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ON A QUASILINEAR FUNCTIONAL INTEGRODIFFERENTIAL EQUATION IN A BANACH SPACE

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This paper is concerned with the study of the existence of a unique strong solution of a quasilinear abstract functional integro-differential equation of the more general type. The main tools employed in our analysis are based on the applications of the method of lines, the well known Banach fixed point theorem and the integral inquality established by Pachpatte.

1. INTRODUCTION

Let X be a real Banach space with norm $\|\cdot\|$. Let PC denote the space of piecewise continuous functions $\psi:[-r,0]\to X$ for a fixed $r \geq 0$. For $\psi \in PC$, $\|\psi\|_{PC} = \sup_{-r\leq\theta\leq 0}\|\psi(\theta)\|$. Let $X_t \in PC$ be defined by $x_t(\theta) = x(t+\theta)$, $\theta\in[-r,0]$. In the present paper we study the quasi-linear abstract functional integro-differential equation of the type

$$x'(t) + A(t, x_t)\,x(t) = f(t, x_t, \int_0^t k(t, s, x_s)\,ds),\ t\in[0, T)$$

$$x_0 = \phi(t),\ -r \leq t \leq 0 \qquad \text{... (1.1)}$$

where $x : [-r, T) \to X$, for each $(t,\psi)\in[0,T)\times PC, A(t,\psi)v$ is m-accretive in v, $k : [0, T) \times PC \to X, f:[0,T)\times PC\times X\to X$ are locally Lipschitzian like functions in their arguments and $\phi:[-r,0]\to X$ is a given Lipschitzian function.

There are many papers written on the various special forms of (1.1) from different points of view[4, 5, 8-11, 13, 14, 19]. In particular, Kartsatos and Parrott[11] have established the existence of the unique strong solution of (1.1) when $k = 0$ and $A(t, x_t) = A(t)$ is m-accretive operator. Pachpatte[18] has studied the stability, asymptotic

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behavior and other properties of the solutions of (1.1) without functional arguments and when $A(t, x_t) = A(t)$ is linear closed operator. In a recent paper Khan and Pachpatte[14] have studied the existence, uniqueness and other properties of the solutions of (1.1) without funcitonal arguments and when the operator $A(t, x_t) = A(t, x(t))$ is in general nonconstant. The special forms of (1.1) represent various mathematical models of physical phenomena arising in many areas of applied mathematics [2, 3, 15, 16]. Equation (1.1) is of more general type and the study of various properties of solutions of (1.1) is more interesting and merits special attention. The aim of the present paper is to establish the existence of a unique strong solution $x(t)$, $t \in [-r, T)$ of (1.1). The well known Banach fixed point theorem, the method of lines and the integral inequality established by Pachpatte[17] are the main tools used in our analysis.

In order to establish the existence of the unique strong solution of (1.1), we first ensure the existence of solution $x(t)$ or $x_u(t)$ of the equation

$$x'(t) + A(t, u_t)x(t) = f(t, x_t, \int_0^t k(t, s, x_s)\, ds),\ t \in [0, T_1]$$

$$x_0 = \phi(t),\ -r \le t \le 0 \qquad \dots (1.2)$$

on an interval $[-r, T_1] \subset [-r, T)$ where u is fixed and belongs to E-a suitable metric space of continuous fucntions, as the uniform limit of "lines" which are the solutions of approximate discrete equations for (1.2). For sufficiently small T_1, we prove that the operator $S : u \to x_u$ maps the space E into itself and is a strict contraction. Then it follows that the fixed point of the operator S is the desired unique strong solution of (1.1).

The paper is organized as follows.

Section 2 deals with the preliminaries and statement of the main result. The proof of the main result is given in Section 3. Finally, in section 4, we give an example to illustrate the application of our results to an interesting model of partial functional integrodifferential equation.

2. Preliminaries and Statement of Result

We first give some preliminaries from Kartsatos and Parrot[10, 11] and hypotheses used in our further discussion.

Let X^* denote the dual space of X. Let $\langle x, x^* \rangle$ be the value of the functional $x^* \in X^*$ at $x \in X$. Define the duality map U on X as follows.

$$U(x) = \{x^* \in X^* : \langle x, x^* \rangle = ||x||^2 = ||x^*||^2\}.$$

By the virtue of Hahn-Banach theorem, the set $U(x)$ is nonempty for each $x \in X$. If X^* is uniformly convex, then the duality mapping U is single valued and uniformly continuous on bounded subset of X. An operator $A : D(A) \subset X \to X$ is said to be accretive if

$$\langle Ax_1 - Ax_2 , \quad U(x_1 - x_2) \rangle \geq 0$$

for every $x_1, x_2 \in D(A)$. An accretive operator A is said to be m-accretive if $R(I + \mu A) = X$ for some (equivalently, for all) $\mu > 0$, where R and I denote the range of $I + \mu A$ and the identity operator on X respectively. If an operator A is m-aceretive then for any $\mu > 0$, the operator $(A + \mu I)^{-1}$ exists and

$$|| (A + \mu I)^{-1} x_1 - (A + \mu I)^{-1} x_2 || \leq (1/\mu) \; || x_1 - x_2 || \qquad \dots (2.1)$$

for every $x_1, x_2 \in X$. (see, Kato[12]). By a strong solution of (1.1) on $[-r, T_1] \subset [-r, T)$ we mean an absolutely continuous X-valued function which, for almost all $t \in [0, T_1)$, is strongly differentiable and satisfies (1.1).

We need the following hypothesis in our subsequent discussion

(H_1) X^* is uniformly convex.

(H_2) The domain of the operator $A_0(., ., .)$ with $A_0(t, \psi, v) = A(t, \psi) v$ is the set $[0, T) \times PC_0 \times D$, where D is a subset of X and PC_0 consists of all $g \in PC$ with $g(t) \in D \bigcup D_0$, $t \in [-r, 0]$ where $D_0 = \{\phi(t) : t \in [-r, 0]\}$.

(H_3) For each $(t, \psi) \in [0, T] \times PC_0$, $A(t, \psi) v$ is m-accretive in v.

(H_4) For each $t, s \in [0, T)$, $\psi_1, \psi_2 \in PC_0$, $v \in D$

$$|| A(t, \psi_1) v - A(s, \psi_2) v ||$$
$$\leq r_0 (|| \psi_1 ||_{PC}, || \psi_2 ||_{PC}, || v ||)$$
$$[\, | t - s | \, (1 + || A(s, \psi_2) v ||) + || \psi_1 - \psi_2 ||_{PC}] \qquad \dots (2.2)$$

where $r_0 : (R^+)^3 \to R^+ = [0, \infty)$ is increasing in all of its arguments.

(H_5) For $t, t_1, t_2, s, s_1, s_2 \in [0, T)$, $\psi, \psi_1, \psi_2 \in PC$

$$|| k(t, s, \psi_1) - k(t, s, \psi_2) || \leq a_1 (t, s) \, || \psi_1 - \psi_2 ||_{PC} \qquad \dots (2.3)$$

$$|| k(t_1, s_1, \psi) - k(t_2, s_2, \psi) || \leq r_1 (|| \psi ||_{PC}) \, [\, | t_1 - t_2 | + | s_1 - s_2 |] \qquad \dots (2.4)$$

where $a_1 : [0, T) \times [0, T) \to R^+$ is continuous function and $r_1 : R^+ \to R^+$ is a non-decreasing function.

(H_6) For $t, s \in [0, T)$, $\psi, \psi_2, \psi_2 \in PC$ and $y, y_1, y_2 \in X$,

$$|| f(t, \psi_1, y_1) - f(t, \psi_2, y_2) ||$$
$$\leq a_2 (t) [\, || \psi_1 - \psi_2 ||_{PC} + || y_1 - y_2 || \,] \qquad \dots (2.5)$$

$$|| f(t, \psi, y) - f(s, \psi, y) ||$$
$$\leq r_2 (|| \psi ||_{PC}, || y ||) \, | t - s | \qquad \dots (2.6)$$

where $a_2 : [0, T) \to R^+$ is continuous and $r_2 : (R^+)^2 \to R^+$ is nondecreasing in both the variables.

(H_7) ϕ is a given Lipschitzian function with Lipschitz constant N on $[-r, 0]$ and $\phi(0) \in D$.

Before we state our main result to be proved in this paper, we discuss in brief the approximation method[11].

Let $M = 1 + \| A(0, \phi(0)) \|$. Choose a positive constant Q such that $Q/M < T$. Let T_1 be such that $0 < T_1 < Q/M$. Define a set E as follows

$$E = \{u : [-r, T_1] \to \overline{D} \bigcup D_0 ;\; u(t) \text{ is continuous}$$

$$u(t) = \phi(t) \text{ for } t \in [-r, 0] \text{ and}$$

$$\| u(t_1) - u(t_2) \| \le M \mid t_1 - t_2 \mid \text{ for } t_1, t_2 \in [0, T_1]\}.$$

Clearly, $E \neq \phi$ because the function $u(t)$ defined by $u(t) = \phi(t)$, $t \in [-r, 0]$ and $u(t) = \phi(0)$ for $t \in [0, T_1]$ belongs to E. Moreover, the set E is a Banach space with supremum norm. Now let $u \in E$ be given and consider the problem (1.2) on the interval $[0, T_1]$. In the following discussion, we assume that the hypotheses (H_1), (H_2), (H_3), (H_7), (2.3) and (2.5) hold.

Consider a partition $\{t_{nj}\}$ of the interval $[0, T_1]$, where $t_{nj} = jh = jT_1/n$, $j = 0, 1, \ldots, n$ ($t_{n0} = 0$, $t_{nn} = T_1$). Choose n sufficiently large so that $h(M_1 T_1 + 1) M_2 < 1$ where $a_1(t, s) \le M_1$ and $a_2(t) \le M_2$ for $t, s \in [0, T_1]$. Define

$$\overline{w}_{n0}(t) = \begin{cases} \phi(t), & t \in [-r, 0] \\ \phi(0), & t \in [0, T_1] \end{cases}$$

and let the space Y_1 be defined by

$$Y_1 = \{x : [-r, T_1] \to X,\; x(t) = \phi(t),\; t \in [-r, 0]$$

and $x(t)$ is constant on $(0, T_1]\}$.

It is easy to see that the space Y_1 is a complete metric space with distance function induced by the norm

$$\| x \|_{T_1} = \sup_{t \in [-r, T_1]} \| x(t) \|. \qquad \ldots (2.7)$$

We now define a mapping L_1 on Y_1 as follows

$$(L_1 x)(t) = \begin{cases} \phi(t), \quad t \in [-r, 0], \\ [A(t_{n0}, u_{t_{n0}}) + (1/h) I]^{-1} \\ \left(w_{n0}/h + f(t_{n0}, x_{t_{n0}}, \int_0^{t_{n0}} k(t_{n0}, s, x_s)\, ds)\right), \\ \qquad t \in (0, T_1] \end{cases} \qquad \ldots (2.8)$$

where $w_{n0} = \phi(0)$. The mapping $L_1 : Y_1 \to Y_1$. For $x, y \in Y_1$, by using (2.1), (2.5), (2.7) and (2.8) we obtain

$$\begin{aligned} || L_1 x - L_1 y ||_{T_1} &\le h \, || f(t_{n0}, x_{t_{n0}}, 0) - f(t_{n0}, y_{t_{n0}}, 0) || \\ &\le h \, a_2(t_{n0}) \, || x_{t_{n0}} - y_{t_{n0}} ||_{PC} \\ &\le hM_2 \sup_{\theta \in [-r, 0]} || x_{t_{n0}}(\theta) - y_{t_{n0}}(\theta) || \\ &= hM_2 \sup_{t \in [t_{n0} - r, t_{n0}]} || x(t) - y(t) || \\ &\le hM_2 \sup_{t \in [-r, T_1]} || x(t) - y(t) || \\ &= hM_2 \, || x - y ||_{T_1} \\ &< || x - y ||_{T_1}. \end{aligned}$$

Thus, the mapping $L_1 : Y_1 \to Y_1$ is a strict contraction. The unique fixed point of L_1 on Y_1 is denoted by $\overline{w}_{n1}$. Let

$$w_{n1} = [A(t_{n0}, u_{t_{n0}}) + (1/h) I]^{-1} \, (w_{n0}/h + f(t_{n0}, \overline{w}_{n1_{t_{n0}}}, 0)). \qquad \text{... (2.9)}$$

Continuing in this way we obtain the spaces y_j, $j = 2, 3, ..., n$, defined by

$$Y_j = [x : [-r, T_1] \to X; \; x(t) = \phi(t) \text{ for } t \in [-r, 0]$$

and $x(t)$ is constant on each interval

$$(0, t_{n1}], \, (t_{n1}, t_{n2}], ..., (t_{n,j-1}, T_1]\}.$$

It is easy to observe that for each $j = 2, 3, ..., n$, Y_j is a complete metric space with the same distance function as on Y_1. We define a mapping L_j on Y_j, $j = 2, 3, ..., n$, as follows.

$$(L_j x)(t) = \begin{cases} \phi(t), & t \in [-r, 0] \\ w_{n1}, & t \in (0, t_{n1}] \\ w_{n2}, & t \in (t_{n1}, t_{n2}] \\ \cdots \\ \cdots \\ \cdots \\ w_{n,j-1}, & t \in (t_{n,j-2}, t_{n,j-1}] \\ [A(t_{n,j-1}, u_{t_{n,j-1}}) + (1/h) I]^{-1} \\ \quad (w_{n,j-1}/h + f(t_{n,j-1}, x_{t_{n,j-1}}, \int_0^{t_{n,j-1}} k(t_{n,j-1}, s, x_s) \, ds), \\ & t \in (t_{n,j-1}, T_1]. \end{cases} \qquad \text{... (2.10)}$$

For each $j = 2, 3, \ldots, n$, $L_j : Y_j \to Y_j$ is a strict contraction on Y_j. In fact, by using (2.1), (2.3), (2.5), (2.7), and (2.10) we have for $x, y \in Y_j$

$$||L_j x - L_j y||_{T_1}$$

$$\leq h \left|\left| f\left(t_{n,j-1}, x_{t_{n,j-1}}, \int_0^{t_{n,j-1}} k(t_{n,j-1}, s, x_s)\, ds\right)\right.\right.$$

$$\left.\left. - f\left(t_{n,j-1}, y_{t_{n,j-1}}, \int_0^{t_{n,j-1}} k(t_{n,j-1}, s, y_s)\, ds\right)\right|\right|$$

$$\leq h a_2(t_{n,j-1}) [\, ||x_{t_{n,j-1}} - y_{t_{n,j-1}}||_{PC}$$

$$+ \int_0^{t_{n,j-1}} a_1(t_{n,j-1}, s)\, ||x_s - y_s||_{PC}\, ds\,]$$

$$\leq hm_2 [\, \sup_{\theta \in [-r,0]} ||x(t_{n,j-1} + \theta) - y(t_{n,j-1} + \theta)||$$

$$+ M_1 \int_0^{t_{n,j-1}} \sup_{\theta \in [-r,0]} ||x(s+\theta) - y(s+\theta)||\, ds\,]$$

$$= hM_2 [\, \sup_{t \in [t_{n,j-1} - r, t_{n,j-1}]} ||x(t) - y(t)||$$

$$+ M_1 \int_0^{t_{n,j-1}} \sup_{\tau \in [s-r, s]} ||x(\tau) - y(\tau)||\, ds\,]$$

$$\leq hM_2 [\, ||x-y||_{T_1} + M_2 ||x-y||_{T_1} \cdot T_1]$$

$$= h(1 + M_1 T_1) M_2 ||x-y||_{T_1} < ||x-y||_{T_1}.$$

We denote the unique fixed point of L_j on Y_j by $\overline{w}_{nj}$ and let

$$w_{nj} = [A(t_{n,j-1}, u_{t_{n,j-1}}) + (1/h) I\,]^{-1}$$

$$(w_{n,j-1}/h + f(t_{n,j-1}, \overline{w}_{nj_{t_{n,j-1}}}, \int_0^{t_{n,j-1}} k(t_{n,j-1}, s, \overline{w}_{nj_s})\, ds)) \qquad \ldots (2.11)$$

$j = 2, 3, \ldots, n$. From (2.9) and (2.11), it is easy to observe that for each $j = 1, 2, \ldots, n$, the points w_{nj} satisfies the equation

$$A(t_{n,j-1}, u_{t_{n,j-1}}) w_{nj} + (w_{nj} - w_{n,j-1})/h$$

$$= f(t_{n,j-1}, \overline{w}_{nj_{t_{n,j-1}}}, \int_0^{t_{n,j-1}} k(t_{n,j-1}, s, \overline{w}_{nj_s})\, ds). \qquad \ldots (2.12)$$

We define the following sequences of functions used in later discussion :

$$w^n(t) = \begin{cases} \phi(t), & t \in [-r, 0] \\ w_{n,j-1} + (t - t_{n,j-1})(w_{nj} - w_{n,j-1})/h, & t \in [t_{n,j-1}, t_{nj}] \end{cases} \quad \text{... (2.13)}$$

and

$$x^n(t) = \overline{w}_{nn}(t), \quad t \in [-r, T_1] \quad \text{... (2.14)}$$

where

$$\overline{w}_{nn}(t) = \begin{cases} \phi(t), & t \in [-r, 0] \\ w_{n1}, & t \in (0, t_{n1}] \\ \cdots \\ \cdots \\ \cdots \\ w_{nj}, & t \in (t_{n,j-1}, t_{nj}] \\ \cdots \\ \cdots \\ w_{nn} = [A(t_{n,n-1}, u_{t_{n,n-1}}) + (1/h) I]^{-1} \\ \quad (w_{n,n-1}/h + f(t_{n,n-1}, \overline{w}_{nn_{t_{n,n-1}}}, \int_0^{t_{n,n-1}} k(t_{n,n-1}, s, \overline{w}_{nn_s})\, ds)) \\ \qquad t \in (t_{n,n-1}, t_{nn} = T_1]. \end{cases}$$

We are now ready to state our main result to be proved in this paper.

Theorem — Assume that the hypotheses (H_1) – (H_7) hold. Then the eqn. (1.1) has a unique strong solution $x(t)$ on $[-r, T_1]$, which is also Lipschitz continuous on $[-r, T_1]$ where $0 < T_1 < T$.

Remark 1 : We note that the authors[13, 14] have studied the existence, uniqueness and other properties of solutions of (1.1) without function arguments when $k = 0$ and $A(t, x_t) = A(t, x(t))$ is the infinitesimal generator of a semigroup in X. However our conditions on $A(t, x_t)$, k, f and approach to the problem are different from those used by the authors[13, 14].

3. Proof of the Theorem

Before we prove our main result we first prepare some lemmas needed in our further discussion.

Lemma 1 — (Pachpatte[17], p.758) — Let $a(t)$, $b(t)$ and $c(t)$ be real-valued nonnegative continuous functions defined on R^+, for which the inequality

$$c(t) \le c_0 + \int_0^t a(s)\, c(s)\, ds + \int_0^t a(s) \left[\int_0^s b(\tau)\, c(\tau)\, d\tau\right] ds,$$

holds for all $t \in R^+$, where c_0 is a nonnegative constant. Then

$$c(t) \le c_0 [1 + \int_0^t a(s) \exp [\int_0^s (a(\tau) + b(\tau)) d\tau] ds],$$

for all $t \in R^+$

In what follows " $\to$ " (" $\rightharpoonup$ " denotes strong (weak) convergence and we suppose that the hypotheses $(H_1)-(H_7)$ hold.

Lemma 2 — The sequence $\{w_{nj}\}$ is uniformly bounded.

Lemma 3 — The sequence $(w_{nj} - w_{n,j-1})/h$ is uniformly bounded.

Lemma 4 — The sequence $\{x^n(t) - w^n(t)\}$ converges to 0 as $n \to \infty$ uniformly on $[-r, T_1]$, where $x^{(n)}$ and $w^n(t)$ are defined by (2.14) and (2.13) respectively.

We note that the proofs of Lemmas 2-4 can be worked out on similar lines as in the proofs of Lemmas in Kartsatos and Parrott[11] with suitable modifications. We omit the details.

In view of Lemmas 2 and 3, there exists constants n_0, n_1, Q_1 and Q_2 with $n_1 \ge n_0$ such that for $n \ge n_0$, we have

$$|| w_{nj} || \le Q_1, \quad j = 0, 1, 2, ..., n \qquad ... (3.1)$$

and for $n \ge n_1$, we have

$$|| w_{nj} - w_{n,j-1} || / h \le Q_2, \quad j = 1, 2, ..., n. \qquad ... (3.2)$$

Remark 2 : It is to be noted that the sequence $\{w^n(t)\}$, $t \in [-r, T_1]$, defined by (2.13), is uniformly Lipschitz continuous with Lipschitz constant $N_1 = \max \{Q_2, N\}$.

Denote the zero function in *PC* by $\overline{O}$ and let M_3 and M_4 be positive real numbers such that $|| k(t, s, \overline{O}) || \le M_3$, $s, t \in [0, T_1]$ and $|| f(t, \overline{O} O) || \le M_4$, $t \in [0, T_1]$. It is easy to obtain the following estimate by using (2.3) – (2.5) and (3.1),

$$|| f(t_{n,j-1}, \overline{w}_{nj_{t_{n,j-1}}}, \int_0^{t_{n,j-1}} K(t_{n,j-1}, s\overline{w}_{nj_s}) ds) ||$$

$$\le M_2 [Q_1 + (M_1 Q_1 + M_3) T_1] + M_4. \qquad ... (3.3)$$

Lemma 5 — The sequence of functions $\{x^n(t)\}$, $t \in [0, T_1]$, converges uniformly as $n \to \infty$ to a strongly continuous function $x(t)$. Moreover, $x(t) \in D$, $t \in [0, T_1]$, $A(t, u_t) x(t)$ is weakly continuous and the strong derivative $x'(t)$ exists and equals

$$- A(t, u_t) x(t) + f(t, x_t, \int_0^t k(t, s, x_s) ds) \text{ a.e. on } [0, T_1].$$

To prove the Lemma 5, we first define operators $A^n(t, u_t)$ and $f^n(t)$ as follows

$$A^n(0, \phi) = A(0, \phi)\, \phi(0)$$

$$A^n(t, u_t) = A(t_{n,j-1}, u_{t_{n,j-1}})\, w_{nj} \quad \text{for} \quad t_{n,j-1} < t \le t_{nj}$$

$$f^n(t) = f\Big(t_{n,j-1}, \overline{w}_{nj_{t_{n,j-1}}}, \int_0^{t_{n,j-1}} k(t_{n,j-1}, s, \overline{w}_{nj_s})\, ds\Big)$$

for $\quad t_{n,j-1} < t \le t_{nj}.$

Using (2.13) and (3.2), we observe that the function $w^n(t)$ is strongly differentiable on $[0, T_1]$ except at a finite number of points at which the strong left derivative $(d^-/dt)\, w^n(t)$ exists. From (2.12) and (2.13) we get a differential equation

$$(d^-/dt)\, w^n(t) + A^n(t, u_t) = f^n(t), \quad t \in (t_{n,j-1}, t_{nj}]. \qquad \dots (3.4)$$

We now prove that $w^n(t)$ converges uniformly to a strongly continuous function $x(t)$ as $n \to \infty$. by virtue of Lemma 4, it follows that $x^n(t)$ also converges uniformly to $x(t)$ as $n \to \infty$. Let $\{t_{nj}\}$ and $\{t_{mi}\}$ be two partitions of $[0, T_1]$, where

$$t_{nj} = jT_1/n, \; j = 0, 1, \dots, n, \; t_{mi} = iT_1/m, \; i = 0, 1, \dots, m.$$

Let $t \in (t_{m,i-1}, t_{mi}] \cap (t_{n,j-1}, t_{nj}]$. Using (3.4), Lemma 1.3 of Kato[12] and Schwarz inequality we obtain

$$\begin{aligned}
&(d^-/dt)\, \| w^m(t) - w^n(t) \|^2 \\
&= 2\, \| w^m(t) - w^n(t) \| \; (d^-/dt)\, \| w^m(t) - w^n(t) \| \\
&= 2 \langle (d^-/dt)\, w^m(t) - (d^-/dt)\, w^n(t), \; U(w^m(t) - w^n(t)) \rangle \\
&= 2 \langle -A^m(t, u_t) + f^m(t) + A^n(t, u_t) - f^n(t), \; U(w^m(t) - w^n(t)) \rangle \\
&\le 2\, \| f^m(t) - f^n(t) \| \; \| w^m(t) - w^n(t) \| \qquad \dots (3.5) \\
&\quad - 2 \langle A^m(t, u_t) - A^n(t, u_t), \; U(w^m(t) - w^n(t)) \rangle.
\end{aligned}$$

Using the definition of an operator $f^n(t)$, (2.5), (2.6) and (3.1), we obtain

$$\begin{aligned}
&\| f^m(t) - f^n(t) \| \\
&\le \Big\| f\Big(t_{m,i-1}, \overline{w}_{mi_{t_{m,i-1}}}, \int_0^{t_{m,i-1}} k(t_{m,i-1}, s, \overline{w}_{mi_s})\, ds\Big) \\
&\quad - f\Big(t_{n,j-1}, \overline{w}_{mi_{t_{m,i-1}}}, \int_0^{t_{m,i-1}} k(t_{m,i-1}, s, \overline{w}_{mi_s})\, ds\Big) \Big\| \\
&\quad + \Big\| f\Big(t_{n,j-1}, \overline{w}_{mi_{t_{m,i-1}}}, \int_0^{t_{m,i-1}} k(t_{m,i-1}, s, \overline{w}_{mi_s})\, ds\Big)
\end{aligned}$$

$$- f(t_{n,j-1}, \overline{w}_{mi_{t_{n,j-1}}}, \int_0^{t_{n,j-1}} k(t_{n,j-1}, s, \overline{w}_{mi_s})\, ds) \|$$

$$+ \| f(t_{n,j-1}, \overline{w}_{mi_{t_{n,j-1}}}, \int_0^{t_{n,j-1}} k(t_{n,j-1}, s, \overline{w}_{mi_s})\, ds)$$

$$- f(t_{n,j-1}, \overline{w}_{nj_{t_{n,j-1}}}, \int_0^{t_{n,j-1}} k(t_{n,j-1}, s, \overline{w}_{nj_s})\, ds) \|$$

$$\leq | t_{m,i-1} - t_{n,j-1} | \, r_2 (Q_1, \| \int_0^{t_{m,i-1}} k(t_{m,i-1}, s, \overline{w}_{mi_s})\, ds \|)$$

$$+ M_2 [\| \overline{w}_{mi_{t_{m,i-1}}} - \overline{w}_{mi_{t_{n,j-1}}} \|_{PC}$$

$$+ \| \int_0^{t_{m,i-1}} k(t_{m,i-1}, s, \overline{w}_{mi_s})\, ds - \int_0^{t_{n,j-1}} k(t_{n,j-1}, s, \overline{w}_{mi_s})\, ds \|]$$

$$+ M_2 [\mathrm{Sup}_{s \in [t_{n,j-1} - r,\, t_{n,j-1}]} \| \overline{w}_{mi}(s) - \overline{w}_{nj}(s) \|$$

$$+ \| \int_0^{t_{n,j-1}} \{ k(t_{n,j-1}, s, \overline{w}_{mi_s}) - k(t_{n,j-1}, s, \overline{w}_{nj_s}) \} ds \|]. \quad \text{... (3.6)}$$

If $t_{n,j-1} < t_{m,i-1}$, then $\overline{w}_{mi}(t_{m,i-1} + \theta) = \overline{w}_{mm}(t_{m,i-1} + \theta)$ and $\overline{w}_{mi}(t_{n,j-1} + \theta) = \overline{w}_{mm}(t_{n,j-1} + \theta)$ for any $\theta \in [-r, 0]$. Since $x^m(t) - w^m(t) \to 0$ uniformly on $[-r, T_1]$, there exists a sequence of positive numbers ε_m such that $\varepsilon_m \to 0$ as $m \to \infty$ and

$$\| w_{mi}(t_{m,i-1} + \theta) - \overline{w}_{mi}(t_{n,j-1} + \theta) \|$$

$$= \| \overline{w}_{mm}(t_{m,i-1} + \theta) - \overline{w}_{mm}(t_{n,j-1} + \theta) \|$$

$$= \| x^m(t_{m,i-1} + \theta) - x^m(t_{n,j-1} + \theta) \|$$

$$\leq \| w^m(t_{m,i-1} + \theta) - w^m(t_{n,j-1} + \theta) \| + \varepsilon$$

$$\leq N_1 | t_{m,i-1} - t_{n,j-1} | + \varepsilon_m. \quad \text{... (3.7)}$$

Here, we have the Lipschitz continuity of $w^m(t)$ on $[-r, T_1]$. Using (2.3) and (3.1), we obtain

$$\| \int_0^{t_{m,i-1}} k(t_{m,i-1}, s, \overline{w}_{mi_s})\, ds \|$$

$$\leq \int_0^{t_{m,i-1}} || k(t_{m,i-1}, s, \overline{w}_{mi_s}) - k(t_{m,i-1}, s, \overline{o}) || \, ds$$

$$+ \int_0^{t_{m,i-1}} || k(t_{m,i-1}, s, \overline{o}) || \, ds$$

$$\leq (M_1 Q_1 + M_3) T_1. \quad \text{... (3.8)}$$

By making use of (2.3), (2.4) and (3.1), we get

$$|| \int_0^{t_{m,i-1}} k(t_{m,i-1}, s, \overline{w}_{mi_s}) \, ds - \int_0^{t_{n,j-1}} k(t_{n,j-1}, s, \overline{w}_{mi_s}) \, ds ||$$

$$\leq \int_0^{t_{n,j-1}} || k(t_{m,i-1}, s, \overline{w}_{mi_s}) - k(t_{n,j-1}, s, \overline{w}_{mi_s}) || \, ds$$

$$+ \int_{t_{n,j-1}}^{t_{m,i-1}} || k(t_{m,i-1}, s, \overline{w}_{mi_s}) - k(t_{m,i-1}, s, \overline{o}) || \, ds$$

$$+ \int_{t_{n,j-1}}^{t_{m,i-1}} || k(t_{m,i-1}, s, \overline{o}) || \, ds.$$

and

$$\leq [r_1(Q_1) T_1 + M_1 Q_1 + M_3] \, | t_{m,i-1} - t_{n,j-1} |, \quad \text{... (3.9)}$$

$$|| \int_0^{t_{n,j-1}} \{ k(t_{n,j-1}, s, \overline{w}_{mi_s}) - k(t_{n,j-1}, s, \overline{w}_{nj_s}) \} \, ds ||$$

$$\leq M_1 \int_0^{t_{n,j-1}} || \overline{w}_{mi_s} - \overline{w}_{nj_s} ||_{PC} \, ds$$

$$\leq M_1 \int_0^{t} || \overline{w}_{mi_s} - \overline{w}_{nj_s} ||_{PC} \, ds$$

$$= M_1 \int_0^{t} \mathrm{Sup}_{\tau \in [-r, s]} || \overline{w}_{mi}(\tau) - \overline{w}_{nj}(\tau) || \, ds. \quad \text{... (3.10)}$$

Here, we have used (2.3). We note that if $t_{m,i-1} < t_{n,j-1}$, then the similar inequalitites hold. Using (3.7) – (3.10) in (3.6) we obtain,

$$|| f^m(t) - f^n(t) ||$$

$$\leq | t_{m,i-1} - t_{n,j-1} | \; [r_2 (Q_1, [M_1 Q_1 + M_3] T_1)$$
$$+ M_2 (N_1 + r_1 (Q_1) T_1 + M_1 Q_1 + M_3)]$$
$$+ M_2 \varepsilon_m + M_2 \operatorname{Sup}_{s \in [-r, t]} || \overline{w}_{mi}(s) - \overline{w}_{nj}(s) ||$$
$$+ M_1 M_2 \int_0^t \operatorname{Sup}_{\tau \in [-r, s]} || \overline{w}_{mi}(\tau) - \overline{w}_{nj}(\tau) || \; ds.$$

It is easy to prove that the sequence $t_{m,i-1} - t_{n,j-1}$ converges to zero uniformly in i, j. There exists a sequence $\varepsilon^1_{mn} = [r_2 (Q_1, [M_1 Q_1 + M_3] T_1) + M_2 (N_1 + r_1 (Q_1) T_1 + M_1 Q_1 + M_3] \; | t_{m,i-1} - t_{n,j-1} | + M_2 \varepsilon_m$ such that $\varepsilon^1_{mn} \to 0$ as $m, n \to \infty$ and

$$|| f^m(t) - f^n(t) ||$$
$$\leq \varepsilon^1_{mn} + M_2 \operatorname{sup}_{s \in [-r, t]} || \overline{w}_{mi}(s) - \overline{w}_{nj}(s) ||$$
$$+ M_1 M_2 \int_0^t \operatorname{sup}_{\tau \in [-r, s]} || \overline{w}_{mi}(\tau) - \overline{w}_{nj}(\tau) || \; ds. \quad \dots (3.11)$$

For any $s \in [-r, t]$, we have

$$|| \overline{w}_{mi}(s) - \overline{w}_{nj}(s) ||$$
$$= || \overline{w}_{mm}(s) - \overline{w}_{nn}(s) ||$$
$$= || x^m(s) - x^n(s) ||$$
$$\leq || w^m(s) - w^n(s) || + || \varepsilon^2_{mn}(s) ||$$

where $\varepsilon^2_{mn}(s) \to 0$ uniformly on $[-r, T_1]$ as $m, n \to \infty$. Therefore, we have

$$\sup_{s \in [-r, t]} || \overline{w}_{mi}(s) - \overline{w}_{nj}(s) ||$$
$$\leq \sup_{s \in [-r, t]} || w^m(s) - w^n(s) || + \varepsilon^2_{mn} \quad \dots (3.12)$$

where the constants $\varepsilon^2_{mn} \to 0$ as $m, n \to \infty$. Using (3.5), (3.11) and (3.12), we obtain

$$(d^- / dt) \, || w^m(t) - w^n(t) ||^2$$
$$\leq 2 [\varepsilon^2_{mn} + M_2 \sup_{s \in [-r, t]} || w^m(s) - w^n(s) ||$$
$$+ M_1 M_2 \int_0^t \sup_{\tau \in [-r, s]} || w^m(\tau) - w^n(\tau) || \, ds] \; || w^m(t) - w^n(t) ||$$
$$- 2 \langle A^m(t, u_t) - A^n(t, u_t), \, U(w^m(t) - w^n(t)) \rangle \quad \dots (3.13)$$

where $\varepsilon^3_{mn} = \varepsilon^1_{mn} + (M_2 + M_1 M_2 T_1)\, \varepsilon^2_{mn}$ and $\varepsilon^3_{mn} \to 0$ as $m, n \to \infty$. By virtue of the uniform continuity of U on bounded subsets of X, we get a sequence of functions $\varepsilon^4_{mn}(t)$ with values in X^* such that $\lim_{m,n,\to\infty} \varepsilon^4_{mn}(t) = 0$ uniformly on $[0, T_1]$ and $U(w^m(t) - w^n(t))$. Hence, we have

$$
\begin{aligned}
&- \langle A^m(t, u_t) - A^n(t, u_t),\ U(w^m(t) - w^n(t)) \rangle \\
&= - \langle A(t_{m,i-1}, u_{t_{m,i-1}}) w_{mi} - A(t_{n,j-1}, u_{t_{n,j-1}}) w_{nj} \\
&\qquad U(w^m(t) - w^n(t)) \rangle \\
&= - \langle A(t_{n,j-1}, u_{t_{n,j-1}}) w_{mi} - A(t_{n,j-1}, u_{t_{n,j-1}}) w_{nj}, \\
&\qquad U(x^m(t) - x^n(t)) \rangle \\
&\quad - \langle A(t_{n,j-1}, u_{t_{n,j-1}}) w_{mi} - A(t_{n,j-1}, u_{t_{n,j-1}}) w_{nj},\ \varepsilon^4_{mn}(t) \rangle \\
&\quad - \langle A(t_{m,i-1}, u_{t_{m,i-1}}) w_{mi} - A(t_{n,j-1}, u_{t_{n,j-1}}) w_{mi}, \\
&\qquad U(w^m(t) - w^n(t)) \rangle \\
&= - \langle A(t_{n,j-1}, u_{t_{n,j-1}}) x^m(t) - A(t_{n,j-1}, u_{t_{n,j-1}}) x^n(t), \\
&\qquad \cdot U(x^m(t) - x^n(t)) \rangle \\
&\quad - \langle A(t_{n,j-1}, u_{t_{n,j-1}}) w_{mi} - A(t_{n,j-1}, u_{t_{n,j-1}}) w_{nj},\ \varepsilon^4_{mn}(t) \rangle \\
&\quad - \langle A(t_{m,i-1}, u_{t_{m,i-1}}) w_{mi} - A(t_{n,j-1}, u_{t_{n,j-1}}) w_{mi}, \\
&\qquad U(w^m(t) - w^n(t)) \rangle \\
&\le [\, \| A(t_{n,j-1}, u_{t_{n,j-1}}) w_{mi} \| + \| A(t_{n,j-1}, u_{t_{n,j-1}}) w_{nj} \| \,] \\
&\qquad \| \varepsilon^4_{mn} \| (t) \| \\
&\quad + r_0 (\| u_{t_{n,j-1}} \|_{PC},\ \| u_{t_{m,i-1}} \|_{PC},\ \| w_{mi} \|) \\
&\quad [\ | t_{n,j-1} - t_{m,i-1} | \ (1 + \| A(t_{m,i-1}, u_{t_{m,i-1}}) w_{mi} \|) \\
&\quad + \| u_{t_{n,j-1}} - u_{t_{m,i-1}} \|_{PC}] \ \| w^m(t) - w^n(t) \| . \qquad \text{... (3.14)}
\end{aligned}
$$

Here we have used the accretiveness of $A(t_{n,j-1}, u_{t_{n,j-1}})$, (2.2) and the Schwarz inequality. Using (2.12), (3.2) and (3.3), we get

$$\| A(t_{n,j-1}, u_{t_{n,j-1}}) w_{nj} \| \le Q_2 + M_2 [Q_1 + (M_1 Q_1 + M_3) T_1] + M_4 = Q_3.$$

Similarly, we have $\| A(t_{m,i-1}, u_{t_{m,i-1}}) w_{mi} \|\ Q_3$.

Moreover,

$$\| A(t_{n,j-1}, u_{t_{n,j-1}}) w_{mi} \|$$
$$\leq \| A(t_{n,j-1}, u_{t_{n,j-1}}) w_{mi} - A(t_{m,i-1}, u_{t_{m,i-1}}) w_{mi} \|$$
$$+ \| A(t_{m,i-1}, u_{t_{m,i-1}}) w_{mi} \|$$
$$\leq r_0 (\| u_{t_{n,j-1}} \|_{PC}, \| u_{t_{m,i-1}} \|_{PC}, \| w_{mi} \|)$$
$$[| t_{n,j-1} - t_{m,i-1} | (1 + \| A(t_{m,i-1}, u_{t_{m,i-1}}) w_{mi} \|)$$
$$+ \| u_{tn,j-1} - u_{t_{m,i-1}} \|_{PC}] + \| A(t_{m,i-1}, u_{t_{m,i-1}}) w_{mi} \|$$
$$\leq r_0 (\| u \|_{T_1}, \| u \|_{T_1}, Q_1)(1 + Q_3 + M) \, | t_{n,j-1} - t_{m,i-1} | + Q_3$$
$$= r_0 (\| u \|_{T_1}, \| u \|_{T_1}, Q_1)(1 + Q_3 + M) \, 2T_1 + Q_3$$
$$= Q_4.$$

Now, we obtain from (3.14),

$$- \langle A^m (t, u_t) - A^n (t, u_t), \, U(w^m (t) - w^n (t)) \rangle$$
$$\leq (Q_3 + Q_4) \| \varepsilon^4_{mn} (t) \|$$
$$+ | t_{n,j-1} - t_{m,i-1} | (1 + Q_3 + M) r_0 (\| u \|_{T_1}, \| u \|_{T}, Q_1)$$
$$\| w^m (t) - w^n (t) \| . \qquad \ldots (3.15)$$

From (3.13) and (3.15), we get

$$(d^- / dt) \| w^m (t) - w^n (t) \|^2$$
$$\leq 2 [\varepsilon^5_{mn} + M_2 \sup_{s \in [-r, t]} \| w^m (s) - w^n (s) \|$$
$$+ M_1 M_2 \int_0^t \sup_{\tau \in [-r, s]} \| w^m (\tau) - w^n (\tau) \| \, ds]$$
$$\times \| w^m (t) - w^n (t) \| + 2 (Q_3 + Q_4) \| \varepsilon^4_{mn} (t) \| \qquad \ldots (3.16)$$

where $\varepsilon^5_{mn} = \varepsilon^3_{mn} + | t_{n,j-1} - t_{m,i-1} | \, r_0 \, (\| u \|_{T_1}, \| u \|_{T_1}, Q_1)(1 + Q_3 + M)$ and since $t_{n,j-1} - t_{m,i-1}$ converges to zero uniformly in i, j, $\varepsilon^5_{mn} \to 0$ as $m, n \to \infty$. Now, using the elementary inequalities $ab \leq \frac{1}{2}(a^2 + b^2)$, $(a + b + c)^2 \leq 3 (a^2 + b^2 + c^2)$ and Schwarz inequality, we observe from (3.16) that

$$(d^- / dt) \| w^m (t) - w^n (t) \|^2$$

$$\leq Q\,[\,\varepsilon_{mn}^{5} + M_2 \sup_{s\in[-r,t]} \|\, w^m(s) - w^n(s) \,\|$$

$$+ M_1 M_2 \int_0^t \sup_{\tau\in[-r,s]} \|\, w^m(\tau) - w^n(\tau) \,\| \, ds\,]^2$$

$$+ \|\, w^m(t) - w^n(t) \,\|^2 + 2\,(Q_3 + Q_4)\, \|\, \varepsilon_{mn}^{4}(t) \,\|$$

$$\leq 3\,[\,(\varepsilon_{mn}^{5})^2 + (M_2 \sup_{s\in[-r,t]} \|\, w^m(s) - w^n(s) \,\|\,)^2$$

$$+ (M_1 M_2 \int_0^t \sup_{\tau\in[-r,s]} \|\, w^m(\tau) - w^n(\tau) \,\| \, ds\,)^2\,]$$

$$+ \sup_{s\in[-r,t]} \|\, w^m(s) - w^n(s) \,\|^2$$

$$+ 2(Q_3 + Q_4)\, \|\, \varepsilon_{mn}^{4}(t) \,\|$$

$$\leq \varepsilon_{mn} + (3M_2^2 + 1) \sup_{s\in[-r,t]} \|\, w^m(s) - w^n(s) \,\|^2$$

$$+ 3M_1^2 M_2^2 \left(\int_0^t ds\right) \left(\int_0^t \sup_{\tau\in[-r,s]} \|\, w^m(\tau) - w^n(\tau) \,\|^2 \, ds\right.$$

$$= \varepsilon_{mn} + M_5 \sup_{s\in[-r,t]} \|\, w^m(s) - w^n(s) \,\|^2$$

$$+ M_6 \int_0^t \sup_{\tau\in[-r,s]} \|\, w^m(\tau) - w^n(\tau) \,\|^2 \, ds. \qquad \ldots (3.17)$$

where $\varepsilon_{mn} = 3\,(\varepsilon_{mn}^{5})^2 + 2(Q_3 + Q_4)\, \|\, \varepsilon_{mn}^{4} \,\|$, $M_5 = 3M_2^2 + 1$, $M_6 = 3\, M_1^2 M_2^2\, T_1$ and $\varepsilon_{mn} \to 0$ as $m, n \to \infty$. Integrating (3.17) we get,

$$\|\, w^m(t) - w^n(t) \,\|^2$$

$$\leq \varepsilon_{mn} T_1 + M_5 \int_0^t \sup_{\tau\in[-r,s]} \|\, w^m(\tau) - w^n(\tau) \,\|^2 \, ds$$

$$+ M_6 \int_0^t \int_0^s \sup_{\mu\in[-r,\tau]} \|\, w^m(\mu) - w^n(\mu) \,\|^2 \, d\tau\, ds.$$

Here we have used $w^m(0) = w^n(0) = \phi(0)$. Since for any $t_1 \in [0, t]$, $t_1 \in (t_{n,j-1}, t_{nj}) \cap (t_{m,i-1}, t_{mi})$ for some m, n, i, j, we have

$$\|\, w^m(t_1) - w^n(t_1) \,\|^2$$

$$\leq \varepsilon_{mn} T_1 + M_5 \int_0^{t_1} \sup_{\tau \in [-r, s]} || w^m(\tau) - w^n(\tau) ||^2 ds$$

$$+ M_6 \int_0^{t_1} \int_0^{s} \sup_{\mu \in [-r, \tau]} || w^m(\mu) - w^n(\mu) ||^2 d\tau ds$$

$$\leq \varepsilon_{mn} T_1 + M_5 \int_0^{t} \sup_{\tau \in [-r, s]} || w^m(\tau) - w^n(\tau) ||^2 ds$$

$$+ M_6 \int_0^{t} \int_0^{s} \sup_{\mu \in [-r, \tau]} || w^m(\mu) - w^n(\mu) ||^2 d\tau ds.$$

Consequently, we obtain,

$$\sup_{s \in [-r, t]} || w^m(s) - w^n(s) ||^2$$

$$\leq \varepsilon_{mn} T_1 + \int_0^{t} Q_5 \operatorname{Sup}_{\tau \in [-r, s]} || w^m(\tau) - w^n(\tau) ||^2 ds$$

$$+ \int_0^{t} Q_5 \int_0^{s} \sup_{\mu \in [-r, \tau]} || w^m(\mu) - w^n(\mu) ||^2 d\tau ds \qquad \ldots (3.18)$$

where $Q_5 = \max\{M_5, M_6\}$. Applying the Lemma 1 to (3.18) with $c(t) = \sup_{s \in [-r, t]} || w^m(s) - w^n(s) ||^2$, we get

$$\sup_{s \in [-r, t]} || w^m(s) - w^n(s) ||^2$$

$$\leq \varepsilon_{mn} T_1 [1 + Q_5 \exp\{ (Q_5 + 1) T_1 \} T_1].$$

Since $\varepsilon_{mn} \to 0$ as $m, n \to \infty$, the sequence $w^m(s) - w^n(s) \to 0$ as $m, n \to \infty$ uniformly on $[0, T_1]$. This implies that $w^n(t) \to x(t)$ uniformly on $[-r, T_1]$, where $x(t)$ is a strongly continuous function. Since each $w^n(t)$ is Lipschitz continuous on $[-r, T_1]$ with Lipschitz constant N_1, the limit function $x(t)$ is Lipschitz continuous on $[-r, T_1]$ with Lipschitz constant N_1. We show that $x(t) \in D$ for all $t \in [0, T_1]$ and $A^n(t, u_t) \to A(t, u_t) x(t)$. We observe that

$$A(t, u_t) x(t)$$

$$A^n(t, u_t) = A(t_{n, j-1}, u_{t_{n, j-1}}) w_{nj} = A(t_{n, j-1}, u_{t_{n, j-1}}) x^n(t)$$

for every $t \in (t_{n, j-1}, t_{nj}]$. For such t's we have from (2.2), (2.14)

$$|| A(t_{n, j-1}, u_{t_{n, j-1}}) x^n(t) - A(t, u_t) x^n(t) ||$$

$$\leq r_0(|| u_{t_{n, j-1}} ||_{PC}, || u_t ||_{PC}, || x^n(t) ||)$$

$$[\,|\,t_{n,j-1}-t\,|\,(1+||\,A(t_{n,j-1},\,u_{t_{n,j-1}})x^n(t)\,||\,)$$

$$+\,||\,u_{t_{n,j-1}}-u_t\,||_{PC}\,]$$

$$\leq r_0(\,||\,u\,||_{T_1},\,||\,u\,||_{T_1},\,Q_1)\,[1+Q_3+M]\,|\,t_{n,j-1}-t\,|$$

$$\leq Q_6\,T_1/n$$

where $\quad Q_6 = r_0(\,||\,u\,||_{T_1},\,||\,u\,||_{T_1},\,Q_1)\,(1+Q_3+M).$

This proves that $A^n(t,u_t)-A(t,u_t)x^n(t) \to 0$ uniformly on $[0,T_1]$ and $A(t,u_t)x^n(t)$ is uniformly bounded. Since $x^n(t)\to x(t)$ and $A(t,u_t)x^n(t)$ is uniformly bounded, $x(t)\in D$ and $A(t,u_t)x^n(t) \to A(t,u_t)x(t)$ for every $t\in[0,T_1]$ (see Lemma 2.5 of Kato[12]). Since

$$A^n(t,u_t)-A(t,u_t)x(t)$$

$$= A(t_{n,j-1},\,u_{t_{n,j-1}})x^n(t)-A(t,u_t)x^n(t)$$

$$+\,A(t,u_t)x^n(t)\ -\ A(t,u_t)x(t)$$

we have actually proved that $A^n(t,u_t) \to A(t,u_t)x(t)$ for each t. The weak continuity of $A(t,u_t)x(t)$ can be proved as in Lemma 4.4 of Kato[12]. For each $g\in X*$, we see that

$$\langle w^n(t),g\rangle = \langle \phi(0),g\rangle - \int_0^t \langle A^n(s,u_s),g\rangle\,ds$$

$$+\int_0^t \langle f^n(s),g\rangle\,ds\,,\ t\in[0,T_1] \qquad \text{... (3.19)}$$

and this shows that $x(t)$ is weakly continuously differentiable on $[0,T_1]$. We observe that for any $\theta\in[-r,0]$, $\overline{w}_{nj}(t_{n,j-1}+\theta) = x^n(t_{n,j-1}+\theta) \to x(s+\theta)$, $t_{n,j-1}<s\leq t_{nj}$. Since $f^n(s)=f(t_{n,j-1},\,\overline{w}_{n_{jtn,j-1}},\,\int_0^{t_{n,j-1}} k(t_{n,j-1},s,\overline{w}_{n_{js}})\,ds$ for such s, we find that $f^n(s)\to f(s,x_s,\int_0^s k(s,\tau,x_\tau)\,d\tau)$ as $n\to\infty$. Using (2.3), (2.5) and (3.1) we get $||\,f^n(s)\,||\leq M_2\,[Q_1+(M_1Q_1+M_3)T_1]+M_4$ and we have $||\,A^n(t,u_t)\,||\leq Q_3$. By applying Lebesque's bounded convergence theorem to (3.19), we obtain

$$\langle x(t),g\rangle = \langle \phi(0),g\rangle - \int_0^t \langle A(s,u_s)x(s),g\rangle\,ds$$

$$+\int_0^t \langle f(s,x_s,\int_0^s k(s,\tau,x_\tau)\,d\tau),g\rangle\,ds.$$

Since the integrands above are continuous in t, we have actually proved that

$\langle\, x(t)\ g \,\rangle$ is continuously differentiable on $[0, T_1]$. The strong derivative of $x(t)$ exists a.e. and equals

$$- A(t, u_t)\, x(t) + f(t, x_t, \int_0^t k(t, s, x_s)\, ds)$$

follows from the Lemma 4.6 of Kato[12] and therefore we omit the details. This completes the proof of Lemma 5.

Thus, for a given $u \in E$, the strong solution $x(t)$ or $x_u(t)$ of (1.2) exists a.e. on $[-r, T_1]$.

Now, we prove the main result. Suppose that $x(t)$ and $y(t)$ are the solutions of the equation (1.2) corresponding to u and v belonging to E. Then by using the Lemma 1.3 of Kato[12], accretiveness of $A(t, u_t)$, Schwarz's inequality, (2.2), (2.3) and (2.5) we have

$$\begin{aligned}
&(d/dt)\, \| x(t) - y(t) \|^2 \\
&= 2\, \| x(t) - y(t) \|\, (d/dt)\, \| x(t) - y(t) \| \\
&= 2 \langle\, (d/dt)\, x(t) - (d/dt)\, y(t),\ U(x(t) - y(t)) \,\rangle \\
&= -\, 2 \,\langle\, A(t, u_t)\, x(t) - A(t, u_t)\, y(t),\ U(x(t) - y(t)) \,\rangle \\
&\quad -\, 2 \,\langle\, A(t, u_t)\, y(t) - A(t, v_t)\, y(t),\ U(x(t) - y(t)) \,\rangle \\
&\quad +\, 2 \langle\, f(t, x_t, \int_0^t k(t, s, x_s)\, ds\,) - f(t, y_t, \int_0^t k(t, s, y_s)\, ds\,),\ U(x(t) - y(t)) \,\rangle. \\
&\le -\, 2 \,\langle A(t, u_t)\, y(t) - A(t, v_t)\, y(t),\ U(x(t) - y(t)\,) \rangle \\
&\quad +\, 2 \,\langle\, f(t, x_t, \int_0^t k(t, s, x_s)\, ds\,) - f(t, y_t, \int_0^t k(t, s, y_s)\, ds\,),\ U(x(t) - y(t)) \,\rangle. \\
&\le 2\, [\ \| A(t, u_t)\, y(t) - A(t, v_t)\, y(t) \| \\
&\quad +\ \| f(t, x_t, \int_0^t k(t, s, x_s)\, ds\,) - f(t, y_t, \int_0^t k(t, s, y_s)\, ds\,) \|\ ] \\
&\qquad \| x(t) - y(t) \| \\
&\le 2\, [r_0(\, \| u \|_{T_1},\ \| v \|_{T_1},\ Q_1)\, \| u_t - v_t \|_{PC} \\
&\quad +\, M_2\, \| x_t - y_t \|_{PC} + M_1 M_2 \int_0^t \| x_s - y_s \|_{PC}\, ds\,] \\
&\qquad \| x(t) - y(t) \|,
\end{aligned}$$

which yields

$$(d/dt)\,||\,x(t)-y(t)\,||$$
$$\leq r_0(\,||\,u\,||_{T_1},\,||\,v\,||_{T_1},\,Q_1)\,||\,u_t-v_t\,||_{PC}$$
$$+\,M_2\,||\,x_t-y_t\,||_{PC}+M_1M_2\int_0^t ||\,x_s-y_s\,||_{PC}\,ds. \qquad \dots (3.20)$$

Integrating (3.20) from 0 to t, we get

$$||\,x(t)-y(t)\,||$$
$$\leq r_0(\,||\,u\,||_{T_1},\,||\,v\,||_{T_1},\,Q_1)\,||\,u-v\,||_{T_1}T_1$$
$$+\,M_2\int_0^t ||\,x_s-y_s\,||_{PC}\,ds+M_1M_2\int_0^t\int_0^s ||\,x_\tau-y_\tau\,||_{PC}\,d\tau\,ds.. \qquad \dots (3.21)$$

Case 1 — Suppose $t\geq r$. Then for every $\theta\in[-r,0]$, we have $t+\theta\geq 0$. For such such θ's, from (3.21) we obtain

$$||\,x(t+\theta)-y(t+\theta)\,||$$
$$\leq T_1r_0(\,||\,u\,||_{T_1},\,||\,v\,||_{T_1},\,Q_1)\,||\,u-v\,||_{T_1}$$
$$+\,M_2\int_0^{t+\theta} ||\,x_s-y_s\,||_{PC}\,ds+M_1M_2\int_0^{t+\theta}\int_0^s ||\,x_\tau-y_\tau\,||_{PC}\,d\tau\,ds$$
$$\leq T_1r_0(\,||\,u\,||_{T_1},\,||\,v\,||_{T_1},\,Q_1)\,||\,u-v\,||_{T_1}$$
$$+\,M_2\int_0^{t} ||\,x_s-y_s\,||_{PC}\,ds+M_1M_2\int_0^{t}\int_0^s ||\,x_\tau-y_\tau\,||_{PC}\,d\tau\,ds.$$

which yields

$$||\,x_t-y_t\,||_{PC}\leq T_1r_0(\,||\,u\,||_{T_1},\,||\,v\,||_{T_1},\,Q_1)\,||\,u-v\,||_{T_1}$$
$$+\,M_2\int_0^t ||\,x_s-y_s\,||_{PC}\,ds$$
$$+\,M_1M_2\int_0^t\int_0^s ||\,x_\tau-y_\tau\,||_{PC}\,d\tau\,ds. \qquad \dots (3.22)$$

Case 2 — Suppose $0\leq t<r$. Then for all $\theta\in[-r,-t]$, we have $t+\theta<0$. For each θ's we have

$$||\,x(t+\theta)-y(t+\theta)\,||=||\,\phi(t+\theta)-\phi(t+\theta)\,||\,=0. \qquad \dots (3.23)$$

For $\theta\in[-t,0]$, $t+\theta\geq 0$. Then from (3.21) we get, as in the first case.

$$||x_t - y_t||_{PC} \le T_1 r_0(||u||_{T_1}, ||v||_{T_1}, Q_1)||u-v||_{T_1} + M_2 \int_0^t ||x_s - y_s||_{PC}\, ds + M_1 M_2 \int_0^t \int_0^s ||x_\tau - y_\tau||_{PC}\, d\tau\, ds. \quad \text{... (3.24)}$$

Thus, for every $t \in [0, T_1]$, we have from (3.22), (3.23) and (3.24)

$$||x_t - y_t||_{PC} \le T_1 r_0(||u||_{T_1}, ||v||_{T_1}, Q_1)||u-v||_{T_1} + \int_0^t Q_7 ||x_s - y_s||_{PC}\, ds + \int_0^t Q_7 \int_0^s ||x_\tau - y_\tau||_{PC}\, d\tau\, ds$$

where $Q_7 = \max\{M_2, M_1 M_2\}$. An application of the Lemma 1 with $f(t) = ||x_t - y_t||_{PC}$ yields

$$||x_t - y_t||_{PC} \le M_7 ||u-v||_{T_1} \quad \text{... (3.25)}$$

where

$$M_7 = T_1 r_0(||u||_{T_1}, ||v||_T, Q_1)[1 + Q_7 \exp\{(Q_7+1)T_1\}\ T_1].$$

Consequently, we get from (3.25)

$$||x-y||_{T_1} \le M_7 ||u-v||_{T_1}.$$

Now, choose T_1 so small such that $M_7 < 1$ then the operator $S : u \to x_u$ is a strict contraction on a complete metric space E. Let $x(t)$, $t \in [-r, T_1]$ be the unique fixed point of S. Then, it follows that $x(t)$ is the desired unique strong solution of eqn (1.1) on $[-r, T_1]$ which is also Lipschitz continuous.

Remark 3 : It is obvious from the proof of above theorem that the interval $[0, T)$ can be replaced by $[T_1, T)$ where $T_1 > 0$. Then the solution $x(t)$ of (1.1) exists beyond T_1 which, in turn, exists on $[-r, T_1)$.

4. An Application

In this section, we give an example to illustrate the application of our main result.

Let Ω be a bounded open subset of R^n with sufficiently smooth boundary $\partial\Omega$ where $R = (-\infty, \infty)$ and $n \ge 2$. For a multiindex $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ of nonnegative integers, we define

$$|\alpha| = \alpha_1 + \dots + \alpha_n, \quad D_i = \frac{\partial}{\partial x_i}$$

$$D^\alpha = D_1^{\alpha_1} \cdot D_2^{\alpha_2} \cdots D_n^{\alpha_n}.$$

Let R^{n_m} be the space of all real vectors of the form $\xi = \{ \xi_\alpha : | \alpha | \le m \}$ i.e. $\xi(u) = \{ D^\alpha u : | \alpha | \le m \}$. We denote the Sobolev space of all real valued functions z such that $D^\alpha z \in L^2(\Omega)$ for every α with $| \alpha | \le m$ by $W^{m,2}(\Omega)$. The Sobolev space $W^{m,2}(\Omega)$ is a separable Hilbert space with inner product

$$\langle z_1, z_2 \rangle_m = \sum_{|\alpha| \le m} \langle D^\alpha z_1 , D^\alpha z_2 \rangle L^2(\Omega).$$

Let $C_0^\infty(\Omega)$ be the space of all $f \in C^\infty(\Omega)$ with compact support. Define $W_0^{m,2}(\Omega) = C_0^\infty(\Omega)$ in $W^{m,2}(\Omega)$. The space $W_0^{m,2}(\Omega)$ is an another separable Hilbert space.

Consider the initial-boundary value problem

$$\frac{\partial z(u,t)}{\partial t} + A(t, u, z(u, t-r), z(u,t))$$

$$= F(t, z(u, t-r), \int_0^t K(t, s, z(u, s-r))\, ds), \quad t \in [0, T), \quad u \in \Omega \quad \text{... (4.1)}$$

with initial-boundary conditions

$$z(u, \theta) = \phi(u, \theta), \ u \in \Omega, \quad \theta \in [-r, 0] \quad \text{... (4.2)}$$

$$D^\alpha z(u,t) = 0, \quad u \in \partial\Omega, \quad t \in (0, T), \quad | \alpha | \le m \quad \text{.. (4.3)}$$

where

$$A(t, u, u_1, u_2) = \sum_{|\alpha| \le m} (-1)^{|\alpha|} D^\alpha b_\alpha(t, u, \xi(u_1)) A_\alpha(u, \xi(u_2))$$

is the elliptic differential operator and $\phi : \Omega \times [-r, 0] \to R$ is a given function.

Assume that the functions involved in (4.1)-(4.3) satisfy the following conditions.

(C_1) For each $\alpha, A_\alpha : \Omega \times R^{n_m} \to R$ is bounded i.e. there exists a constant $Q > 0$ such that

$$| A_\alpha(u, \xi) | \le Q. \quad \text{... (4.4)}$$

(C_2) for $u \in \Omega$ and $\xi, \xi' \in R^{n_m}$ we have

$$\sum_{|\alpha| \le m} [A_\alpha(u, \xi) - A_\alpha(u, \xi')] (\xi_\alpha - \xi'_\alpha) \ge 0. \quad \text{... (4.5)}$$

(C_3) For each $\alpha, b_\alpha : [0, T) \times \Omega \times R^{n_m} \to R_+$ is defined and continuous. There are constants $Q_1 > 0$, $Q_2 > 0$ such that

$$| b_\alpha(t, u, \xi) - b_\alpha(t', u, \xi') |$$

$$\le Q_1 | t - t' | + Q_2 | \xi - \xi' | \quad \text{... (4.6)}$$

for every $t, t' \in [0, T)$, $u \in \Omega$, $\xi, \xi' \in R^{n_m}$.

(C_4) The function $K : [0, T) \times [0, T) \times R \to R$ satisfies

$$| D^\alpha K(t, s, v) - D^\alpha K(t, s, \overline{v}) |$$
$$\leq Q_2 | D^\alpha v - D^\alpha \overline{v} |, \quad \text{... (4.7)}$$

$$| D^\alpha K(t_1, s_1, v) - D^\alpha K(t_2, s_2, v) |$$
$$\leq | D^\alpha v | [| t_1 - t_2 | + | s_1 - s_2 |], \quad \text{... (4.8)}$$

for $t, t_1, t_2, s, s_1, s_2 \in [0, T)$, $v, \overline{v} \in R$ where Q_2 is nonnegative constant and $| \alpha | \leq m$.

(C_5) The function $F : [0, T) \times R \times R \to R$ satisfies

$$D^\alpha F(t, v_1, v_2) - D^\alpha F(t, \overline{v}_1, \overline{v}_2) |$$
$$\leq Q_3 [| D^\alpha v_1 - D^\alpha \overline{v}_1 | + | D^\alpha v_2 - D^\alpha \overline{v}_2 |], \quad \text{... (4.9)}$$

$$| D^\alpha F(t_1, v_1, v_2) - D^\alpha F(t_2, \overline{v}_1, \overline{v}_2) |$$
$$\leq [| D^\alpha v_1 | + | D^\alpha v_2 |] | t_1 - t_2 |, \quad \text{... (4.10)}$$

for $t, t_1, t_2, t_2 \in [0, T)$, $v_1, v_2 \in R$ where $Q_3 \geq 0$ is a constant and $| \alpha | \leq m$.

Let $X = W_0^{m,2}(\Omega)$. Define an operator $\overline{A}(t, u_1) u_2$ on X from the equation

$$\langle \overline{A}(t, u_1) u_2, u_3 \rangle_m$$
$$= \sum_{|\alpha| \leq m} \int_\Omega b_\alpha(t, u, \xi(u_1(u)) A_\alpha(u, \xi(u_2(u))) D^\alpha u_3(u)\, du. \quad \text{... (4.11)}$$

We observe that $\overline{A}(t, u_1) u_2$ is continuous monotone and bounded in u_2 and satisfies the following Lipschitz condition

$$| | \overline{A}(t, u_1) u_2 - \overline{A}(\overline{t}, \overline{u}_1) u_2 | |_{m,2}$$
$$\leq Q_4 | t - \overline{t} | + Q_5 | | u_1 - \overline{u}_1 | |_{m,2}, \quad \text{... (4.12)}$$

for all $t, \overline{t} \in [0, T)$, $u_1, \overline{u}_1$, $u_2 \in X$ where Q_4, Q_5 are positive constants.

We now define an operator $A(t, \psi) v : [0, T) \times PC \times X \to X$, mappings $k : [0, T) \times [0, T) \times PC \to X$ and $f : [0, T) \times PC \times X \to X$ as follows

$$A(t, \psi) v = \overline{A}(t, \psi(-r)) v, \quad \text{... (4.13)}$$

$$k(t, s, \psi)(u) = K(t, s, \psi(-r)(u)), \quad \text{... (4.14)}$$

$$f(t, \psi, y)(u) = F(t, \psi(-r)(u), y(u)). \quad \text{... (4.15)}$$

By using (4.13) – (4.15), the problem (4.1) – (4.3) can be formulated abstractly as

$$x'(t) + A(t, x_t)\, x(t) = f(t, x_t, \int_0^t k(t, s, x_s)\, ds\,)\; t \in [0, T)$$

$$x_0 = \phi(t)\,, \quad -r \le t \le 0. \qquad \text{... (4.16)}$$

From the conditions on the functions involved in (4.1) – (4.3), we observe that the hypothesis of the Theorem are satisfied and hence by virtue of the Theorem, a strong solution $z(u,\ t) = x(t)u$, $u \in \Omega$, $t \in [0, T)$ of (4.1) – (4.3) exists and is unique.

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