge \delta(x)$. Therefore $\gamma(x) = \delta(x)$, which shows that $\gamma = \delta$. This completes the proof. □

Let S and T be semigroups. By a homomorphism we mean a mapping $f : S \to T$ satisfying the identity $f(xy) = f(x) f(y)$ for all $x, y \in S$. Throughout, Aut (S) will denote the set of all automorphisms of S.

Definition 4 — An interior ideal A of a semigroup S is called a characteristic interior ideal of S if $f(A) = A$ for all $f \in \mathrm{Aut}(S)$.

Definition 5 — A fuzzy interior ideal δ of a semigroup S is called a fuzzy characteristic interior ideal of S if $\delta(f(x)) = \delta(x)$ for all $x \in S$ and all $f \in \mathrm{Aut}(S)$.

Theorem 8 — If δ is a fuzzy characteristic interior ideal of a semigroup S, then each level interior ideal of δ is a characteristic interior ideal of S.

PROOF : Let $t \in \mathrm{Im}(\delta)$, $f \in \mathrm{Aut}(S)$ and $x \in \delta_t$. Since δ is a fuzzy characteristic interior ideal, we obtain $\delta(f(x)) = \delta(x) \geq t$. It follows that $f(x) \in \delta_t$ and hence $f(\delta_t) \subset \delta_t$. Now let $x \in \delta_t$ and $y \in S$ be such that $f(y) = x$. Then $\delta(y) = \delta(f(y)) = \delta(x) \geq t$, whence $y \in \delta_t$. It follows that $x = f(y) \in f(\delta_t)$, so that $\delta_t \subset f(\delta_t)$. Thus $\delta_t, t \in \mathrm{Im}(\delta)$, is a characteristic interior ideal of S. □

The following lemma is obvious and we omit the proof.

Lemma 3 — Let S be a semigroup and let $x \in S$. If δ is a fuzzy interior ideal of S, then $\delta(x) = t$ if and only if $x \in \delta_t$ and $x \notin \delta_s$ for all $s > t$.

Now we consider the converse of Theorem 8.

Theorem 9 — Let δ be a fuzzy interior ideal of a semigroup S. If each level interior ideal of δ is a characteristic interior ideal of S, then δ is a fuzzy characteristic interior ideal of S.

PROOF : Let $f \in \mathrm{Aut}(S)$ and let $x \in S$ be such that $\delta(x) = t$. Then $x \in \delta_t$ and $x \notin \delta_s$ for all $s > t$, by Lemma 3. Since $f(\delta_t) = \delta_t$ by hypothesis, we get $f(x) \in \delta_t$ and hence $\delta(f(x)) \geq t$. Let $s = \delta(f(x))$. If possible, let $s > t$. Then $f(x) \in \delta_s = f(\delta_s)$. Since f is one-one, it follows that $x \in \delta_s$, which is a contradiction. Hence $\delta(f(x)) = t = \delta(x)$, showing that δ is a fuzzy characteristic interior ideal of S. □

Lemma 4 — If δ is a fuzzy interior ideal of a semigroup S, then so is δ^α for every real number $\alpha \geq 0$, where δ^α is defined by $\delta^\alpha(x) = (\delta(x))^\alpha$ for all $x \in S$.

PROOF : Let x, a, $y \in S$ and let $\alpha \geq 0$ be any real number. If $\delta(x) \leq \delta(y)$, then $\delta(xy) \geq \delta(x)$ and $\delta(y) \geq \delta(x)$. This implies that $\delta^\alpha(xy) = (\delta(xy))^\alpha \geq (\delta(x))^\alpha = \delta^\alpha(x)$ and $\delta^\alpha(y) = (\delta(y))^\alpha \geq (\delta(x))^\alpha = \delta^\alpha(x)$. Hence $\delta^\alpha(xy) \geq \min\{\delta^\alpha(x), \delta^\alpha(y)\}$. The argument is similar if $\delta(x) \geq \delta(y)$. Therefore δ^α is a fuzzy subsemigroup of S. Since $\delta(xay) \geq \delta(a)$, clearly $\delta^\alpha(xay) = (\delta(xay))^\alpha \geq (\delta(a))^\alpha = \delta^\alpha(a)$. This completes the proof. □

Let δ be a fuzzy characteristic interior ideal of a semigroup S. Let $x \in S$ and $f \in \mathrm{Aut}(S)$. Then $\delta(f(x)) = \delta(x)$, which implies that $\delta^\alpha(f(x)) = (\delta(f(x)))^\alpha = (\delta(x))^\alpha = \delta^\alpha(x)$ for every real number $\alpha \geq 0$. Thus we have the following theorem.

Theorem 10 — If δ is a fuzzy characteristic interior ideal of a semigroup S, then δ^α is also a fuzzy characteristic interior ideal of S for every real number $\alpha \geq 0$.

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