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MAINTENANCE OF OSCILLATION OF NEUTRAL DIFFERENTIAL EQUATIONS UNDER THE EFFECT OF A FORCING TERM

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Sufficient conditions have been obtained under which all solutions of nth order ($n \geq 1$) neutral differential equation (NDE)

$$\left[x(t) + \sum_{i=1}^{s} p_i(t)\, x(r_i(t))\right]^{(n)} + \sum_{j=1}^{m} q_j(t)\, h_j(x(\sigma_j(t))) = 0$$

oscillate or tend to zero as $t \to \infty$, and all solutions of the associated forced equation

$$\left[x(t) + \sum_{i=1}^{s} p_i(t)\, x(r_i(t))\right]^{(n)} + \sum_{j=1}^{m} q_j(t)\, h_j(x(\sigma_j(t))) = f(t)$$

oscillate or tend to zero as $t \to \infty$, when $\lim_{t \to \infty} F(t) = 0$ or $F(t)$ is τ-periodic with $r(t) = t - \tau$, $\tau > 0$, where F is an n-times continuously differentiable function such that $F^{(n)}(t) = f(t)$.

1. INTRODUCTION

Consider nth order ($n \geq 1$) nonlinear neutral delay differential equations of the type

$$\left[x(t) + \sum_{i=1}^{s} p_i(t)\, x(r_i(t))\right]^{(n)} + \sum_{j=1}^{m} q_j(t)\, h_j(x(\sigma_j(t))) = 0 \qquad \ldots (1)$$

and

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$$\left[x(t) + \sum_{i=1}^{s} p_i(t)\, x(r_i(t))\right]^{(n)} + \sum_{j=1}^{m} q_j(t)\, h_j(x(\sigma_j(t))) = f(t) \qquad \ldots (2)$$

where the following conditions are assumed to hold throughout this work :

(i) $p_i, q_j, r_i, \sigma_j \in C([\sigma_0, \infty), R)$, $\sigma_0 \in R$, $r_i(t) \le t$, $\lim_{t\to\infty} r_i(t) = \infty$, $\sigma_j(t) \le t$ and $\lim_{t\to\infty} \sigma_j(t) = \infty$, $i = 1, \ldots, s$, $j = 1, \ldots, m$;

(ii) $h_j \in C(R, R)$, $h_j(u)$ is nondecreasing and $uh_j(u) > 0$ if $u \neq 0$ for each $j = 1, \ldots, m$;

(iii) $f \in C([\sigma_0, \infty), R)$ and there exists $F \in C^{(n)}([\sigma_0, \infty), R)$ such that $F^{(n)}(t) = f(t)$.

The object of this work is to obtain sufficient conditions for oscillatory and strongly monotone behaviour of solutions of the homogeneous equation (1), and then to show that the behaviour of solutions of (1) is maintained under the influence of certain classes of forcing functions. We restrict our attention here to sublinear equations, viz, the cases where each h_j statisfies the condition

(A_1) $\int_0^c \frac{du}{h_j(u)} < \infty$ for every $c > 0$.

Such problems for delay-differential equations with a single variable delay have been studied by Kusano and Onose[4] and Foster[1] with sublinear conditions stronger than (A_1). On the other hand, Graef *et al.*[2] and Jaros and Kusano[3] have studied this problem for the homogeneous equation (1) with $n = 1$, while Parhi and Mohanty[6] have studied the nonhomogeneous equation (2) with $n = 2$.

The following assumptions are made for our use in the sequel :

(A_2) $P_i(t) \ge 0$, $i = 1, \ldots, s$, and $\sum_{i=1}^{s} p_i(t) \le p_1 < 1$, where p_1 is a constant;

(A_3) $q_j(t) \ge 0$, $j = 1, \ldots, m$, and

$$\int_{\sigma_0}^{\infty} q_j(t)\, h_j((\sigma_j(t))^{n-1})\, dt = \infty \text{ for some } j \in \{1, \ldots, m\};$$

(A_4) $h_{j_0}(uv) \ge h_{j_0}(u)\, h_{j_0}(v)$ for all $u > 0$, v large and some $j_0 \in \{1, \ldots, m\}$;

(A_5) $h_j(-u) = -h_j(u)$, $j = 1, \ldots, m$.

In what follows, by a solution of (1) (or (2)) on $[T_x, \infty)$, $T_x \ge \sigma_0$, we mean that $x : (-\infty, \infty) \to R$ is a continuous function such that $x(t) + \sum_{i=1}^{s} p_i(t)\, x(r_i(t))$ is n-times continuously differentiable and eqn. (1) (or (2)) is satisfied for $t \ge T_x$. Such a solution is said to be oscillatory if it has arbitrarily large zeros in $[T_x, \infty)$, and it is said to be non-oscillatory otherwise.

2. UNFORCED OSCILLATION

In this section we study oscillation of solutions of (1). The following lemma due to Kiguradze[5] (Lemma 5.2.1) will be used in the sequel.

Lemma 2.1 — Let $u(t)$ together with its first $(n-1)$ derivatives be continuous and of constant sign on the interval $[t_1, \infty)$, $t_1 \geq 0$, and $u(t)\, u^{(n)}(t) \leq 0$. Then there exists an integer l, $0 \leq l \leq n-1$, which is odd if n is even and is even if n is odd, such that for $t \geq t_1$

(i) $u(t)\, u^{(k)}(t) > 0$, $k = 0, 1, \ldots, l$ and

$$(-1)^{l+k}\, u(t)\, u^{(k)}(t) > 0, \quad k = l+1, \ldots, n-1;$$

and

(ii) $$|u(t)| \geq \frac{(t-t_1)^{n-1}}{(n-1)\ldots(n-l)} |u^{(n-1)}(2^{n-l-1}t)|.$$

Theorem 2.2 — Let the assumptions (A_1)-(A_5) hold. Then every solution of (1) oscillates if n is even and oscillates or tends to zero as $t \to \infty$ if n is odd.

PROOF : Let $x(t)$ be a nonoscillatory solution of (1) on $[T_x, \infty)$. We may assume, without any loss of generality, that $x(t) > 0$ for $t \geq t_0 > \max\{0, T_x\}$. So there is a $t_1 > t_0$ such that $x(r_i(t)) > 0$ and $x(\sigma_j(t)) > 0$, $i = 1, \ldots, s$, $j = 1, \ldots, m$, for $t \geq t_1$. Setting, for $t \geq t_1$,

$$z(t) = x(t) + \sum_{i=1}^{s} p_i(t)\, x(r_i(t)) \qquad \ldots (3)$$

we obtain $z(t) \geq x(t) > 0$ and

$$z^{(n)}(t) = -\sum_{j=1}^{m} q_j(t)\, h_j(x(\sigma_j(t))) \leq 0. \qquad \ldots (4)$$

Thus $z^{(n-1)}(t) > 0$ for large t. From Lemma 2.1, it follows that there exists an integer l, $0 \leq l \leq n-1$, which is odd if n is even and is even if n is odd, such that, for $t \geq t_1$,

$$z^{(k)}(t) > 0, \quad k = 0, 1, \ldots, l, \ (-1)^{l+k} z^{(k)}(t) > 0, \ k = l+1, \ldots, n-1$$

and

$$z(t) \geq \frac{(t-t_1)^{n-1}}{(n-1)\ldots(n-l)} z^{(n-1)}(2^{n-l-1}t). \qquad \ldots (5)$$

Let n be even. Hence $z'(t) > 0$ for $t \geq t_1$. From (3) we get, for $t \geq t_2 > t_1$,

$$z(t) - \sum_{i=1}^{s} p_i(t)\, z(r_i(t)) \leq x(t). \qquad \ldots (6)$$

So

$$0 < (1-p_1)z(t) \le \left(1 - \sum_{j=1}^{s} p_j(t)\right) z(t) \le x(t).$$

But (5) yields, for $t \ge t_3 > t_2\, 2^{n-l-1}$,

$$\begin{aligned} z(t) &\ge z(2^{l+1-n}\, t) \\ &\ge \frac{(2^{l+1-n}\, t - t_1)^{n-1}}{(n-1)\ldots(n-l)}\, z^{(n-1)}(t) \\ &\ge k_1\, t^{n-1}\, z^{(n-1)}(t) \end{aligned}$$

where $0 < k_1 < 1$. Thus, for $t \ge t_3$,

$$x(t) \ge k_1\,(1-p_1)\, t^{n-1}\, z^{(n-1)}(t).$$

Now there exists a $t_4 > t_3$ such that, for $t \ge t_4$,

$$x(\sigma_{j_0}(t)) \ge k_2\,(\sigma_{j_0}(t))^{n-1}\, z^{(n-1)}(\sigma_{j_0}(t)) \ge k_2\,(\sigma_{j_0}(t))^{n-1}\, z^{(n-1)}(t)$$

$j = j_0 \in \{1, \ldots, m\}$, where $0 < (1-p_1)\, k_1 = k_2 < 1$. Hence, by using the properties of h_{j_0}, from (4), we obtain that

$$z^{(n)}(t) + q_{j_0}(t)\, h_{j_0}(k_2\,(\sigma_{j_0}(t))^{n-1}\, z^{(n-1)}(t)) \le 0$$

for $t \ge t_4$, that is,

$$z^{(n)}(t) + q_{j_0}(t)\, h_{j_0}((\sigma_{j_0}(t))^{n-1})\, h_{j_0}(k_2\, z^{(n-1)}(t)) \le 0.$$

An integration of the above inequality yields

$$\begin{aligned} \int_{t_4}^{t} q_{j_0}(\theta)\, h_{j_0}((\sigma_{j_0}(\theta))^{n-1})\, d\theta &\le -\int_{t_4}^{t} \frac{z^{(n)}(\theta)}{h_{j_0}(k_2\, z^{(n-1)}(\theta))}\, d\theta \\ &\le \frac{1}{k_2} \int_{k_2 z^{(n-1)}(t)}^{k_2 z^{(n-1)}(t_4)} \frac{du}{h_{j_0}(u)}. \end{aligned}$$

We may note that $0 \le \lim_{t \to \infty} z^{(n-1)}(t) \le z^{(n-1)}(t_4)$. Thus, letting $t \to \infty$ and using (A_1), we obtain

$$\int_{t_4}^{\infty} q_{j_0}(\theta)\, h_{j_0}((\sigma_{j_0}(\theta))^{n-1})\, d\theta < \infty$$

which contradicts (A_3). Hence $x(t)$ is oscillatory.

Let n be odd. Then $l \ge 0$ is even. If $l > 0$, then $z'(t) > 0$ for $t \ge t_1$ and one may proceed as above to obtain the required contradiction. If $l = 0$, then $z'(t) < 0$ for

$t \geq t_1$. Furthermore, $z'(t) \leq 0$ for $t \geq t_1$ when $n = 1$. Consequently, $\lim_{t \to \infty} z(t) = \lambda$, $0 \leq \lambda < \infty$. Suppose $\lambda > 0$. Then, for $0 < \eta < (1-p_1)\lambda/2(1+p_1)$, there exists a $t_5 > t_1$ such that

$$\lambda - \eta < z(t) < z(r_i(t)) < \lambda + \eta, \quad i = 1, \ldots, s$$

for $t \geq t_5$. Thus (6) yields, for $t \geq t_5$, that $x(t) > \lambda - \eta - p_1(\lambda + \eta) = \lambda(1-p_1) - \eta(1+p_1) > 0$, that is,

$$x(t) > \lambda_1 z(t) \qquad \ldots (7)$$

where $\lambda_1 = \{\lambda(1-p_1) - \eta(1+p_1)\} / (\lambda + \eta)$. Furthermore, for $t \geq t_6 > t_5 2^{n-l-1}$,

$$z(t) = \frac{z(t)}{z(2^{l+1-n} t)} z(2^{l+1-n} t) > \lambda_2 z(2^{l+1-n} t)$$

where $0 < \lambda_2 = (\lambda - \eta)/(\lambda + \eta) < 1$. Thus, (5) yields

$$z(t) > \frac{\lambda_2 (2^{l+1-n} t - t_1)^{n-1}}{(n-1) \ldots (n-l)} z^{(n-1)}(t)$$

$$> \lambda_3 t^{n-1} z^{(n-1)}(t) \qquad \ldots (8)$$

for $t \geq t_6$, where $0 < \lambda_3 < 1$. Consequently, from (7) and (8) for $t \geq t_7 > t_6$ we obtain,

$$x(\sigma_j(t)) > \lambda_1 \lambda_3 (\sigma_j(t))^{n-1} z^{(n-1)}(\sigma_j(t))$$

$$> \lambda_1 \lambda_3 (\sigma_j(t))^{n-1} z^{(n-1)}(t).$$

Then proceeding as in the case, when n is even we arrive at a contradiction. If $\lambda = 0$, then $z(t) \geq x(t)$ implies that $\lim_{t \to \infty} x(t) = 0$. This completes the proof of the theorem.

Remark 1 : Clearly, Theorem 2.2 does not hold for functions $p_i(t)$ such that $\limsup_{t \to \infty} \sum_{i=1}^{s} p_i(t) = 1$. To accommodate this case the condition (A_2) is weakened to

$$(A_2') \quad p_i(t) \geq 0, \ i = 1, \ldots, s \text{ and } \sum_{i=1}^{s} p_i(t) < 1$$

and the condition (A_3) is strengthened to

$$(A_3') \quad q_j(t) \geq 0, \ j = 1, \ldots, m, \text{ and}$$

$$\int_{\sigma_0}^{\infty} q_j(t) h_j\left(\left(1 - \sum_{i=1}^{s} p_i(\sigma_j(t))\right)(\sigma_j(t))^{n-1}\right) dt = \infty$$

for some $j = j_0 \in \{1, \ldots, m\}$.

Indeed, we have the following theorem.

Theorem 2.3 — Let (A_1), (A_2'), (A_3'), (A_4) and (A_5) hold. Then every solution of (1) oscillates if n is even.

The proof is omitted as it is similar to that of Theorem 2.2.

Remark 2 : We may note that, for $n = 1$, (A_3) reduces to (A_6) $q_j(t) \geq 0$, $j = 1, \ldots, m$ such that $\int_{\sigma_0}^{\infty} q_j(t)\,dt = \infty$ for some $j = j_0 \in \{1, \ldots, m\}$. For $n \geq 2$, (A_6) implies (A_3). The assumptions (A_4) and (A_5) may be dropped if we assume (A_6) and (A_1) is replaced by

$$(A_1') \quad \int_0^c \frac{du}{h_j(u)} < \infty \text{ and } \int_0^{-c} \frac{du}{h_j(u)} < \infty$$

for every $c > 0$ and $j = 1, \ldots, m$. Indeed, we have the following result.

Theorem 2.4 — Suppose that (A_1'), (A_2) and (A_6) hold. Then every solution of (1) oscillates if n is even and oscillates or tends for zero as $t \to \infty$ if n is odd.

PROOF : For n even, we may proceed as in Theorem 2.2 to obtain

$$x(t) \geq k_1 (1 - p_1)\, t^{n-1}\, z^{(n-1)}(t),$$

where $0 < k_1 < 1$. For sufficiently large t, we have $x(t) > z^{(n-1)}(t)$. The rest of the proof is similar to that of Theorem 2.2.

For $n > 2$ and odd, we have as in Theorem 2.2,

$$x(t) > \lambda_1 \lambda_3\, t^{n-1}\, z^{(n-1)}(t) > z^{(n-1)}(t)$$

for large t, where $0 < \lambda_1, \lambda_3 < 1$. Then following the proof of Theorem 2.2, the required contradiction is obtained.

3. FORCED OSCILLATION

The purpose of this section is to study the effect of a small or periodic forcing term on oscillatory behaviour of solutions of (2). We may note that $\lim_{t \to \infty} f(t) = 0$ if $\lim_{t \to \infty} F(t) = 0$ and $f(t)$ is τ-periodic when $F(t)$ is τ-periodic.

Theorem 3.1 — In addition to the hypotheses of Theorem 2.2. assume that

(A_7) $F(t)$ is oscillatory and $\lim_{t \to \infty} F(t) = 0$.

Then every solution of (2) oscillates if n is even and oscillates or tends to zero as $t \to \infty$ if n is odd.

PROOF : Suppose that $x(t)$ is an eventually positive solution of (2). Then there exists a $t_0 > \max\{0, T_x\}$ such that $x(t) > 0$, $x(r_i(t)) > 0$, $i = 1, \ldots, s$ and $x(\sigma_j(t)) > 0$, $j = 1, \ldots, m$, for $t \geq t_0$. The case when $x(t)$ is eventually negative is similar.

For $t \geq t_0$, set

$$z(t) = x(t) + \sum_{i=1}^{s} p_i(t)\, x(r_i(t)) - F(t). \qquad \ldots (9)$$

Thus

$$z^{(n)}(t) = -\sum_{j=1}^{m} q_j(t)\, h_j(x(\sigma_j(t))) \leq 0. \qquad \ldots (10)$$

Consequently, $z(t) > 0$ or < 0 for $t \geq t_1 > t_0$. But $z(t) < 0$ implies that $0 < x(t) < F(t)$ for $t \geq t_1$, which is not possible in view of oscillation of the $F(t)$. Thus $z(t) > 0$ for $t \geq t_1$ and hence $z^{(n-1)}(t) > 0$ for large t. From Lemma 2.1 it follows that there is an integer l, $0 \leq l \leq n-1$, which is odd if n is even and is even if n is odd such that

$$z^{(k)}(t) > 0,\ k = 0, 1, \ldots, l,\ (-1)^{l+k} z^{(k)}(t) > 0,\ k = l+1, \ldots, n-1 \qquad \ldots (11)$$

and

$$z(t) \geq \frac{(t-t_1)^{n-1}}{(n-1)\ldots(n-l)}\, z^{(n-1)}(2^{n-l-1}\, t) \qquad \ldots (12)$$

for $t \geq t_1$.

Let n be even. Then $z'(t) > 0$ for $t \geq t_1$. Clearly, for $0 < \varepsilon < (1-p_1)\, z(t_1)$, there exists a $t_2 > t_1$ such that $|F(t)| < \varepsilon/2$ for $t \geq t_2$. From (9) we get $x(t) \leq z(t) + F(t)$ and hence

$$z(t) - \sum_{i=1}^{s} p_i(t)\, z(r_i(t)) \leq x(t) - F(t) + \sum_{i=1}^{s} p_i(t)\, F(r_i(t))$$

$$< x(t) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \sum_{i=1}^{s} p_i(t), \qquad \ldots (13)$$

that is,

$$0 < (1-p_1)\, z(t_1) - \varepsilon < (1-p_1)\, z(t) - \varepsilon < x(t)$$

for $t \geq t_3 > t_2$. Setting $y(t) = (1-p_1)\, z(t) - \varepsilon$ for $t \geq t_3$, we have $0 < y(t) < x(t)$, $y'(t) > 0$ and

$$y^{(n)}(t) = -(1-p_1) \sum_{j=1}^{m} q_j(t)\, h_j(x(\sigma_j(t))) \leq 0.$$

Then proceeding as in the proof of Theorem 2.2, we again obtain a contradiction, and so $x(t)$ is oscillatory.

Let n be odd. Then $l \geq 0$ is even. If $l > 0$, then the required contradiction may be obtained by proceeding as above. If $l = 0$, then $z'(t) < 0$ for $t \geq t_1$ by (11), and if $n = 1$, then $z'(t) \leq 0$ for $t \geq t_1$. Thus $\lim_{t \to \infty} z(t) = \alpha$, $0 \leq \alpha < \infty$. If $\alpha > 0$, then we choose $0 < \varepsilon < (1-p_1)\alpha/2$ such that $| F(t) | < \varepsilon/2$ for $t \geq t_4 > t_1$. For $0 < \delta < \varepsilon/(1+p_1)$, there exists a $t_5 > t_4$ such that

$$\alpha - \delta < z(t) \leq z(r_i(t)) < \alpha + \delta$$

for $t \geq t_5$. Thus (13) yields that

$$\begin{aligned} x(t) &> z(t) - \sum_{i=1}^{s} p_i(t)\, z(r_i(t)) - \varepsilon \\ &> (\alpha - \delta) - p_1(\alpha + \delta) - \varepsilon \\ &> (1-p_1)\alpha - \delta(1+p_1) - \varepsilon = \alpha_1 > 0. \end{aligned}$$

Hence $x(t) > (\alpha_1/(\alpha+\delta))\, z(t)$ for $t \geq t_5$. The rest of the proof is similar to that of Theorem 2.2 and hence is omitted. If $\alpha = 0$, then $x(t) \leq z(t) + F(t)$ implies that $\lim_{t \to \infty} x(t) = 0$.

Hence the theorem is proved.

Theorem 3.2 — Let the assumptions of Theorem 2.4 and (A_7) hold. Then the conclusions of Theorem 3.1 remain true.

The proof is similar to that of Theorem 3.1 and is omitted.

Theorem 3.3 — If the conditions of Theorem 2.2 (Theorem 2.4) are satisfied and $\lim_{t \to \infty} F(t) = 0$, then every solution of (2) oscillates or tends to 0 as $t \to \infty$.

PROOF : The proof for the case $z(t) > 0$ for large t is same as that of Theorem 3.1 (Theorem 3.2). The case $z(t) < 0$ for large t yields $x(t) < F(t)$ eventually and hence $\lim_{t \to \infty} x(t) = 0$.

Next consider the equation

$$[x(t) + p(t)\, x(t-\tau)]^{(n)} + \sum_{j=1}^{m} q_j(t)\, h_j(x(\sigma_j(t))) = f(t) \qquad \ldots (2^*)$$

where $p(t)$ is a real valued continuous function on $[\sigma_0, \infty)$, $\tau > 0$ is a constant and $q_j(t)$, $\sigma_j(t)$, $h_j(u)$ and $f(t)$ are as defined earlier.

Theorem 3.4 — Suppose that the assumptions (A_1), (A_3), (A_4) and (A_5) hold. Let $0 \leq p(t) \leq p_2 < 1$ for some constant p_2 and $F(t)$ be τ-periodic. Then the following statements hold :

(i) Every solution of (2^*) oscillates if n is even;

(ii) if $\lim_{t \to \infty} \inf F(t) = d_1 < 0$, $\lim_{t \to \infty} \sup F(t) = d_2 > 0$,

and n is odd, then every solution of (2*) either (a) oscillates or (b) satisfies $\liminf_{t\to\infty} x(t) = 0$ and

$$\frac{d_2}{1+p_2} \le \limsup_{t\to\infty} x(t) \le \frac{d_2 - d_1}{1-p_2}.$$

PROOF : Let $x(t)$ be a nonoscillatory solution of (2*) on $[T_x, \infty)$ and assume that $x(t) > 0$ eventually. The case $x(t) < 0$ eventually is similar. Thus $x(t) > 0$, $x(t-\tau) > 0$ and $x(\sigma_j(t)) > 0$ for $t \ge t_0 > \max\{0, T_x\}$. For $t \ge t_0$, letting

$$z(t) = x(t) + p(t)\, x(t-\tau) - F(t) \qquad \ldots (14)$$

we obtain (10). Hence $z(t) > 0$ or < 0 for $t \ge t_1 > t_0$.

We claim that $x(t)$ is bounded. If not, then $x(t)$ is unbounded. Since $0 < x(t) \le z(t) + F(t)$ and $F(t)$ is bounded, $z(t)$ is unbounded. Further, $z(t) < 0$ implies that $x(t) < F(t)$ and hence $x(t)$ is bounded, a contradiction. Thus $z(t) > 0$ for $t \ge t_1$. For $n = 1$, we obtain from (10) that $z'(t) \le 0$, $t \ge t_1$. Hence $z(t)$ is bounded a contradiction. Thus our claim holds, that is, $x(t)$ is bounded for $n = 1$. Let $n \ge 2$. Clearly, $z'(t) > 0$ for large t, because $z'(t) < 0$ eventually implies that $z(t)$ is bounded, which is a contradiction. From (14) we have, for $t \ge t_2 > t_1$,

$$z(t) - p(t)\, z(t-\tau) = x(t) - F(t) - p(t)\, p(t-\tau)\, x(t-2\tau) + p(t)\, F(t-\tau)$$

that is,

$$(1 - p(t))\, z(t) \le x(t) - (1-p(t))\, F(t)$$

or,

$$0 < (1-p_2)\,(z(t) + F(t)) \le x(t). \qquad \ldots (15)$$

Since $F(t)$ is τ-periodic, there exist two real numbers c_1 and c_2 and two increasing sequences $\langle t_r' \rangle$ and $\langle t_r'' \rangle \subset [t_2, \infty)$ such that $\lim_{r\to\infty} t_r' = \infty = \lim_{r\to\infty} t_r''$, $F(t_r') = c_1$, $F(t_r'') = c_2$ and $c_1 \le F(t) \le c_2$. Hence for $t \ge t_r'$, $r \ge 1$,

$$z(t) + c_1 \ge z(t_r') + c_1 = z(t_r') + F(t_r') \ge x(t_r') > 0.$$

Consequently, for $t \ge t_r'$,

$$0 < (1-p_2)\,(z(t) + c_1) \le (1-p_2)\,(z(t) + F(t)) \le x(t)$$

by (15). Setting $y(t) = (1-p_2)\,(z(t) + c_1)$ for $t \ge t_r'$, any $r \ge 1$, we obtain $0 < y(t) \le x(t)$, $y'(t) > 0$ and

$$y^{(n)}(t) = -(1-p_2) \sum_{j=1}^{m} q_j(t)\, h_j(x(\sigma_j(t))) \le 0.$$

Now applying Lemma 2.1 and proceeding as in Theorem 2.2 we arrive at a contradiction. Thus our claim holds, that is, $x(t)$ is bounded for $n \ge 2$. This in turn

implies that $z(t)$ is bounded. Consequently (10) implies

$$(-1)^{n+k} z^{(k)}(t) < 0, \quad k = 1, 2, \ldots, n-1 \qquad \ldots (16)$$

for $t \geq t_3 > t_1$.

If n is even, then from (16) it follows that $z'(t) > 0$ for $t \geq t_3$. We may note that the derivation of (15) is independent of the sign of $z(t)$ because $z(t) + F(t) > 0$. Then proceeding as above we arrive at a contradiction. Hence $x(t)$ has to be oscillatory. Let n be odd. For $n = 1$, (10) yields that $z'(t) \leq 0$ for $t \geq t_0$. For $n > 2$, $z'(t) < 0$ for $t \geq t_3 > t_1$ due to (16). Now $z(t) < 0$ for $t \geq t_1$ implies that $0 < x(t) \leq F(t)$ and so $0 \leq \liminf_{t \to \infty} F(t) = d_1 < 0$, which is a contradiction. Thus $z(t) > 0$ for $t \geq t_1$, and so $\lim_{t \to \infty} z(t) = \beta$, $0 \leq \beta < \infty$. If $\beta = 0$, then, for $0 < \eta < -d_1$, there exists a $t_4 > t_3$ such that $z(t) < \eta$ for $t \geq t_4$. From (14) we have

$$0 < x(t_r') \leq z(t_r') + F(t_r') < \eta + c_1 \leq \eta + d_1 < 0$$

for $t_r' > t_4$, which is again a contradiction. Thus $\beta > 0$. Consequently, for $0 < \varepsilon < \beta$, we may find $t_5 > t_4$ such that $\beta - \varepsilon < z(t) < z(t-\tau) < \beta + \varepsilon$ for $t \geq t_5$. Since $x(t)$ is bounded, $\liminf_{t \to \infty} x(t) = b$, $0 \leq b < \infty$. We claim that $b = 0$. If not, $x(t) > b - \varepsilon' > 0$ for $t \geq t_6 > t_5$ for some $t_6 > t_5$. Then $x(t) > \alpha z(t)$ for $t \geq t_6$, where $\alpha = (b - \varepsilon')/(\beta + \varepsilon)$. Proceeding as in Theorem 2.2 when n is odd, one may arrive at a contradiction. Thus our claim holds. Hence, from (14) we obtain

$$\begin{aligned}
\limsup_{t \to \infty} (x(t) - F(t)) &\leq \lim_{t \to \infty} z(t) \\
&\leq \liminf_{t \to \infty} (x(t) + p(t)\, x(t-\tau) - F(t)) \\
&\leq \liminf_{t \to \infty} x(t) + \limsup_{t \to \infty} (p(t)\, x(t-\tau) - F(t)) \\
&\leq p_2 \limsup_{t \to \infty} x(t-\tau) - \liminf_{t \to \infty} F(t),
\end{aligned}$$

that is,

$$\begin{aligned}
\limsup_{t \to \infty} x(t) + \liminf_{t \to \infty} (-F(t)) &\leq \limsup_{t \to \infty} (x(t) - F(t)) \\
&\leq p_2 \limsup_{t \to \infty} x(t) - d_1,
\end{aligned}$$

or $$(1 - p_2) \limsup_{t \to \infty} x(t) \leq d_2 - d_1.$$

On the other hand,

$$\begin{aligned}
\limsup_{t \to \infty} [x(t) + p(t)\, x(t-\tau)] &\leq \limsup_{t \to \infty} x(t) + p_2 \limsup_{t \to \infty} x(t-\tau) \\
&\leq (1 + p_2) \limsup_{t \to \infty} x(t)
\end{aligned}$$

and

$$\limsup_{t\to\infty}[x(t)+p(t)\,x(t-\tau)] = \limsup_{t\to\infty}[z(t)+F(t)]$$

$$\geq \liminf_{t\to\infty} z(t) + \limsup_{t\to\infty} F(t)$$

$$\geq d_2$$

imply that $(1+p_2)\limsup_{t\to\infty} x(t) \geq d_2$. Thus,

$$\frac{d_2}{1+p_2} \leq \limsup_{t\to\infty} x(t) \leq \frac{d_2-d_1}{1-p_2}$$

as desired. Hence the theorem is proved.

Theorem 3.5 — Let (A_1') and (A_6) hold, let $0 \leq p(t) \leq p_2 < 1$, for some constant p_2, and let $F(t)$ be τ-periodic. Then the conclusions of Theorem 3.4 hold true.

The proof is similar to that of Theorem 3.4 and is omitted.

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