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INVARIANT STATISTICAL CONVERGENCE AND *A*-INVARIANT STATISTICAL CONVERGENCE

FATIH NURAY AND EKREM SAVAS

Department of Mathematics, Firat University, Elaziğ / Turkey

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In this paper we introduce the concepts of σ-statistical convergence and *A*-σ-statistical convergence and give some inclusion relations. Also we obtain necessary and sufficient conditions to characterize the matrices that map some related sequence classes into one another.

1. INTRODUCTION

Let σ be a mapping of the of positive integers into itself. A continuous linear functional φ on *m*, the space of real bounded sequences, is said to be an invariant mean or a σ-mean if and only if

(1) $\phi(x) \geq 0$ when the sequence $x = (x_n)$ has $x_n \geq 0$ for all *n*,

(2) $\phi(e) = 1$ where $e = (1, 1, 1, ...)$, and

(3) $\phi(x_{\sigma(n)}) = \phi(x)$ for all $x \in m$.

The mappings φ are assumed to be one-to-one and such that $\sigma^m(n) \neq n$ for all positive integers *n* and *m*, where $\sigma^m(n)$ denotes the *m*th iterate of the mapping σ at *n*. Thus φ extends the limit functional on *c*, the space of convergent sequences, in the sense that $\phi(x) = \lim x$ for all $x \in c$. In case σ is the translation mapping $\sigma(n) = n + 1$, the σ-mean is often called a Banach limit and V_σ, the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences[5].

If $x = (x_n)$, set $Tx = (Tx_n) = (x_{\sigma(n)})$. It can be shown[16] that

$$V_\sigma = \{x = (x_n) : \lim_m t_{mn}(x) = Le, \text{ uniformly in } n, \; L = \sigma\text{-lim } x\}$$

where

$$t_{mn}(x) = (x_n + Tx_n + \ldots + T^m x_n)/(m+1).$$

Let $V_{\sigma 0}$ denote the set of all sequences which are σ-convergent to zero.

Several authors including Schaefer[16], Mursaleen[7], Savas[13] and others have studied invariant convergent sequences.

Recently, Mursaleen[8] defined strongly σ-convergent sequences by saying that $x_k \to L\,[V_\sigma]$ if and only if

$$\lim_n 1/n \sum_{k=1}^{n} | x_{\sigma^k(m)} - L | \to 0, \text{ uniformly in } m.$$

By $[V_\sigma]$, we denote the set of all strongly σ-convergent sequences. For $\sigma(m) = m + 1$, the space $[V_\sigma]$ is the space of strongly almost convergent sequences. Quite recently, the concept of strong σ-convergence was generalized by Savas[12] as below

$$[V_\sigma]_p = \{x = (x_k) : \lim_n 1/n \sum_{k=1}^{n} | x_{\sigma^k(m)} - L |^p \to 0, \text{ uniformly in } m\}.$$

If $p = 1$, then $[V_\sigma]_p = [V_\sigma]$. It is known that $l_\infty \supset [V_\sigma]_p$, where l_∞ is the set of all bounded sequences. Furthermore, strongly invariant A-summable sequences are defined by Savas[15] as the following :

$$w_0(A_\sigma) = \{x = (x_k) : \lim_n \sum_k a_{nk} | x_{\sigma^k(m)} | = 0, \text{ uniformly in } m\}.$$

Let $f : [0, \infty) \to [0, \infty)$. Then f is called a modulus if

(i) $f(x) = 0$ if and only if $x = 0$,

(ii) $f(x + y) \leq f(x) + f(y)$ for all $x \geq 0$, $y \geq 0$,

(iii) f is increasing

(iv) f is continuous from the right at 0.

By using a modulus function f and a nonnegative regular matrix $A = (a_{nk})$, we defined the sequence space $w(A_\sigma, f)$ as follows :

$$w(A_\sigma, f) = \{x = (x_k) : \lim_n \sum_k a_{nk} f(| x_{\sigma^k(m)} - L |) = 0, \text{ uniformly in } m\}.$$

If $L = 0$, then, in this case, we write $w_0(A_\sigma, f)$ for $w(A_\sigma, f)$. If $x \in w(A_\sigma, f)$ then we say that x is strongly invariant A-summable to L with respect to the modulus f, in the case $A = (a_{nk})$ = Cesaro matrix $(C, 1)$, $w(A_\sigma, f)$ reduces to

$$[V_\sigma(f)] = \{x = (x_k) : \lim_n 1/n \sum_{k=1}^{n} f(| x_{\sigma^k(m)} - L |) \to 0, \text{ uniformly in } m\}.$$

A complex number sequence $x = (x_k)$ is said to be statistically convergent to the number L if for every $\varepsilon > 0$,

$$\lim_n 1/n \; | \{k \leq n : | x_k - L | \geq \varepsilon \} | = 0$$

where the vertical bars indicate the number of elements in the enclosed set. In this case we write S-lim $x = L$ or $x_k \to L(S)$ and we define $S = \{x = (x_k) :$ for some L,

S-Lim $x = L\}$, $S_0 = \{x = (x_k) : S\text{-lim } x = 0\}$.

The idea of the statistical convergence of real numbers was introduced by Fast[3]. Schoenberg[17] studied statistical convergence as a summability method and listed some of the elementary properties of statistical convergence. Both of these authors noted that if a bounded sequence is statistically convergent to L then it is Cesaro summable to L. Statistical convergence also arises as an example of "convergence in density" as introduced by Buck[1].

Subsequently, stastistically convergent sequences have been discussed in Salat[10], Fridy[4], Maddox[6], Savas[14] and others independently.

In this paper we shall introduce the concepts of σ-statistical convergence and A-invariant statistical convergence, and we give some inclusion relations. Also we shall obtain necessary and sufficient conditions to characterize the matrices of classes $(S_0 \bigcap l_\infty, V_{\sigma 0})$ and $(S \bigcap l_\infty, V_\sigma)$, which will fill a gap in the existing literature.

2. Main Results

Definition 2.1.[11] — A set E of positive integers is said to have uniform invariant density of zero if and only if the number of elements of E which lie in the set $\{\sigma(m), \sigma^2(m), \ldots, \sigma^n(m)\}$ is $o(n)$ as $n \to \infty$, uniformly in m.

By using the concept of uniform invariant density, we can give the following definition.

Definition 2.2. — A complex number sequence $x = (x_k)$ is said to be σ-statistically convergent to the number L if for every $\varepsilon > 0$,

$$\lim_n 1/n \mid \{k \le n : \mid x_{\sigma^k(m)} - L \mid \ge \varepsilon\} \mid = 0, \text{ uniformly in } m = 1, 2, \ldots$$

In this case we write S_σ-lim $x = L$ or $x_k \to L(S_\sigma)$ and we define

$$S_\sigma = \{x = (x_k) : \text{for some } L,\ S_\sigma - \lim x = L\}.$$

Now, we shall give some inclusion relations between $[V_\sigma]_p$-convergence and S_σ-convergence and show that these are equivalent for bounded sequences. We also study the relation between S_σ-convergence and $[V_\sigma(f)]$-convergence.

Theorem 2.1 — (i) $x_k \to L([V_\sigma]_p)$, $0 < p < \infty$, implies $x_k \to L\ (S_\sigma)$.

(ii) $x \in l_\infty$ and $x_k \to L(S_\sigma)$ imply $x_k \to L([V_\sigma]_p)$.

(iii) $S_\sigma \bigcap l_\infty = [V_\sigma]_p$.

Proof : If $\varepsilon > 0$ and $x_k \to L([V_\sigma]_p)$, $0 < p < \infty$, we can write

$$\sum_{k=1}^{n} \mid x_{\sigma^k(m)} - L \mid^p \ge \mid \{k \le n : \mid x_{\sigma^k(m)} - L \mid \ge \varepsilon\} \mid \varepsilon^p$$

for each m. It follows that if $x_k \to L([V_\sigma]_p)$ then $x_k \to L(S_\sigma)$.

Now suppose that x is bounded and σ-statistically convergent to L. For each $m \geq 1$ set

$$G = \sup_k \mid x_{\sigma^k(m)} \mid + \mid L \mid .$$

Let $\varepsilon > 0$ be given and select N_ε such that

$$1/n \mid \{k \leq n : \mid x_{\sigma^k(m)} - L \mid \geq (\varepsilon/2)^{1/p}\} \mid < \varepsilon/2G^p$$

for all m and $n > N_\varepsilon$ and set $L_{nm} = \{k \leq n : \mid x_{\sigma^k(m)} - L \mid \geq (\varepsilon/2)^{1/p}\}$.

Now for all m and $n > N_\varepsilon$ we have that

$$1/n \sum_{k=1}^{n} \mid x_{\sigma^k(m)} - L \mid^p = 1/n \Big(\sum_{k \in L_{nm}} \mid x_{\sigma^k(m)} - L \mid^p + \sum_{\substack{k \notin L_{nm} \\ k \leq n}} \mid x_{\sigma^k(m)} - L \mid^p \Big)$$

$$< 1/n \ (n\,\varepsilon/2G^p)\, G^p + 1/n\,(\varepsilon/2)n = \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Hence x is strongly invariant p-convergent to L.

Theorem 2.2 — Let f be any modulus function and x be a sequence; then

(i) $x_k \to L([V_\sigma(f)])$ implies $x_k \to L(S_\sigma)$.

(ii) f is bounded and $x_k \to L(S_\sigma)$ imply $x_k \to L([V_\sigma(f)])$.

(iii) $[V_\sigma(f)] = S_\sigma$ if f is bounded.

The proof of the theorem is similar to the proof of Theorem 2.1 and we omitted it.

If A is nonnegative regular matrix, we can make some connections between strong invariant A-summability with respect to a modulus and A-invariant statistcal convergence. The following definition is an extension of the definition of σ-statistical convergence.

Let $\| x \|_\infty = \sup_k \mid x_k \mid$ and given $\varepsilon > 0$, $S(x;\varepsilon) = \{k \in N : \mid x_k \mid \geq \varepsilon\}$.

If $N \supset S$, we let X_S denote the charateristic function of S.

Definition 2.3 — Let A be a nonegative regular matrix and x be a sequence. Then x is said to be A-invariant statistically convergent to L if $X_{S(x-Le;\,\varepsilon)} \in w_0(A_\sigma)$ for every $\varepsilon > 0$.

Now, we give some inclusion relations between A-invariant statisticall convergence and $w(A_\sigma, f)$-convergence. Before giving these inclusion relations we give two lemmas and a theorem which will be used in the proof of Theorem 2.4.

Lemma 2.1[14] — Let f be a modulus and $\alpha > 0$ be a given constant. Then there exists a constant $c > 0$ such that $f(x) > cx$, $(0 < x < \alpha)$.

Lemma 2.2[9] — Let A be a nonegative regular matrix and f be a modulus; then $w_0(A_\sigma, f) \supset w_0(A_\sigma)$.

The proof of the following theorem follows immediately from Lemma 2.1 and

Lemma 2.2.

Theorem 2.3 — Let x be a bounded sequence, f be a modulus and A be a nonnegative regular matrix. Then x is strongly invariant A-summable to 0 with respect to the modulus f if and only if x is strongly invariant A-summable to 0, i.e., $w_0(A_\sigma) \cap l_\infty = w_0(A_\sigma, f) \cap l_\infty$.

Theorem 2.4 — Let A be a nonnegative regular matrix and f be a modulus.

(i) If x is strongly invariant A-summable to L with respect to the modulus f, then x is A-invariant statistically convergent to L.

(ii) If x is bounded and A-invariant statistically convergent to L, then x is strongly invariant A-summable to L with respect to the modulus f.

PROOF : (i) If $x \in w_0(A_\sigma, f)$ and $y \in l_\infty$ then $xy \in w_0(A_\sigma, f)$. Now suppose that $x \in w_0(A_\sigma, f)$ and $\varepsilon > 0$ has been given. Define $y \in l_\infty$ by $y_k = 1/x_k$ if $|x_k| \geq \varepsilon$ and$y_k = 0$ otherwise. Consequently,

$$xy = X_{S(x;\,\varepsilon)} \in w_0(A_\sigma, f) \cap l_\infty$$

and from Theorem 2.3. we have $x_{S(x;\,\varepsilon)} \in w_0(A_\sigma) \cap l_\infty$, hence x is A-invariant statistically convergent to 0. The remainder of the claim follows immediately.

(ii) Now suppose that $x \in l_\infty$ and x is A-invariant statistically convergent to L, then the definition yields that

$$X_{S(x-Le;\,\varepsilon)} \in w_0(A_\sigma) \cap l_\infty \text{ for every } \varepsilon > 0.$$

Note that if $x \in l_\infty$ and $X_{S(x-Le;\,\varepsilon)} \in w_0(A_\sigma) \cap l_\infty$ then

$$\| x - Le - (x - Le)X_{S(x-Le;\,\varepsilon)} \|_\infty < \varepsilon.$$

It follows that if $X_{S(x-Le;\,\varepsilon)} \in w_0(A_\sigma) \cap l_\infty$ for all $\varepsilon > 0$, then $x - Le$ is in the closure of $w_0(A_\sigma) \cap l_\infty$. Since $w_0(A_\sigma) \cap l_\infty$ is closed, $x - Le \in w_0(A_\sigma) \cap l_\infty$, and from Theorem 2.3., we write $x - Le \in w_0(A_\sigma, f) \cap l_\infty$, i.e., $x - Le\ w_0(A_\sigma, f)$.

3. MATRIX TRANSFORMATIONS

Let E and F be two nonempty subsets of the space w of complex sequences, and $A = (a_{nk})$, $(n,\ k = 1, 2, 3, \ldots)$ be an infinite matrix of complex numbers. We write $Ax = (A_n(x))$ if $A_n(x) = \sum_k a_{nk} x_k$ converges for each n. If $x = (x_k) \in E$ implies $Ax \in F$, we say that A defines a matrix transformation from E to F and we denote it by $A : E \to F$. By (E, F) we mean the class of matrices A such that $A : E \to F$. We shall use the notation $a(n, k)$ to denote the element a_{nk} of the matrix. c_0 will denote the space of null sequences.

We now characterize the matrices in the classes $(S_0 \cap l_\infty, V_{\sigma 0})$ and $(S \cap l_\infty, V_\sigma)$. We write

$$t_{mn}(Ax) = \sum_k a(n, k, m)\, x_k$$

where

$$a(n, k, m) = \sum_{k=0}^{m} a(\sigma^j(n), k)/(m+1).$$

Definition 3.1[2] — Let $s = (s_j)$ be a strictly increasing sequence of integers with $s_1 \geq 1$. We say that $s \in S$ if

$$\lim (S_{2m})^{-1} \sum_{t=1}^{m} (s_{2t} - s_{2t-1}) = 0.$$

We let θ denote a divergent sequence of 0's and 1's and s denote a strictly increasing sequence of natural numbers. For a given $s = (s_k)$ we define $\theta^{(s)}$ by

$$\theta_k(s) = 1 \quad \text{if} \quad s_{2t-1} \leq k < s_{2t} \text{ and}$$

$$\theta_k(s) = 0 \quad \text{if} \quad s_{2t} \leq k < s_{2t+1} \text{ or } k < s_1.$$

Given $\theta = (\theta_k)$, we define $s^{(\theta)}$ by the relations :

if $s^{(\theta)}{}_{2t-1} \leq k < s^{(\theta)}{}_{2t}$ then $\theta_k = 1$ and

if $s^{(\theta)}{}_{2t} \leq k < s^{(\theta)}{}_{2t+1}$ or $k < s_1$, then $\theta_k = 0$ for $t = 1, 2, \ldots$

Lemma 3.2[2] — Let s be a strictly increasing sequence of natural numbers. Then $s^{(\theta)}$ is statistically null if and only if $s \in S$.

Lemma 3.3[2] — If $x \in w$ statistically convergent to L, then there is a convergent sequence y and a statistically null sequence z such that y is convergent to L, $x = y + z$ and $\lim_n 1/n \mid \{k \leq n : z_k \neq 0\} \mid = 0$. Moreover if x is bounded then z is also bounded and $\| z \|_\infty \leq \| x \|_\infty + | L |$.

Theorem 3.1 — $A \in (S_0 \cap l_\infty, V_{\sigma 0})$ if and only if

(i) $A \in (c_0, V_{\sigma 0})$ and

(ii) $\lim_m \sum_{t=1}^{\infty} \sum_{k=s_{2t-1}}^{s_{2t}} | a(n, k, m) | = 0$ for every $s \in S$, uniformly in n.

PROOF : *Necessity* — Since null sequences are bounded and statistically null, the necessity of (i) is clear. Now suppose that the condition (ii) does not hold for some $s \in S$, then there exist a $\delta > 0$ and a natural numbers sequence $\{n_j\}$ and increasing natural numbers sequence $\{m_j\}$ such that

$$\sum_{t=1}^{\infty} \sum_{k=s_{2t-1}}^{s_{2t}} | a(n_j, k, m_j) | \geq 2\delta$$

for every $j \in N$. Using the fact that A maps c_0 into $V_{\sigma 0}$ and a sliding hump construction, it is possible to find two sequences of natural numbers (p_j) and (q_j)

such that $p_j < q_j < p_{j+1}$ for all $j \in N$ and

$$\sum_{t=1}^{p_j} \sum_{k=s_{2t-1}}^{s_{2t}} | a(n_j, k, m_j) | < \delta/2$$

$$\sum_{t=q_{j+1}}^{\infty} \sum_{k=s_{2t-1}}^{s_{2t}} | a(n_j, k, m_j) | < \delta/2, \text{ and}$$

$$\sum_{t=p_j}^{q_j} \sum_{k=s_{2t-1}}^{s_{2t}} | a(n_j, k, m_j) | > \delta.$$

Define a sequence (z_k) by $(z_k)(a(n_j, k, m_j)) = | a(n_j, k, m_j) |$ if $s_{2t} \le k < s_{2t+1}$ and $p_j < t < q_j$ and $z_k = 0$ elsewhere. Observe that by construction, $| t_{m_j n_j}(Az) | > \delta$ for all $j \in N$ and also note that z has been constructed in such a fashion that if $\theta_k^{(s)} = 0$ then $z_k = 0$ and hence z is statistically null. This contradicts the hypothesis that A takes bounded statistically null sequences into $V_{\sigma 0}$.

Sufficiency — Suppose that conditions (i) and (ii) hold and $x \in S_0 \cap l_\infty$. Apply Lemma 3.2 to write $x = y + z$ where y is a null sequence and $\lim_n (1/n) | \{k \le n : z_k \ne 0\} | = 0$. We assume, without loss of generality, that $|| x ||_\infty \le 1$ and hence $|| z ||_\infty \le 1$. It is known that any sequence whose support is contained in the support of z is also statistically null. We now claim that $Az \in V_{\sigma 0}$..

First note that $t_{mn}(Az)$ exists for each n and m, this follows from (i) and that z is bounded. Now we define a sequence θ by $\theta_k = 1$ if $z_k \ne 0$ and 0 otherwise. As remarked above, θ is also statistically null and $| z_k | \le \theta_k < 1$ for all $k \in N$. It follows that

$$| t_{mn}(Az) | = | \sum_k a(n, k, m) z_k | \le \sum_k | a(n, k, m) z_k |$$

$$\le \sum_k | a(n, k, m) | \theta_k = \sum_{t=1}^{\infty} \sum_{k=s_{2t-1}}^{s_{2t}} | a(n, k, m) |$$

where $s = s^{(\theta)}$. Since θ is statistically null $s^{(\theta)} \in S$ and by (ii), $\lim_m t_{mn}(Az) = 0$ uniformly in n, and hence we have $Az \in V_{\sigma 0}$ and consequently $Ax = Ay + Az \in V_{\sigma 0}$, and this completes the proof.

By using Lemma 3.2., we can prove following theorem :

Theorem 3.2 — Let A be a σ-regular matrix and $A \in (S_0, V_{\sigma 0})$. If x is a bounded sequence which is statistically convergent to L, then x is invariant A-summable to L, i.e., $A \in (S \cap l_\infty, V_\sigma)$.

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