SOME FIXED POINT THEOREMS IN
METRIC SPACES

NIKOLA JOTIC

Institute of Mathematics, Knez Mihaila 35, 11000 Belgrade Serbia,
Yugoslavia

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Let \((X, d)\) be a metric space and \(f\) a self-mapping which satisfies the condition (12) below with variable coefficients. If coefficients satisfy the conditions (13)-(15) below and \(X\) is complete, then \(f\) has a unique fixed point. This result generalizes corresponding results of Ćirić, Hardy and Rogers, Hicks and Rhoades, Ivanov, Pal and Maiti, Zamfirescu and many other known generalizations of Banach contraction principle.

In this paper we shall prove three theorems which contain many known theorems as special cases.

**Theorem 1** — Let \((X, d)\) be a complete metric space, \(f : X \rightarrow X\) a self-mapping and \(\varphi : [0, \infty) \rightarrow [0, \infty)\) a real function such that

\[
\varphi(r) < r \quad \text{for} \quad r > 0, \quad \ldots \quad (1)
\]

\[
\lim_{t \rightarrow r^+} \sup \varphi(t) < r \quad \text{for} \quad r > 0. \quad \ldots \quad (2)
\]

If \(f : X \rightarrow X\) satisfies the condition

\[
d[f(x), f^2(x)] \leq \varphi[d(x, f(x))] \quad \ldots \quad (3)
\]

for every \(x \in X\), then \(\{f^n(x)\}\) is a Cauchy sequence. Furthermore, if \(X\) is complete and if a mapping \(G(x) = d(x, f(x))\) is lower semi-continuous at a limit point of \(\{f^n(x)\}\), say \(x^*\), then \(x^*\) is a fixed point of \(f\).

**PROOF** : Let \(x_0 \in X\) be arbitrary \(X\) and define

\[
a_n = d[f^n(x), f^{n+1}(x)]. \quad \ldots \quad (4)
\]

We may assume that \(a_n = 0\) for all \(n \in \mathbb{N}\), since otherwise the assertion of the theorem trivially holds. Then by (1) and (3) we get

\[
a_{n+1} = d(f^{n+1}(x), f^{n+2}(x)) \leq \varphi(d(f^n(x), f^{n+1}(x))) < a_n.
\]
Therefore \( \{a_n\} \) decreases and hence has a limit \( a \geq 0 \). If we suppose that \( a > 0 \), then from
\[
a_{n+1} \leq \varphi(a_n) < a \tag{5}
\]
it follows that, by (2), that we get
\[
a \leq \limsup_{t \to a^+} \varphi(t) < a \tag{6}
\]
which is a contradiction. Therefore \( a = 0 \), that is
\[
\lim_{n \to \infty} d(f^n(x), f^{n+1}(x)) = \lim_{n \to \infty} a_n = 0. \tag{7}
\]

Now we shall show that \( \{f^n(x)\} \) is a Cauchy sequence. Suppose that \( \{f^n(x)\} \) is not a Cauchy sequence. Then there exist an \( \varepsilon > 0 \) and a sequence of integers \( m_k, n_k \), with \( m_k > n_k \geq k \), and such that
\[
h_k = d[f^{m_k}(x), f^{n_k}(x)] \geq \varepsilon \tag{8}
\]
for \( k = 1, 2, \ldots \).

We may assume that
\[
d[f^{m_k-1}(x), f^{n_k}(x)] < \varepsilon, \tag{9}
\]
by choosing \( m_k \) to be the smallest number exceeding \( n_k \) for which (8) holds. Now recalling (4), we have
\[
h_k = d(f^{m_k}(x), f^{n_k}(x)) \leq d(f^{m_k}(x), f^{m_k-1}(x)) + d(f^{m_k-1}(x), f^{n_k}(x))
\]
\[
\leq a_{m_k-1} + \varepsilon \leq a_{n_k} + \varepsilon. \tag{10}
\]
Hence, by (7), \( h_k \to \varepsilon + \) as \( k \to \infty \). But now using the triangle inequality, (3), (4), (5) and (8), we have
\[
h_k = d(f^{m_k}(x), f^{n_k}(x)) \leq d(f^{m_k}(x), f^{m_k+1}(x)) + d(f^{m_k+1}(x), f^{n_k+1}(x))
\]
\[
+ d(f^{n_k+1}(x), f^{n_k}(x))
\]
\[
= a_{m_k} + d(f(f^{m_k}(x)), f(f^{n_k}(x))) + a_{n_k} \leq 2a_{n_k} + \varphi(h_k). \tag{11}
\]
Hence, using (2), (7) and (10), we get
\[
\varepsilon \leq \limsup_{t \to \varepsilon^+} \varphi(t) < \varepsilon
\]
which is a contradiction. Therefore, \( \{f^n(x)\} \) is a Cauchy sequence.

To prove the rest of the theorem, assume that \( X \) is complete. Then
\[
\lim_{n \to \infty} f^n(x) = x^* \text{ for some } x^* \in X, \text{ as } f^n(x) \text{ is a Cauchy sequence. If } G \text{ is lower semi-continuous at } x^*, \text{ then we have, by (7),}
\]
\[
G(x^*) = d(x^*, f(x^*)) \leq \liminf_{n \to \infty} d(f^n(x), f^{n+1}(x)) = 0.
\]
Hence \( f(x^*) = x^* \), which completes the proof of the theorem.

**Remark 1**: If in our Theorem \( \varphi \) is defined by \( \varphi(r) = h \cdot r \) for each \( r \geq 0 \), where \( 0 < h < 1 \), then Theorem 1 reduces to the theorem of Hicks and Rhoades\(^{12} \).

Now we shall prove the following main result.

**Theorem 2**: Let \((X, d)\) be a complete metric space and \( f \) a self-mapping of \( X \). If there exists real functions \( \alpha, \beta, \gamma, \delta : (0, \infty) \rightarrow (-\infty, +\infty) \) which are continuous from the right and such that for each \( x, y \in X \) and \( r = d(x, y) \) the following inequalities hold:

\[
\alpha(r) \cdot d(x, y) + \beta(r) \cdot d(f(x), f(y)) + \gamma(r) \cdot [d(x, f(x)) + d(y, f(y))] \\
+ \delta(r) \cdot [d(x, f(y)) + d(y, f(x))] \geq 0, \quad \ldots \; (12)
\]

\[
\alpha(r) + \beta(r) + 2\gamma(r) + \delta(r) + |\delta(r)| < 0 \quad \ldots \; (13)
\]

\[
\beta(r) + \gamma(r) + \delta(r) < 0 \quad \ldots \; (14)
\]

\[
\alpha(r) + \beta(r) + 2\delta(r) < 0 \quad \ldots \; (15)
\]

then \( f \) has a unique fixed point in \( X \).

**Proof**: Let \( x \in X \) be arbitrary and let \( y = f(x) \). Put \( d_x = d(x, y) = d[x, f(x)] \). Then by (12) we have

\[
\alpha(d_x) \cdot d_x + \beta(d_x) \cdot d(f(x), f^2(x)) + \gamma(d_x) \cdot (d_x + d(f(x), f^2(x))) \\
+ \delta(d_x) \cdot d(x, f^2(x)) \geq 0, \quad \ldots \; (16)
\]

as \( d(f(x), f(x)) = 0 \).

We shall consider two cases:

**Case 1**: \( \delta(d_x) \geq 0 \). By the triangle inequality and \( \delta(d_x) \geq 0 \) we get:

\[
\delta(d_x) \cdot d(x, f^2(x)) \leq \delta(d_x) \cdot d(x, f(x)) + \delta(d_x) \cdot d(f(x), f^2(x)) \\
= \delta(d_x) \cdot d_x + \delta(d_x) \cdot d(f(x), f^2(x)).
\]

Then by (16) we have

\[
[\alpha(d_x) + \gamma(d_x) + \delta(d_x)] \cdot d_x + [\beta(d_x) + \gamma(d_x) + \delta(d_x)] \cdot d(f(x), f^2(x)) \geq 0,
\]

and hence

\[
(\alpha(d_x) + \gamma(d_x) + \delta(d_x)) \cdot d_x \geq - (\beta(d_x) + \gamma(d_x) + \delta(d_x)) \cdot d(f(x), f^2(x)). \quad \ldots \; (17)
\]

From this inequality and (14) it follows that, as \( \delta(d_x) = |\delta(d_x)| \),

\[
\alpha(d_x) + \gamma(d_x) + |\delta(d_x)| \geq 0. \quad \ldots \; (18)
\]

So from (17) we have

\[
d(f(x), f^2(x)) \leq - \frac{\alpha(d_x) + \gamma(d_x) + |\delta(d_x)|}{\beta(d_x) + \gamma(d_x) + \delta(d_x)} \cdot d_x. \quad \ldots \; (19)
\]
From (13) for $\delta(d_x) > 0$ we have

$$[\alpha(d_x) + \gamma(d_x) + \delta(d_x)] < [-\beta(d_x) + \gamma(d_x) + \delta(d_x)].$$

Now (14) and (18) implies

$$0 < -\frac{\alpha(d_x) + \gamma(d_x) + |\delta(d_x)|}{\beta(d_x) + \gamma(d_x) + \delta(d_x)} < 1. \tag{20}$$

Define a function $\varphi : [0, \infty) \to [0, \infty)$ as follows that

$$\varphi(t) = -\frac{\alpha(t) + \gamma(t) + |\delta(t)|}{\beta(t) + \gamma(t) + \delta(t)} \tag{21}$$

Then from (19) and (20) we have, for all $x \in X$,

$$d(f(x), f^2(x)) \leq \varphi(d(x, f(x))). \tag{22}$$

**Case 2:** Suppose now that $\delta(d_x) < 0$. By the triangle inequality

$$d(x, f^2(x)) \leq d(f(x), f^2(x)) - d(x, f(x))$$

and $\delta(d_x) < 0$ we get

$$\delta(d_x) d(x, f^2(x)) \leq \delta(d_x) d(f(x), f^2(x)) - \delta(d_x) dx.$$ 

Now from (16) we obtain

$$\alpha(d_x) dx + \beta(d_x) d(f(x), f^2(x)) + \gamma(d_x) (d_x + d(f(x), f^2(x))) + \delta(d_x) (d(f(x), f^2(x)) - d(x, f(x))) = 0$$

and hence

$$[\alpha(d_x) + \gamma(d_x) - \delta(d_x)] d_x + [\beta(d_x) + \gamma(d_x) + \delta(d_x)] d[f(x), f^2(x)] \geq 0.$$ 

Hence, using (14) and as $-\delta(d_x) = \delta(d_x)$, we get

$$d(f(x), f^2(x)) \leq -\frac{\alpha(d_x) + \gamma(d_x) + |\delta(d_x)|}{\beta(d_x) + \gamma(d_x) + \delta(d_x)} dx.$$ 

From $\delta(d_x) < 0$ and (13) it follows that

$$\alpha(d_x) + \gamma(d_x) + |\delta(d_x)| + \beta(d_x) + \gamma(d_x) + \delta(d_x) < 0.$$ 

Hence we obtain the relation (20) again:

$$0 \leq -\frac{\alpha(d_x) + \gamma(d_x) + |\delta(d_x)|}{\beta(d_x) + \gamma(d_x) + \delta(d_x)} < 1.$$ 

If we now, define a function $\varphi : [0, \infty) \to [0, \infty)$, as before by formula (21), we obtain the relation (22). Therefore, in both cases $\delta(d_x) \geq 0$ and $\delta(d_x) < 0$ we have the relation (22), where $\varphi$ is defined by (21).
Put \( x_0 = x \) and let sequence \( \{x_n\}_{n \in \mathbb{N}} \) be defined by \( x_n = f(x_{n-1}) \).

From (20) we conclude that the function \( \varphi(t) \) defined by (21) satisfies (1). Since \( \alpha, \beta, \gamma \) and \( \delta \) are continuous from the right and from (14) \( \beta(r) + \gamma(r) + \delta(r) \neq 0 \), it follows that

\[
\lim_{t \to r^+} \varphi(t) = -\frac{\alpha(r) + \gamma(r) + |\delta(r)|}{\beta(r) + \gamma(r) + \delta(r)} r = \varphi(r)
\]

so \( \varphi(t) \) is continuous from the right. Therefore, for \( r > 0 \) we have

\[
\limsup_{t \to r^+} \varphi(t) = \lim_{t \to r^+} \varphi(t) = \varphi(r) < r
\]

and we conclude that the function \( \varphi(t) \) satisfies (2). From (22) we see that \( \varphi(t) \) also satisfies (3). Therefore, from the first part of Theorem 1 we conclude that \( \{x_n\} \) is a convergent sequence in \( X \).

Now we shall prove that \( \lim_{n \to \infty} x_n = x^* \) implies \( x^* = f(x^*) \). Let \( d_n = d(x_n, x^*) \).

By (12) we obtain

\[
\alpha(d_n) d(x_n, x^*) + \beta(d_n) d(f(x_n), f(x^*)) + \gamma(d_n) [d(x_n, f(x_n)) + d(x^*, f(x^*))] + \delta(d_n) [d(x_n, f(x^*)) + d(x^*, f(x_n))] \geq 0.
\]

From this inequality we get:

\[
\alpha(d_n) d(x_n, x^*) + \beta(d_n) d(x_n, x_{n+1}) + \gamma(d_n) [d(x_n, x_{n+1}) + d(x^*, f(x^*))] + \delta(d_n) [d(x_n, f(x^*)) + d(x^*, x_{n+1})] \geq 0.
\]

If now we let \( n \) tend to infinity, we get

\[
(\beta(0) + \gamma(0) + \delta(0)) d(x^*, f(x^*)) \geq 0
\]

because \( \alpha, \beta, \delta, \gamma \) are continuous from the right. By this inequality and (14) it follows that \( d(x^*, f(x^*)) = 0 \). Hence \( x^* = f(x^*) \).

Now we shall prove that this fixed point is unique. Let \( y_1 = f(y_1), y_2 = f(y_2) \) and \( r = d(y_1, y_2) \). From (12) it follows that

\[
\alpha(r) d(y_1, y_2) + \beta(r) d(f(y_1), f(y_2)) + \gamma(r) [d(y_1, f(y_1)) + d(y_2, f(y_2))] + \delta(r) [d(y_1, f(y_2)) + d(y_2, f(y_1))] \geq 0
\]

which implies

\[
(\alpha(r) + \beta(r) + 2\delta(r))d \geq 0.
\]

By (15) it follows that \( d = 0 \) i.e. \( y_1 = y_2 \). The proof is complete.

Corollary 1 — Let \( (X, d) \) be a complete metric space and \( f : X \to X \) a self-mapping. If \( \alpha, \gamma, \delta : (0, +\infty) \to (-\infty, +\infty) \) are functions as in Theorem 2 and such that the inequalities (12)-(15) are satisfied with \( \beta = -1 \), then \( f \) has a unique fixed point in \( X \).
Remark 2: If in Corollary 1 functions $\alpha, \gamma, \delta$ are nonnegative, then Corollary 1 reduces to the theorem which contains Theorem 2 of Hardy and Rogers\textsuperscript{11}.

Remark 3: If in Corollary 1 $\alpha, \beta$ and $\gamma$ are constants, then Corollary 1 reduces to theorems of Ćirić\textsuperscript{4}, Zamfirescu\textsuperscript{22} and Theorem 1 of Hardy and Rogers\textsuperscript{11}.

Theorem 3 — Let $(X, d)$, $f$ and $\alpha, \beta, \gamma, \delta$ be as in Theorem 2 such that the conditions (12)-(14) are satisfied. Then $f$ has at least one fixed point in $X$.

PROOF: Condition (15) was used in the proof of the Theorem 1 only in the part of the uniquenes of the fixed point.

Remark 4: If in Theorem 3:

(A) $\beta = \delta = 0$ and $\gamma = -1$, or

(B) $\beta = 0$, $\gamma = -1$ and $\alpha = \gamma$, or

(C) $\alpha = 0$, $\beta = \gamma = -1$,

then Theorem 3 reduces to corresponding Theorem of Pal and Maiti\textsuperscript{16}.

Note that similar results can be found in the papers of Achari\textsuperscript{1}, Basu\textsuperscript{2}, Ćirić\textsuperscript{4-10}, Fisher\textsuperscript{13}, Pachpatte\textsuperscript{15}, Pal \textit{et al.}\textsuperscript{17}, Ray\textsuperscript{18} and Tasković\textsuperscript{19-21}.

REFERENCES