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# CANCELLABLE H-GROUPOIDS AND APPLICATIONS OF ACTS TO H-GROUPOIDS

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An H-groupoid is a nonempty Hausdorff space with continuous multiplication satisfying $xy \cdot yz = xz$. It is shown that any such groupoid for which any one of the two cancellation laws hold is iseomorphic to direct product of an idempotent H-groupoid and a principal H-groupoid. As a consequence of this it follows that if any compact connected H-groupoid has codimension one and is left cancellable (or right cancellable) then it is either principal or consists of idempotents only. Also it is proved that if $M$ is an H-groupoid in which the left (or right) cancellation law holds and if $eMe$ is trivial for an idempotent $e \in M$, then $M$ is an idempotent H-groupoid. Two congruences **F** and **G** are defined on an H-groupoid $M$. It is shown that the corresponding quotient H-groupoids $M/\mathbf{F}$ and $M/\mathbf{G}$ can be morphically imbedded into $M$. The proof uses the concept of acts. (An act is a continuous function $S \times X \to X$ such that $S$ is a semigroup, $X$ is a nonempty Hausdorff space and $s(tx) = (st)x$ for all $s, t \in S$ and $x \in X$, all operations being denoted by juxtaposition the context being clear.) We also show the existence of canonical actions.

## INTRODUCTION

Recently Chinda and Agarwal[4] introduced H-groupoids as nonempty Hausdorff spaces with continuous multiplication satisfying the identity $xy \cdot yz = xz$. There the authors studied basic properties of these groupoids, defined some congruences and developed the theory of H-groupoids. The authors[5] have given a complete description of all topological groupoids on the real line satisfying the identity $xy \cdot yz = xz$. This paper is a continuation of that work.

The paper is divided into five sections. In Section 1 some definitions and preliminary results are given. Section 2 is devoted to the study of cancellable H-groupoids. It is shown that any H-groupoid with either cancellation law is iseomorphic to the direct product of two of its H-subgroupoids out of which one is principal and the other consists of all of its idempotents. It then follows that if a compact connected H-groupoid has codimension one and is left cancellable or right cancellable then it is either principal or consists of idempotents only. Also it is proved that an H-groupoid $M$ with any of the two cancellation laws is an idempotent H-groupoid if $eMe = e$ for some idempotent $e$ of $M$.

In section 3, we study the two relations on an H-groupoid $M$, which are defined as follows :

$$\mathbf{F} = \{(x, y) \in M \times M : xa = ya \text{ for some } a \in M\}$$

$$\mathbf{G} = \{(x, y) \in M \times M : ax = ay \text{ for some } a \in M\}.$$

The corresponding quotient H-groupoids $M/\mathbf{F}$ and $M/\mathbf{G}$ are characterised. It is proved that if $M$ is compact then $M/\mathbf{F}$ is iseomorphic to $Me$ and $M/\mathbf{G}$ is iseomorphic to $eM$, where $e$ is an arbitrary idempotent of $M$. As a corollary it is obtained that for any $a, b \in M$, $aM$ and $bM$ are iseomorphic ($Ma$ and $Mb$ are iseomorphic) and hence they have the same cardinality.

Acts and their applications to H-groupoids are discussed in Section 4. It is shown that $M/\mathbf{F}$ and $M/\mathbf{G}$ can be morphically imbedded into $M$, where $\mathbf{F}$ and $\mathbf{G}$ are relations defined earlier in Section 3. Also an interesting example of morphic imbedding is given.

Section 5 deals with canonical actions. Bednarek and Wallace[1], while discussing equivalences on Machine state space, have shown the existence of a unique canonical action under certain conditions. The congruences $\mathbf{F}$ and $\mathbf{G}$ defined above give rise to a congruence which together with $\mathbf{F}$ and $\mathbf{G}$ satisfies those conditions and gives rise to a unique map which is an act on the relevant spaces and so is a canonical action. This paper in that sense therefore serves the purpose of showing the existence of topological groupoids on which certain acts are indeed canonical actions.

## 1. PRELIMINARIES

Throughout we will write $x$ for $\{x\}$. $AB$ will denote the collection of all elements of the form $ab$ with $a \in A$ and $b \in B$. Thus if $A = \{a\}$ we write $aB$ and $Ba$ instead of $\{a\}B$ and $B\{a\}$. Also sometimes we will omit parentheses or use dot as a separator in products when the meaning is clear. We recall some definitions and theorem from Badnarek and Wallace[2] and Chinda and Aggarwal[4] first.

*Definitions* 1.1 (Chinda and Aggarwal[4]) — (a) An H-groupoid is a nonempty Hausdorff space $M$ with a continuous multiplication $M \times M \to M$ denoted by juxtaposition such that $(xy)(yz) = xz$ for all $x, y, z \in M$. An idempotent is an element $x \in M$ such that $x^2 = x$. An H-groupoid such that each of its elements is an idempotent called an idempotent H-groupoid.

(b) Let $M$ be an H-groupoid. A subset $S \subseteq M$ is an H-subgroupoid of $M$ if it is closed with respect to multiplication defined on $M$.

The following are some examples of H-groupoids.

*Example* I — Let $M$ be a nonempty Hausdorff space and define $xy = x$ for all $x, y \in M$. This multiplication is known as left-zero or left-trivial. Clearly this multiplication is continuous and satisfies the defining identity of an H-groupoid because

$$(xy)(yz) = xy = x = xz$$

for all $x, y, z \in M$. Hence $M$ is an H-groupoid. Since $xx = x$ for each $x \in M$ therefore, $M$ is an idempotent H-groupoid.

Similarly one can see that any nonempty Hausdorff space with right-zero (right-trivial) multiplication $xy = y$ is an idempotent H-groupoid.

*Example* II — Let $G$ be a topological group in which every element other than the identity element 1, is of order two. Then $G$ becomes an H-groupoid with respect to the group multiplication because

$$(ab)\ (bc) = a\ (bb)\ c = a\ 1\ c = ac$$

for all $a, b, c \in G$.

Here identity element is the only idempotent element.

We can make use of this example to construct more examples of H-groupoids as follows :

*Example* III — Let $G$ be a topological group in which every element other than the identity element 1, is of order two and let $f : G \to G$ be a morphism satisfying the property that $f(f(x)) = f(x)$ for all $x \in G$. Define multiplication on $G$ by $x \circ y = x \cdot f(y)$ for all $x, y \in G$. Then it is easy to verify that the identity $(xy)\ (yz) = xz$ holds.

Hence $G$ with this multiplication is an H-groupoid.

In particular if $f(x) = x$ we get H-groupoid of Example II.

This example will be meaningless and vaccuous unless the existence of a nontrivial function satisfying the condition specified in the example is demonstrated. We do precisely that in what follows :

Let $A$ and $B$ be any two Hausdorff spaces which are topological groups and let $X = A \times B$. Then $X$ is a Hausdorff space and also a topological group. Define a function $f : X \to X$ by $f(a, b) = (a, 1)$, 1 being the identity of $B$. Then clearly $f$ is continuous and

$$f(f(a, b)) = f(a, 1) = (a, 1) = f(a, b)$$

for all $(a, b) \in X$.

Further since

$$\begin{aligned} f((a, b)\ (c, d)) &= f(ac, bd) = (ac, 1) \\ &= (a, 1)\ (c, 1) \\ &= f(a, b)\ f(c, d), \end{aligned}$$

therefore, $f$ is a morphism.

Hence $f$ satisfies all the conditions required in the above example and $f$ is nontrivial.

*Example* IV — Let $X$ be any Hausdorff space and let $M = X \times X$. Define multiplication on $M$ as follows :

For $(a, b)\ (c, d) \in M$ let

$$(a, b)\ (c, d) = (c, b).$$

Then this multiplication is continuous. It is straightforward to see that multiplication satisfies the defining identity of an H-groupoid. Thus $M$ with this multiplication is an H-groupoid.

It may be noted that this is an idempotent H-groupoid as $(a, b)\ (a, b) = (a, b)$ for all $(a, b) \in M$.

The above example can be generalized in the following manner.

*Example* V — Let $X$ be a nonempty Hausdorff space. Let $f : X \to X$ be a continuous map satisfying the property $f(f(x)) = f(x)$ for all $x \in X$. Let $M = X \times X$ and define multiplication on $M$ as follows :

For $a, b, c, d \in X$

$$(a, b)\ (c, d) = (c, f(b)).$$

Then one can check that $M$ is an H-groupoid.

In particular if $f(x) = x$ for all $x \in X$ we get H-groupoid of Example (IV).

*Theorem* 1.2 (Badnarek and Wallace[2]) — Let $M$ be an H-groupoid and $E$ be the set of all idempotents of $M$. Then

(i) mediality prevails, that is $(xy)\ (uv) = (xu)\ (yv)$ for all $x, y, u, v \in M$.

(ii) $x^2 \in E$ for each $x \in M$ and $E^2 = E$.

(iii) $e \in E$ implies $(xe)\ e = xe$, $e\ (ey) = ey$ for all $x, y \in M$.

(iv) $(xy)\ z = x\ (yz)$ for all $x, y, z \in M$, that is $M$ is associative.

(v) $xey = xy$ for all $x, y \in M$ and $e \in E$.

In view of 1.2 (iv) $(xM)x = x(Mx)$, so we can write either of them equal to $xMx$.

*Definition* 1.3 (Chinda and Aggarwal[4]) — An H-groupoid $M$ is said to be principal if $xMx = M$ for some $x \in M$.

*Theorem* 1.4 (Chinda and Aggarwal[4]) — Let $M$ be an H-groupoid and $E$ be the set of all idempotents of $M$. Then

(i) The square of each element of $xMx$ is $x^2$ and $xMx = x^2 Mx^2$ for each $x \in M$.

(ii) If $M$ is a principal H-groupoid then $xMx = M$ for each $x \in M$.

(iii) $aM = a^2M$; $Ma = Ma^2$ and $aM$, $Ma$ are H-subgroupoids of $M$ for any $a \in M$.

(iv) for any $a \in M$, $aMa$ is a principal H-subgroupoid of $M$.

(v) $M$ is a principal H-groupoid iff it is a topological group with only elements (other than the identity element) of order two.

*Definition* 1.5 (Chinda and Aggarwal[4]) — Let $M$ and $M'$ be H-groupoids. A function $\phi : M \to M'$ is a morphism if it is continuous and $\phi(xy) = \phi(x)\,\phi(y)$ for all $x, y \in M$, a surmorphism if it is also onto, and an iseomorphism if it is also a homeomorphism.

Note that if the spaces are compact then any continuous bijection is a homeomorphism.

*Definition* 1.6 (Chinda and Aggarwal[4]) — A congruence on an H-groupoid $M$ is an equivalence relation $\mathbf{R}$ on $M$ such that $(x, y) \in \mathbf{R}$ implies $(xa, ya)$ and $(ax, ay) \in \mathbf{R}$ for all $a \in M$.

*Lemma* 1.7 (Chinda[3] and Chinda and Aggarwal[4]) — Let $M$ be a compact or discrete H-groupoid. If $\mathbf{R}$ is a closed congruence on $M$, if $\phi : M \to M/\mathbf{R}$ is the natural map and if $M/\mathbf{R}$ has the quotient topology then $M/\mathbf{R}$ is an H-groupoid and $\phi$ is a surmorphism. Conversely if $h : M \to M'$ is a surmorphism of H-groupoids and if $\mathbf{R} = \{(x, y) : h(x) = h(y)\}$ then $\mathbf{R}$ is a closed congruence and the induced map $\bar{h} : M/\mathbf{R} \to M'$ defined by $\bar{h}(\bar{x}) = h(x)$ is an iseomorphism.

The proof is categorical, the hypothesis compact or discrete is used to show that $M/\mathbf{R}$ is Hausdorff.

From now onwards unless otherwise specified, throughout the paper the letter $M$ will denote an H-groupoid and $E$ will denote the set of all idempotents of $M$.

## 2. Cancellable H-Groupoids

We now define Cancellable H-groupoids as follows :

*Definition* 2.1 — An H-groupoid is left cancellable if for all $a, b, c \in M$,

$$ab = ac \Rightarrow b = c \qquad \ldots (1)$$

and right cancellable if for all $a, b, c \in M$,

$$ac = bc \Rightarrow a = b. \qquad \ldots (2)$$

An H-groupoid is cancellable if it is both right and left cancellable.

The laws (1) and (2) respectively will be called as the left cancellation law and the right cancellation law.

*Remark* : It may be noted that an H-groupoid having left-zero multiplication is right cancellable and an H-groupoid having right-zero multiplication is left cancellable.

*Theorem* 2.2 — Let $M$ be an H-groupoid and $E$ be the set of all idempotents of $M$. If $M$ is either left or right cancellable then $M$ is iseomorphic to $E \times eMe$ where $e \in E$ is arbitrary.

Proof : $E$ and $eMe$ are H-subgroupoids of $M$, by Theorem 1.4 (iii) and 1.4 (iv). Since direct product of H-groupoids is again an H-groupoid therefore it follows that $E \times eMe$ is an H-groupoid. Define $\psi : M \to E \times eMe$ by

$$\psi(m) = (m^2, eme) \qquad \forall\ m \in M.$$

Since the multiplication on $M$ is continuous therefore the map $\Psi$ is continuous.

$\psi$ *is one-to-one* : Assume $\psi(m_1) = \psi(m_2)$. Then $em_1 e \cdot m_1^2 = em_2 e \cdot m_2^2$. This implies $em_1 = em_2$ because $em_1 e \cdot m_1^2 = em_1 \cdot m_1^2 = em_1 \cdot m_1 m_1 = em_1$ and similarly $em_2 e \cdot m_2^2 = e m_2$ using Theorem 1.2 (v) and the defining identity of an H-groupoid. If $M$ is left cancellable then $em_1 = em_2$ implies $m_1 = m_2$ and hence $\psi$ is one-to-one. Instead if $M$ is right cancellable then we consider $m_1^2 \cdot em_1 e = m_2^2 \cdot em_2 e$, which implies $m_1 = m_2$. Thus $\psi$ is one-to-one.

$\psi$ *is onto* : Let $(e', eme) \in E \times eMe$. Take $m' = e' me'$. Then, using Theorem 1.2 (v) and associativity it is easy to show that $\psi(m') = (e', eme)$.

$\psi^{-1}$ *is continuous* : Clearly $\psi^{-1} : E \times eMe \to M$ is the map defined by $\psi^{-1}(e', eme) = e'me'$ for all $(e', eme) \in E \times eMe$. We show that $\psi^{-1}$ is composite of two continuous maps. Let $(e', eme) \in E \times eMe$ then $\psi^{-1}(e', eme) = e'me' = e' \cdot eme \cdot e'$ by Theorem 1.2 (v) because $e$ is an idempotent. Now,

$$E \times eMe \to E \times M \to M$$

$$(e', eme) \to (e', eme) \to e' \cdot eme \cdot e'.$$

Thus $\psi^{-1}$ is the composite of the above two maps both of which are continuous. Thus $\psi$ is a homeomorphism. We now show that $\psi$ preserves multiplication from which it follows that $M$ is iseomorphic to $E \times eMe$.

$$\begin{aligned}\psi(m_1 m_2) &= ((m_1 m_2)^2, e \cdot m_1 m_2 \cdot e) \\ &= (m_1 m_1 \cdot m_2 m_2, em_1 e \cdot em_2 e) \\ &= (m_1^2, em_1 e)(m_2^2, em_2 e) \\ &= \psi(m_1)\psi(m_2)\end{aligned}$$

Thus the theorem is proved.

*Corollary* 2.3 — Let $M$ be an H-groupoid satisfying any one of the two cancellation laws. If $eMe = \{e\}$ for some $e \in E$ then $M$ consists of idempotents only.

PROOF : Let $f \in E$ be such that $fMf = \{f\}$. By Theorem 2.2, $M$ is iseomorphic to $E \times eMe$ for any arbitrary $e \in E$. In particular this is true for $f \in E$. This gives $M$ is iseomorphic to $E \times fMf$. So $M$ is iseomorphic to $E \times f$. It follows that $M = E$ and hence $M$ consists of idempotents only.

*Remark* : The above result is not true without cancellation law, as can be seen from the following.

Let $M$ be an H-groupoid having at least three elements with constant multiplication, i.e., $xy = c$ for all $x, y \in M$, where $c$ is a fixed element of $M$. Then $M$ is neither left cancellable nor right cancellable. Here $c$ is the only idempotent of $M$ and $cMc = c$. However $M \neq E$.

It is now possible to derive another corollary to Theorem 2.2 which completely describes compact connected H-groupoids having codimension one. To arrive at this corollary we need the following definition.

For each $n$ write $H^n(X)$ for the $n$th Alexander-Wallace-Spanier cohomology groups of $X$ with arbitrary but fixed coefficient group.

*Definition* 2.4 — Let $X$ be a nonempty compact Hausdorff space. We define cohomological dimension or more briefly the codimension of $X$ to be

$$cd(X) = \inf \{n : n \text{ is a non-negative integer and } H^{n+1}(X, A) = 0 \text{ for every closed subset } A \text{ of } X\}.$$

We define $cd(\Phi) = -1$. For an arbitrary Hausdorff space $X$, we then define the codimension of $X$ to be

$$cd(X) = \sup \{cd(K) : K \text{ is a compact subset of } X\}.$$

The following result is known[3].

If $X$ and $Y$ are Hausdorff spaces with $cd(X) = m$ and $cd(Y) = 1$, then $cd(X \times Y) = m + 1$, $X$, $Y$ being continua.

*Corollary* 2.5 — Let $M$ be a compact connected right or left cancellable H-groupoid having codimension one. Then either $M$ is principal or $M$ is an idempotent H-groupoid.

PROOF : $M$ is iseomorphic to $E \times eMe$ by Theorem 2.2. Since $cd(M) = 1$, at least one of $E$, $eMe$ must have codimension less than one. In case $cd(eMe)$ is less than one, since $e \in eMe$, $eMe \neq \Phi$, and hence $cd(eMe) = 0$. This impkies $eMe = \{e\}$ because $eMe$ is compact and connected. Thus $M$ is iseomorphic to $E \times e$, from which it follows that $M = E$. Hence $M$ is an idempotent H-groupoid.

On the other hand if $cd(E)$ is less than one, then since $E \neq \Phi$, $cd(E) = 0$. As before since $E$ is compact and connected, $E$ is singleton. In that case $M$ is iseomorphic to $e \times eMe$, which implies $M = eMe$. Thus $M$ is principal.

*Theorem* 2.6 — An H-groupoid is principal iff it is cancellable.

PROOF : Let $M$ be a principal H-groupoid. From Theorem 1.4 (v) it follows that $M$ is a topological group, therefore it is right as well as left cancellable. Conversely let $M$ be an H-groupoid in which both cancellation laws hold. Let $e$ be an idempotent of $M$ and $x$ be any arbitrary element of $M$. Since $ex = ee \cdot x = e \cdot ex$ and $M$ is left cancellable therefore $x = ex$.

Similarly $xe = x \cdot ee = xe \cdot e$ implies $x = xe$ as $M$ is right cancellable. Thus $x = ex = e \cdot xe \in eMe$. This shows $M \subseteq eMe$, which implies $M = eMe$. Hence $M$ is principal by Definition 1.3.

*Theorem* 2.7 — Let $M$ be an H-groupoid and $E$ be the set of all idempotents of $M$. Then

(i) If for any two elements $x, y \in M$, there exists $a \in M$ such that $axa = aya$ then $bxb = byb$ for all $b \in M$.

(ii) The cardinality of $aMa$ and the cardinality of $bMb$ are same for any $a, b \in M$.

(iii) If $eMe = \{e\}$ for some $e \in E$ then $fMf = \{f\}$ for each $f \in E$.

PROOF : (i) Let $x$, $y$ be any two elements of $M$ and $a \in M$ be such that $axa = aya$. Let $b \in M$ be arbitrary then $bxb = ba \cdot axb = ba \cdot axa \cdot ab = ba \cdot aya \cdot ab = b \cdot ya \cdot ab = byb$, using associativity and the defining identity of an H-groupoid.

(ii) Define $h : aMa \to bMb$ by

$$h(ama) = bmb \qquad \forall\ m \in M.$$

Then using (i) it is straightforward to check that $h$ is a bijection.

(iii) If $eMe = \{e\}$ then it follows that the cardinality of $fMf$ is equal to one, by (ii) above. Since $f \in fMf$ therefore $fMf = \{f\}$.

It may be noted here that the Corollary 2.3 can also be derived from the above theorem as follows.

Let $e \in E$ be such that $eMe = \{e\}$. Then by Theorem 2.7 above $fMf = \{f\}$ for each $f \in E$. Let $x \in M$ then $x^2Mx^2 = x^2$ because $x^2 \in E$. This gives $x^2xx^2 = x^2$ and hence $x^3 = x^2$ using associativity and Theorem 1.2(v). By cancellation law (left or right) we get $x^2 = x$. Thus $x$ is an idempotent. Hence $M$ consists of idempotents only.

## 3. THE RELATIONS F AND G

The relation **F** has been studied by Chinda[3] on a hemigroup. Here we study the same relation on an H-groupoid.

*Definition* 3.1 — Let $M$ be an H-groupoid. On $M$ define a relation **F** as follows: For $x, y \in M$,

$$(x, y) \in \mathbf{F} \text{ iff } xa = ya \text{ for some } a \in M. \qquad \text{(The relation F)}$$

*Proposition* 3.2 — Let $M$ be an H-groupoid. Then $(x, y) \in \mathbf{F}$ iff $xa = ya$ for all $a \in M$.

PROOF : Clearly if $xa = ya$ for all $a \in M$ then $(x, y) \in \mathbf{F}$. Conversely let $(x, y) \in \mathbf{F}$, then $xb = yb$ for some $b \in M$. Let $a \in M$ be arbitrary. We have

$$xa = xb \cdot ba = yb \cdot ba = ya$$

using the defining identity of an H-groupoid. Therefore $xa = ya$ for all $a \in M$.

*Theorem* 3.3 — Let $M$ be a compact H-groupoid. Then **F** is a closed congruence. So $M/\mathbf{F}$ is an H-groupoid and the natural map $\phi : M \to M/\mathbf{F}$ is a surmorphism.

PROOF : Using Proposition 3.2 it is easy to check that **F** is an equivalence relation.

To show **F** is a congruence, let $(x, y) \in \mathbf{F}$ and $a \in M$, we will prove that $(xa, ya)$ and $(ax, ay) \in \mathbf{F}$. Since $(x, y) \in \mathbf{F}$ therefore $xa = ya$. This implies $(xa, ya) \in \mathbf{F}$, since **F** is reflexive being an equivalence relation. Now let $b$ be any element of $M$ then $xb = yb$ as $(x, y) \in \mathbf{F}$. Then $a(xb) = a(yb)$ which implies $(ax)b = (ay)b$ as multiplication is associative. By Definition 3.1 we get $(ax, ay) \in \mathbf{F}$.

Next we show that **F** is closed. We prove that $(M \times M)\backslash\mathbf{F}$ is open. Let $(c, d) \in (M \times M)\backslash\mathbf{F}$. Then there exists at least one $m \in M$ such that $cm \neq dm$. Since we are in a Hausdorff space, therefore there exist open sets $U$ and $V$ such that $cm$

$\in U$ and $dm \in V$ and $U \bigcap V = \Phi$. Also since multiplication on $M$ is continuous, therefore there exist open sets $U_c, U_m$ and $V_d, V_m$ such that $c \in U_c$, $m \in U_m \bigcap V_m$ and $d \in V_d$ and $U_c U_m \subseteq U$ and $V_d V_m \subseteq V$. We now claim that $(c, d) \in U_c \times V_d \subseteq (M \times M) \setminus \mathbf{F}$. Assume on the contrary that $(U_c \times V_d) \bigcap \mathbf{F} \neq \Phi$. Let $(x, y) \in U_c \times V_d$ be such that $(x, y) \in \mathbf{F}$ also. Then by Theorem 3.2 above, $xm = ym$. Since $x \in U_c$ and $m \in U_m$ therefore $xm \in U$. Also $y \in V_d$ and $m \in V_m$ therefore $ym \in V$. But that means $U \bigcap V \neq \Phi$, which is a contradiction. Hence $(U_c \times V_d) \bigcap \mathbf{F} = \Phi$. Thus $(M \times M) \setminus \mathbf{F}$ *is open, i.e., F is closed.*

These facts together with Lemma 1.7 establish the required result.

*Theorem* 3.4 — Let $M$ be a compact H-groupoid, and let $e$ be an arbitrary idempotent of $M$. Then

(i) $Me$ is an H-subgroupoid of $M$.

(ii) $f : M \to Me$ defined by $f(x) = xe$ is a morphism.

(iii) $\hat{\phi} = \phi \mid Me$ is the unique iseomorphism of $Me$ onto $M/\mathbf{F}$ such that $\hat{\phi} \circ f = \phi$, where $\phi$ is the natural map defined in Theorem 3.3.

PROOF : (i) By Theorem 1.4 (iii), $Me$ is an H-subgroupoid of $M$.

(ii) Clearly $f$ is continuous. Since

$$f(xy) = xy \cdot e = x \cdot ye = xe \cdot ye = f(x)\, f(y),$$

using Theorem 1.2 (v) and associativity, therefore $f$ is a morphism.

(iii) $\hat{\phi}$ is a morphism because it is the restriction of the map $\phi$ (defined in Theorem 3.3) on $Me$. Let $\hat{\phi}(xe) = \hat{\phi}(ye)$ then $\phi(xe) = \phi(ye)$. This gives $(xe, ye) \in \mathbf{F}$. So $xe \cdot e = ye \cdot e$, and since $e^2 = e$ therefore using associativity we get $xe = ye$. Hence $\hat{\phi}$ is one-to-one.

Now let $\bar{z} \in M/\mathbf{F}$, since $\phi$ is onto therefore $\bar{z} = \phi(x)$ for some $x \in M$. But $\phi(x) = \phi(xe)$ because $xe = xe \cdot e$ implying $(x, xe) \in \mathbf{F}$. Thus $\bar{z} = \phi(x) = \phi(xe) = \hat{\phi}(xe)$. Hence $\hat{\phi}$ is onto.

Thus $\hat{\phi}$ is a continuous bijection, therefore a homeomorphism. Also $\hat{\phi}$ is a morphism, therefore it follows that $\hat{\phi}$ is an iseomorphism.

For any $x \in M$, $(\hat{\phi} f)(x) = \hat{\phi}(f(x)) = \hat{\phi}(xe) = \phi(xe) = \phi(x)$. Thus $\hat{\phi} f = \phi$. It can be easily seen that $\hat{\phi}$ is unique.

*Corollary* 3.5 — If $M$ is a compact H-groupoid, then $Ma$ is iseomorphic to $Mb$ for any $a, b \in M$. Hence they have the same cardinality.

PROOF : From Theorem 3.4 (iii), it follows that $Me$ is iseomorphic to $Mf$ for any $e, f \in E$. Also for any $a, b \in M$ we have $Ma = Ma^2$ and $Mb = Mb^2$ from Theorem 1.4 (iii), and since $a^2, b^2$ are idempotents therefore we get $Ma$ is iseomorphic to $Mb$.

*Corollary* 3.6 — If $Me = \{e\}$ for one $e \in E$, then $Mf = \{f\}$ for each $f \in E$.

Now using left-right duality we define another relation $\mathbf{G}$ on $M$ as follows.

*Definition* 3.7 — Let $M$ be an H-groupoid. For $x, y \in M$,

$(x, y) \in \mathbf{G}$ iff $ax = ay$ for some $a \in M$. (The relation **G**)

The following are some results for relation **G**, similar to the results for relation **F**, and can be proved easily. Therefore we only state them.

*Proposition* 3.8 — Let $M$ be an H-groupoid. Then $(x, y) \in \mathbf{G}$ iff $ax = ay$ for all $a \in M$.

*Theorem* 3.9 — Let $M$ be a compact H-groupoid. Then **G** is a closed congruence. So $M/\mathbf{G}$ is an H-groupoid and the natural map $\psi : M \to M/\mathbf{G}$ is a surmorphism.

*Theorem* 3.10 — Let $M$ be a compact H-groupoid and let $e$ be an arbitrary idempotent element of $M$. Then

(i) $eM$ is an H-subgroupoid of $M$.

(ii) $g : M \to eM$ defined by $g(x) = ex$ is a morphism.

(iii) $\hat{\psi} = \psi \mid eM : eM \to M/\mathbf{G}$ is the unique iseomorphism of $eM$ onto $M/\mathbf{G}$ such that $\hat{\psi} \circ g = \psi$.

*Corollary* 3.11 — If $M$ is a compact H-groupoid, then $aM$ is iseomorphic to $bM$ for any $a, b \in M$. Hence they have the same cardinality.

*Corollary* 3.12 — If $eM = \{e\}$ for one $e \in E$, then $fM = \{f\}$ for each $f \in E$.

It follows from Theorem 2.2 of Chinda[3] that the relations **F** and **G** are one and the same relation on a hemigroup. Hence it is natural to ask,

(i) Are **F** and **G** two different names of the same relation on an H-groupoid ?

If not,

(ii) Is there any relation between **F** and **G** in the sense of containment ?

We answer both these questions in the negative. The following examples will suffice.

*Examples* — (I) Let $M$ be an H-groupoid with left-zero multiplication i.e. $xy = x$ for all $x, y \in M$. We show that $\mathbf{F} = \Delta$ where $\Delta = \{(x, x) : x \in M\}$, the diagonal of $M$. Let $(x, y) \in \mathbf{F}$ then $xa = ya$ for any $a \in M$. Since multiplication is left-zero therefore $xa = x$ and $ya = y$. This gives $x = y$, which implies $(x, y) \in \Delta$. Thus $\mathbf{F} \subseteq \Delta$. Conversely $\Delta \subseteq \mathbf{F}$, since **F** is reflexive being an equivalence relation. Hence $\mathbf{F} = \Delta$. Here $\mathbf{G} = M \times M$. For if $(x, y) \in M \times M$ then $xx = x = xy$ as multiplication is left-zero, and hence $(x, y) \in \mathbf{G}$. Hence $\mathbf{F} \neq \mathbf{G}$ if $M$ contains more than one element.

(II) Let $M$ be an H-groupoid with right-zero multiplication. It can be shown as above that $\mathbf{F} = M \times M$ and $\mathbf{G} = \Delta$.

(III) Let $X$ be a nonempty Hausdorff space and $M = X \times X$. Then $M$ is an H-groupoid with multiplication defined by $(a, b) \cdot (c, d) = (a, d)$ for all $a, b, c, d \in X$ (Chinda and Aggarwal[4]).

Here

$$\mathbf{F} = \{((a, b), (c, d)) \in M \times M : (a, b)(x, y) = (c, d)(x, y) \text{ for some } (x, y) \in M \times M\}$$

$$= \{((a, b), (c, d)) \in M \times M : (a, y) = (c, y) \text{ for some } y \in M\}$$

$$= \{((a, b), (c, d)) \in M \times M : a = c, y = y \text{ for some } y \in M\}$$

$$= \{((a, b), (c, d)) \in M \times M : a = c\}$$

$$= \{((a, b), (a, d)) \in M \times M\}.$$

Similarly one can show that $\mathbf{G} = \{((a, b), (c, b)) \in M \times M\}$.

So $\mathbf{F} \neq \mathbf{G}$, if $M$ contains more than one element. Note that in this example neither $\mathbf{F}$ is contained in $\mathbf{G}$ nor $\mathbf{G}$ is contained in $\mathbf{F}$.

(IV) Let $M$ be an H-groupoid with constant multiplication, i.e., $xy = c$ for all $x, y \in M$, where $c$ is a fixed element of $M$. Then $\mathbf{F} = M \times M$ and $\mathbf{G} = M \times M$ because $xa = c = ya$ and $ax = c = ay$ for all $x, y, a \in M$.

Note that in the examples (I), (II), (III) above, $M = M^2$ and $\mathbf{F} \cap \mathbf{G} = \Delta$. This suggests the following :

*Theorem* 3.13 — If $M$ is an H-groupoid with $M = M^2$ then $\mathbf{F} \cap \mathbf{G} = \Delta$, where $\mathbf{F}$ and $\mathbf{G}$ are the relations defined above and $\Delta$ is the diagonal of $M$.

PROOF : Obviously $\Delta \subseteq \mathbf{F} \cap \mathbf{G}$. Now let $(x, y) \in \mathbf{F} \cap \mathbf{G}$, then $xa = ya$ and $ax = ay$ for all $a \in M$ by Propositions 3.2 and 3.8. In particular when $a = x$ we get $xx = yx$ and $xx = xy$. This gives $xy = yx$ and so

$$x^2 = xx = xy \cdot yx = yx \cdot xy = yy = y^2$$

using mediality and the defining identity of an H-groupoid. Again since $(x, y) \in \mathbf{G}$ and $y \in M$. Therefore $yx = yy$, which implies

$$x^3 = x^2 \cdot x = y^2 \cdot x = yy \cdot x = y \cdot yx = y \cdot yy = y^3.$$

Now we show that $x^3 = x$ for each $x \in M$. Let $x \in M$, then $x = ab$ for some $a$, $b \in M$, because $M = M^2$. So $x^3 = (ab)^3 = ab \cdot (ab)^2 = ab \cdot (ab \cdot ab) = (a \cdot ba)\ (ba \cdot b)$ $= ab = x$ using associativity and defining identify of H-groupoids.

It follows that $x = x^3 = y^3 = y$ and hence $(x, y) \in \Delta$. Thus $\mathbf{F} \cap \mathbf{G} \subseteq \Delta$. This proves the equality.

## 4. ACTS AND MORPHIC IMBEDDING

In this section we show that the quotient H-groupoids $M/\mathbf{F}$ and $M/\mathbf{G}$ can be morphically imbedded into $M$.

We begin by defining a left act.

*Definition* 4.1 — Let $T$ be a semigroup i.e. a nonempty Hausdorff space with continuous associative multiplication $T \times T \to T$ denoted by juxtaposition. Let $X$ be any Hausdorff space. Then $T$ is said to act on the left on $X$ if there exists a continuous function $\mu : T \times X \to X$ satisfying

$$\mu(t, \mu(t', x)) = \mu(tt', x) \qquad \forall\ t, t' \in T \text{ and } x \in X.$$

Denoting the value of the function $\mu$ at the place $(t, x)$ by $tx$ we can also write above identity as

$$t(t'x) = (tt')x.$$

The map $\mu$ is called a left act.

Here $T$ is called the input semigroup and $X$ is called the state space. If $T$ and $X$ are compact we shall refer to the act as a compact act.

We give below some examples of left act.

*Examples* — (1) Let $T$ be a semigroup and $X$ be a Hausdorff space. Define $\mu : T \times X \to X$ by

$$\mu(t, x) = x \qquad \forall\ t \in T, x \in X.$$

Then $\mu$ is continuous. Moreover

$$\mu(t, \mu(t', x)) = \mu(t, x) = x$$

and $$\mu(tt', x) = x.$$

Therefore

$$\mu(t, \mu(t', x)) = \mu(tt', x) \qquad \forall\ t, t' \in T, x \in X.$$

Hence $\mu$ is a left act.

(II) Let $\mathcal{R}$ be the space of real numbers with usual topology. If multiplication on $\mathcal{R}$ is left-zero (i.e. $xy = x$ for all $x, y \in \mathcal{R}$), then $\mathcal{R}$ becomes a semigroup.

Define $\mu : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ by

$$\mu(t, x) = t \qquad \forall\ t, x \in \mathcal{R}.$$

Then $\mu$ is continuous. Also

$$\mu(t, \mu(t', x)) = \mu(t, t') = t$$

and $$\mu(tt', x) = \mu(t, x) = t.$$

Hence $$\mu(t, \mu(t', x) = \mu(tt', x) \qquad \forall\ t, t' \in \mathcal{R},\ x \in \mathcal{R}.$$

This shows $\mu$ is a left act.

(III) Let $X$ be any Hausdorff space and let $T = X \times X$. Define multiplication on $T$ as follows :

$$(a, b)\ (c, d) = (a, d) \qquad \forall\ a, b, c, d \in X.$$

Then this multiplication is continuous and for any $(a, b)$, $(c, d)$, $(e, f) \in T$.

$$[(a, b)\ (c, d)]\ (e, f) = (a, d)\ (e, f) = (a, f)$$

and $$(a, b)\ [(c, d)\ (e, f)] = (a, b)\ (c, f) = (a, f).$$

Thus multiplication is associative and hence $T$ becomes a semigroup.

Define $\mu : T \times X \to X$ by

$$\mu((a, b), c) = a \qquad \forall\ a, b, c \in X.$$

Clearly $\mu$ is continuous. Also for any $(a, b)$, $(a', b') \in T$ and $x \in X$

$$\mu((a, b)(a', b'), x) = \mu((a, b'), x) = a$$

and $$\mu((a, b), \mu((a', b'), x)) = \mu((a, b), a') = a.$$

Thus $\mu$ is a left act.

Similarly if we define $\mu : T \times X \to X$ by

$$\mu((a, b), c) = c \qquad \forall\ a, b, c \in X$$

then $\mu$ becomes a left act.

*Remark* : Let $M$ be an H-groupoid with $\hat{m} : M \times M \to M$ as continuous multiplication. Since the multiplication is associative it follows immediately that the function $\hat{m}$ is a left act with $T = X = M$. It may be noted that for any subspace $S$ of $M$ which is closed under multiplication, the restriction map $\hat{m} \mid S \times M \to M$ is also a left act.

*Proposition* 4.2 — Let $M$ be a compact H-groupoid and $\mathbf{R}$ be a closed congruence on $M$. Then the map $\mu : M \times M/\mathbf{R} \to M/\mathbf{R}$ defined by $\mu(x, \bar{y}) = \overline{xy}$ is a left act, where $\bar{y}$ denotes the equivalence class of $y$.

PROOF : $\mu$ is well defined : It is enough to show that if $\bar{y}_1 = \bar{y}_2$ then $\overline{xy}_1 = \overline{xy}_2$. Since $\bar{y}_1 = \bar{y}_2$ therefore $\bar{x}\bar{y}_1 = \bar{x}\bar{y}_2$. This implies $\overline{xy}_1 = \overline{xy}_2$ because the natural map is a morphism (by Lemma 1.7). Hence $\mu$ is well defined.

Clearly $\mu$ is continuous.

We now show that

$$\mu(x, \mu(x', \bar{y})) = \mu(xx', \bar{y}) \qquad \forall\ x, x' \in M \text{ and } \bar{y} \in M/\mathbf{R}.$$

We have

$$\mu(x, \mu(x', \bar{y})) = \mu(x, \overline{x'y}) = \overline{x(x'y)} = \overline{(xx')y} = \mu(xx', \bar{y})$$

using associativity of $M$.

Hence $\mu$ is a compact left act.

*Definition* 4.3 (Badnarek and Wallace[1]) — If $T$ acts on the left on $X$, then a continuous function $\phi$ from $X$ into the Hausdorff space $Y$ will be called a $T$-morphism if $T$ acts on the left on $Y$ and if $\phi \circ \mu = \nu \circ (i \times \phi)$, where $i$ denotes the identity map on $T$ and $\mu, \nu$ are respectively the action maps on $X$ and $Y$.

If $\phi$ is a homeomorphism into as well as morphism then we call $\phi$ a morphic imbedding of $X$ in $Y$, and say in such a case that $X$ can be morphically imbedded into $Y$.

*Theorem* 4.4 — Let $M$ be a compact H-groupoid and $\mathbf{F}$ be the closed congruence defined earlier in Definition 3.1. Then $\alpha : M/\mathbf{F} \to M$ defined by $\alpha(\bar{x}) = xe$ is a

morphic imbedding, where $e$ is a fixed idempotent and $\bar{x}$ is the equivalence class of $x$ in the relation $\mathbf{F}$.

PROOF : By Proposition 4.2 the map $\mu : M \times M/\mathbf{F} \to M/\mathbf{F}$ defined by $\mu(x, \bar{y}) = \overline{xy}$ is a left act. The map $\hat{m} : M \times M \to M$ is also a left act, by the Remark before Proposition 4.2. Let $\alpha : M/\mathbf{F} \to M$ be a function defined by

$$\alpha(\bar{x}) = xe,$$

where $e$ is a fixed idempotent of $M$. We will show that $\alpha$ is a homeomorphism into as well as morphism and $\alpha \circ \mu = \hat{m} \circ (i \times \alpha)$.

$\alpha$ *is well defined* : Let $\bar{x}, \bar{y} \in M/\mathbf{F}$ be such that $\bar{x} = \bar{y}$. Then $(x, y) \in \mathbf{F}$ which implies $xe = ye$ by Proposition 3.2. Hence $\alpha(\bar{x}) = \alpha(\bar{y})$.

$\alpha$ *is one-to-one* : Let $\alpha(\bar{x}) = \alpha(\bar{y})$. Then $xe = ye$ which implies $(x, y) \in \mathbf{F}$ by Definition 3.1. Hence $\bar{x} = \bar{y}$. Thus $\alpha$ is one-to-one.

$\alpha^{-1} \mid$ *Image* $\alpha$ *is continuous* : Let $\hat{\alpha} : M/\mathbf{F} \to Me$ be a map defined by $\hat{\alpha}(\bar{x}) = \alpha(\bar{x})$ for all $\bar{x} \in M/\mathbf{F}$. Then $\hat{\alpha}$ is one-to-one and onto. We claim that $\hat{\alpha}^{-1} = \hat{\phi}$ where $\hat{\phi} : Me \to M/\mathbf{F}$ is the map defined in Theorem 3.4, i.e., restriction of the natural map $M \to M/\mathbf{F}$ to $Me$.

For any $x \in M$,

$$(\hat{\alpha} \circ \hat{\phi})(xe) = \hat{\alpha}(\hat{\phi}(xe)) = \hat{\alpha}(\overline{xe}) = \hat{\alpha}(\bar{x}) = \alpha(\bar{x}) = xe$$

and $$(\hat{\phi} \circ \hat{\alpha})(\bar{x}) = \hat{\phi}(\hat{\alpha}(\bar{x})) = \hat{\phi}(\alpha(\bar{x})) = \hat{\phi}(xe) = \overline{xe} = \bar{x}.$$

Thus $\hat{\phi} \circ \hat{\alpha}$ and $\hat{\alpha} \circ \hat{\phi}$ are the identity maps on the respective spaces and hence $\hat{\phi} = \hat{\alpha}^{-1}$.

Note that $\alpha^{-1} \mid \text{Image } \alpha = \hat{\alpha}^{-1} = \hat{\phi}$ and $\hat{\phi}$ is continuous by Theorem 3.4.

$\alpha$ *is continuous* : Note that $\alpha = j \circ \hat{\alpha}$, where $j : Me \hookrightarrow M$ is the inclusion map. Now apply the fact that $\hat{\alpha}$ is continuous.

$\alpha$ *is a morphism* : We have $\alpha(\bar{x}\,\bar{y}) = \alpha(\overline{xy}) = (xy)e = (xey)e = (xe)(ye) = \alpha(\bar{x})\,\alpha(\bar{y})$

Thus $\alpha$ is a homeomorphism into as well as morphism. It is easy to check that $\alpha \circ \mu = \hat{m} \circ (i \times \alpha)$. It follows that $\alpha$ is a morphic imbedding. This proves $M/\mathbf{F}$ can be morphically imbedded into $M$.

*Example of morphic imbedding*

Let $\mathcal{R}$ be the space of real numbers with usual topology and let $M = \mathcal{R} \times \mathcal{R}$, the cartesian product. On $M$ define multiplication as follows :

$$(a, b)(c, d) = (a, d) \qquad \forall\ a, b, c, d \in \mathcal{R}.$$

Then $M$ is an H-groupoid (Chinda and Aggarwal[4]). Moreover $(a, b)^2 = (a, b)$ for all $(a, b) \in M$. Hence $M$ is an idempotent H-groupoid. Also $((a, b), (c, d)) \in \mathbf{F}$ if and only if $a = c$ as shown in Example (III) on page 978. The equivalence class of $(a, b)$ corresponding to the relation $\mathbf{F}$ is the vertical line through $(a, b)$. Thus

$M/\mathbf{F}$ is the collection of all these distinct vertical lines, each vertical line being an element of $M/\mathbf{F}$. Then $M/\mathbf{F}$ can be morphically imbedded into $M$ and a morphic imbedding can be described as follows :

Since each element of $M$ is an idempotent. We can take $e$ of the Theorem 4.4 to be any element of $M$ say $(e, f)$. Then $\alpha(\overline{x}) = xe$ becomes $\alpha((\overline{a, b})) = (a, b)(e, f) = (a, f)$, which is a continuous section of the canonical map $M \to M/\mathbf{F}$ and is represented by the horizontal line through $(e, f)$. Note that $\alpha$ is only one of the infinite number of morphic imbeddings, for we have one for each value of $f \in \mathcal{R}$ in $(e, f)$. Thus $M/\mathbf{F}$ can be morphically imbedded in $M$ onto any horizontal line.

We now state, without proof, that $M/\mathbf{G}$ can also be morphically imbedded into $M$. The proof uses the concept of right act, and the corresponding notion of morphic imbedding which are analogous to Definitions 4.1 and 4.3.

*Proposition* 4.5 — Let $M$ be a compact H-groupoid and $\mathbf{R}$ be a closed congruence on $M$. Then the map $\mu : M/\mathbf{R} \times M \to M/\mathbf{R}$ defined by $\mu(\overline{x}, y) = \overline{xy}$ is a right act.

*Theorem* 4.6 — Let $M$ be a compact H-groupoid and $\mathbf{G}$ be a closed congruence defined in Definition 3.7. Then the map $\beta : M/\mathbf{G} \to M$ defined by $\beta(\overline{x}) = ex$ where $e$ is a fixed idempotent, is a morphic imbedding. Hence $M/\mathbf{G}$ can be morphically imbedded into $M$.

## 5. Canonical Actions

Badnarek and Wallace[1] have defined canonical actions. In this section we derive such maps. We begin with a definition.

*Definition* 5.1 (Badnarek and Wallace[1]) — If $T$ acts on the left on $X$, then an equivalence $\mathbf{R}$ on $X$ is called a $T$-equivalence if $T\mathbf{R} \subseteq \mathbf{R}$, that is, if for any $(x, y) \in \mathbf{R}$ and any $t \in T$ it follows that $(tx, ty) \in \mathbf{R}$.

*Theorem* 5.2 (Badnarek and Wallace[1]) — If $f : T \times X \to X$ is a left compact act and if $\mathbf{C}$ is a closed congruence on $T$ and $\mathbf{R}$ is a closed $T$-equivalence on $X$ satisfying $\mathbf{CR} \subseteq \mathbf{R}$, then there exists one and only one continuous function $\alpha$ such that $\alpha \circ (\psi \times \phi) = \phi \circ f$ ($\psi$ and $\phi$ are natural maps). Thus $T/\mathbf{C}$ acts on the left on $X/\mathbf{R}$, and $\alpha$ is called the canonical action.

*Theorem* 5.3 — Let $T$ be a compact semigroup with continuous multiplication $\hat{t} : T \times T \to T$ denoted by juxtaposition. Let $\mathbf{A}$ and $\mathbf{B}$ be closed congruences such that $\mathbf{A} \subseteq \mathbf{B}$, then $\mathbf{AB} \subseteq \mathbf{B}$. Moreover, the map $f : T/\mathbf{A} \times T/\mathbf{B} \to T/\mathbf{B}$ defined by

$$f(\eta_a(x), \eta_b(y)) = \eta_b(xy) \qquad \ldots (*)$$

is a left compact act, and the one and only one continuous function satisfying (*). Thus $f$ is the canonical action.

PROOF : First we show that $\mathbf{AB} \subseteq \mathbf{B}$. Let $(x, y) \in \mathbf{A}$ and $(x', y') \in \mathbf{B}$. Since $\mathbf{A}$ is a congruence and $(x, y) \in \mathbf{A}$ therefore $(xx', yx') \in \mathbf{A}$. This implies $(xx', yx') \in \mathbf{B}$. Further since $\mathbf{B}$ is a congruence and $(x', y') \in \mathbf{B}$ therefore $(yx', yy') \in \mathbf{B}$. Since $\mathbf{B}$ is

transtitive and $(xx', yx'), (yx', yy')$ both are in $\mathbf{B}$ therefore $(xx', yy') \in \mathbf{B}$. Thus $\mathbf{AB} \subseteq \mathbf{B}$.

Note that remark on page 981 and definition of congruence imply, $\mathbf{B}$ is a closed $M$-equivalence on $M$. The result follows from Theorem 5.2.

We now show the existence of congruences on H-groupoids which satisfy the conditions of the theorem and thus give rise to canonical actions.

Let $\mathbf{P} = \mathbf{F} \cap \mathbf{G}$ where $\mathbf{F}$ and $\mathbf{G}$ are closed congruences defined earlier. Since $\mathbf{F}$ and $\mathbf{G}$ are closed congruences therefore $\mathbf{P}$ is also a closed congruence. By Theorem 5.3 we obtain the following.

*Corollary* 5.4 — Let $M$ be a compact H-groupoid with continuous multiplication $\hat{m} : M \times M \to M$, denoted by juxtaposition. Then $\hat{m}$ is a left compact act and the map $\phi : M/\mathbf{P} \times M/\mathbf{F} \to M/\mathbf{F}$ such that $\phi \circ (\eta_p \times \eta_f) = \eta_f \circ \hat{m}$ is the canonical action, $\eta_p, \eta_f$ being the natural maps.

*Corollary* 5.5 — Let $M$ be a compact H-groupoid with continuous multiplication $\hat{m} : M \times M \to M$ denoted by juxtaposition. Then $\hat{m}$ is a left compact act and the map $\Psi : M/\mathbf{P} \times M/\mathbf{G} \to M/\mathbf{G}$ such that $\Psi \circ (\eta_p \times \eta_g) = \eta_g \circ \hat{m}$ is the canonical action, $\eta_p, \eta_g$ being the natural maps.

Similar results can be proved for right acts.

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