Indian J. pure appl. Math., 25(3); 299-307, March 1994

CR-SUBMANIFOLDS OF A TRANS-SASAKIAN MANIFOLD - II

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(Received 19 *March* 1993; *accepted* 1 *June* 1993)

In this paper the author studies *CR*-submanifolds of a trans-Sasakian manifold which generalizes both α-Sasakian and β-Kenmotsu structure.

It was in 1978 that A Bejancu introduced the notion of *CR*-Submanifold of a Kaehler manifold[1]. Since then a number of authors extensively studied these Submanifolds[4-6] etc. On the other hand *CR*-Submanifolds of a Sasakian manifold have been studied by Kobayashi[13], Yano and Kon[16], Pak[15] and the present author[9]. Moreover, *CR*-Submanifolds of a Kenmotsu manifold have been studied by Papaghuic[3]. More general, one has the notion of α-Sasakian structure and β-Kenmotsu structure[11]. Oubina[14] introduced a new class of almost contact Riemannian manifold known as trans-Sasakian manifold.

The purpose of this paper is to study *CR*-Submanifolds of a trans-Sasakian manifold which generalizes both α-Sasakian and β-Kenmotsu structure. This paper is in continuation to the author's earlier paper[10].

1. PRELIMINARIES

Let $\overline{M}$ be a $(2m+1)$-dimensional almost contact metric manifold with structure tensors (ϕ, ξ, η, g). Then they satisfy[5].

$$\phi^2 = -I + \eta(x)\,\xi, \eta(\xi) = 1, \quad \phi\xi = 0, \eta\,\xi = 0 \qquad \text{... (1.1)}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\,\eta(Y) \qquad \text{... (1.2)}$$

where X, Y are vector fields on $\overline{M}$.

An almost contact metric structure (ϕ, ξ, η, g) on $\overline{M}$ is called trans-Sasakian[14].

$$(\overline{\nabla}_X \phi)(Y) = \alpha\,\{g(X, Y)\,\xi - \eta(Y)X\} + \beta\,\{g(\phi X, Y)\xi - \eta(Y)\,\phi X\} \qquad \text{... (1.3.)}$$

for α and β non zero constant and we say that the trans-Sasakian structure is of type (α, β). In particular it is normal. From the above formula one easily obtain

$$\overline{\nabla}_X \xi = -\alpha\phi X + \beta \{X - \eta(X)\xi\}. \qquad \text{... (1.4)}$$

Let M be a n-dimensional isometrically immersed submanifold of $\overline{M}$ and tangent to ξ. Let g be the metric tensor field on $\overline{M}$ as well as the induced metric on M. We denote by $\overline{\nabla}$ the covariant differentiation in $\overline{M}$ and by ∇ the covariant differentiation in M determined by the induced metric. Let $T(\overline{M})$ (resp. $T(M)$) be the Lei-Algebra of vector fields in $\overline{M}$ (resp. in M) and $T^{\perp}M$ the set of all vector fields normal to M.

The Gauss-Weingarten formulas are given by :

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y)\,;\ \overline{\nabla}_X N = -A_N X + \nabla_X N \qquad \text{... (1.5)}$$

$X, Y \in T(M)$, $N \in T^{\perp}(M)$ where $\overline{\nabla}$ is the connection in the nromal bundle, h is the second fundamental form of M, A_N the Weingarten endomorphism associated with N satisfying

$$g(A_N X, Y) = g(h(X, Y), N). \qquad \text{... (1.6)}$$

Definitions — An n-dimensional Riemennian submanifold M of a trans-Sasakian manifold $\overline{M}$ is called a CR-submanifold if ξ is tangent to M and there exists on M a differentiable distribution $D{:}x \rightarrow D_x \subset T_x M$ satisfying the following conditions :

(i) D_x is invariant under ϕ i.e. $\phi D_x \subset D_x$ for each $x \in M$

(ii) the complementary orthogonal distribution

$D : x \rightarrow D_x^{\perp} \subset T_x M$ is totally real under ϕ i.e. $\phi D_x^{\perp} \subset T_x^{\perp}$ M for each $x \in M$.

If dim. $D_x = 0$ (resp. dim. $D_x = 0$) then the CR-submanifold is called an invariant (resp. totally real) submanifold. A CR-Submanifold is called a proper CR-submanifold if it is neither invariant nor totally real. The pair $(D, D^{\perp})$ is called ξ-horizontal (ξ-vertical) if $\xi_x \in D_x$ (resp. $\xi_x \in D_x$) for each $x \in M$ (Kobayashi[13]).

For a vector field X tangent to M, we put

$$\phi X = PX + FX \qquad \text{... (1.7)}$$

where PX and FX are the tangential and the normal component of ϕX respectively. Then P is an endomorsymbol fsm of the tangent bundle TM and F is a normal-bundle valued 1-form on TM.

Also, for a vector field N normal to M, we put

$$\phi N = tN + fN \qquad \text{... (1.8)}$$

where tN (resp. fN) denotes the tangential (resp. normal) component of ϕN. Then f is an endomorsymbol fsm of TM and t is a tangent-bundle-valued 1-form on $T^{\perp}M$.

If we denote the orthogonal component of ϕD in $T^{\perp}M$ by μ, then we have

$$T^{\perp}M = \phi D^{\perp} \oplus \mu.$$

it is obvious that $\phi\mu = \mu$.

2. Covariant Differentiations

Let P, f, F and t be the endomorsymbol fsm and the vector-bundle-valued 1-forms defined in (1.7) and (1.8), respectively. Let us define the covariant differentiations of P, f, t and f as follows :

$$(\overline{\nabla}_X P)(Y) = \nabla_X PY - P\nabla_X Y \quad \text{... (2.1)}$$

$$(\overline{\nabla}_X F)(Y) = D_X^{\perp}(FY) - F\nabla_X Y \quad \text{... (2.2)}$$

$$(\overline{\nabla}_X t)(N) = \nabla_X tN - t\nabla_X^{\perp} N \quad \text{... (2.3)}$$

$$(\overline{\nabla}_X f)(N) = \nabla_X^{\perp}(fN) - f\nabla_X^{\perp} N \quad \text{... (2.4)}$$

for any vector fields X and Y tangent to M and any vector field N normal to M.

The endomorsymbol fsm P (resp. the endomorsymbol fsm f, the 1-forms F and t) is parallel if $\overline{\nabla} P = 0$ (resp. $\overline{\nabla} f = 0$, $\overline{\nabla} F = 0$ and $\overline{\nabla} t = 0$).

Now from (1.3) and (1.5) - (1.8), one can easily prove the following :

Proposition 2.1 — For the covariant differentiations defined above we have :

$$(\overline{\nabla}_x P(Y) = A_{FY}X + th(X,Y) + \alpha\,\{g(X,Y)\xi - \eta(Y)X\}$$
$$+\ \beta\{g(PX,Y)\xi - \eta(Y)PX\} \quad \text{... (2.5)}$$

$$(\overline{\nabla}_X F)(Y) = fh(X,Y) - h(X,PY) - \beta\eta(Y)FX \quad \text{... (2.6)}$$

$$(\overline{\nabla}_X t)(N) = A_{fN}X - PA_N X + \beta g(\phi X, N)\xi \quad \text{... (2.7)}$$

$$(\overline{\nabla}_X f)(N) = -h(X, tN) - FA_N X \quad \text{... (2.8)}$$

for any vector fields X and Y tangent to M and any vector field N normal to M.

Now we prove :

Lemma 2.2 — Let M be a CR-submanifold of a trans-Sasakian manifold $\overline{M}$. Then we have

$$A_{\phi Z}W - A_{\phi W}Z = \alpha\,\{\eta(W)Z - \eta(Z)W\} \quad \text{... (2.9)}$$

for any $W, Z \in D^{\perp}$.

Proof : For $W, Z \in D^{\perp}$, $Y \in TM$, using (1.3), (1.5) and (1.6) we have

$$g(A_{\phi W}Z) = g(h(Y,Z), \phi W) = g(\overline{\nabla}_Y Z, \phi W) = -g(\phi\,\overline{\nabla}_Y Z, W)$$
$$= -g(\overline{\nabla}_Y \phi Z, W) - \alpha\, g(Y,Z)\eta(W) + \alpha\,\eta(Z)\,g(Y,Z)$$
$$= g(A_{\phi Z}Y, W) - \alpha\,\eta(W)\,g(Y,Z) + \alpha\,\eta(Z)\,g(Y,W)$$
$$= g(A_{\phi Z}W, Y) - \alpha\,\eta(W)\,g(W,Z) + \alpha\,\eta(Z)\,g(Y,Z)$$

from which our assertion follows.

Corollary 2.3. — Let M be ξ-horizontal CR-submanifold of a trans-Sasakian manifold $\bar{M}$. Then we have

$$A_{\phi Z} W = A_{\phi W} Z$$

for any $W, Z \in D^{\perp}$.

Next we have :

Lemma 2.4 — Let M be a CR-submanifold of a trans-Sasakian manifold $\bar{M}$. Then we have

$$\bar{\nabla}^{\perp}_{W} \; FZ - \bar{\nabla}^{\perp}_{Z} \; FW + \beta \; \{\eta(Z) FW - \eta(W) FZ\} \in \phi D \qquad \text{... (2.10)}$$

for any $W, Z \in D$.

PROOF : From (2.5), (2.6) and using (2.1) and (2.2), we obtain

$$P \nabla_X Z = - A_{FZ} X - th(X, Z) + \alpha \; \{\eta(Z) X - g(X, Z) \xi\}$$

$$+ \beta \; \{\eta(Z) PX - g(PX, Z) \xi\} \qquad \text{... (2.11)}$$

$$F \nabla_X Z = \nabla^{\perp}_{X} \; FZ - fh(X, Z) - \beta \, \eta(Z) FX \qquad \text{... (2.12)}$$

for any X tangent to M and $Z \in D$.

Putting $X = W \in D^{\perp}$ in (2.11) and taking account of (2.9) we have

$$P[Z, W] = P\nabla_Z W - P\nabla_W Z$$

$$= - A_{FW} Z + \alpha \, \eta(W) Z + A_{FZ} W - \alpha \, \eta(Z) W = 0.$$

Also from (2.12) we have

$$\phi[Z, W] = F[Z, W] = F\nabla_Z W - F\nabla_W Z$$

$$= \bar{\nabla}^{\perp}_{Z} \; FW - \bar{\nabla}^{\perp}_{W} \; FZ + \beta \; \{\eta(Z) FW - \eta(W) FZ\} \in \phi D^{\perp}.$$

Corollary 2.5 — Let M be a ξ-horizontal CR-submanifold of a trans-Sasakian manifold $\bar{M}$. Then we have

$$\bar{\nabla}^{\perp}_{W} \; FZ - \bar{\nabla}^{\perp}_{Z} \; FW \in \phi D^{\perp}.$$

For any $W, Z \in D$.

For ξ-horizontal CR-submanifold of a trans-Sasakian manifold, we have from (1.3)

$$\bar{\nabla}_W \phi Z - \phi \bar{\nabla}_W Z = \alpha \, g(W, Z) \xi$$

for any $W, Z \in D^{\perp}$.

On account of (1.5), we obtain

$$- A_{\phi Z} W + \nabla_W \; \phi Z = \phi \; (\nabla_W Z + h(W, Z)) + \alpha \, g(W, Z) \xi$$

from which

$$\phi \; ([Z, W]) = A_{\phi Z} W - A_{\phi W} Z + \bar{\nabla}^{\perp}_{Z} \; \phi W - \bar{\nabla}^{\perp}_{W} \; \phi Z.$$

By virtue of above equation and Lemma 2.2, Lemma 2.5 we have :

Proposition : 2.6 — Let M be a ξ-horizontal CR-submanifold of a trans-Sasakian manifold $\bar{M}$. Then the vertical distribution $D^{\perp}$ is integrable.

For the horizontal distribution D, we prove :

Proposition 2.7 — Let M be a CR-submanifold of a trans-Sasakian manifold $\bar{M}$. Then the horizontal distribution D is integrable if and only if

$$g(h(X, PY) - h(Y, PX), FZ) = \eta(\nabla_X Z)\,\eta(Y) - \eta(\nabla_Y Z)\,\eta(X) + 2\eta\,(Z)\;g(X, PY)$$

for any $X,\ Y\ \in D$ and $Z \in D$.

PROOF : Making an innerproduct with PY in (2.11) we have

$$g(\nabla_X Z, Y) - \eta(\nabla_X Z)\,\eta(Y) = -\,g(h(X, PY), FZ) + \alpha\,\eta(Z)\,g(X, PY) + \beta\,\eta\,(Z)\,(g(X, Y) - \eta\,(X)\,\eta\,(Y))$$

which further means

$$\begin{aligned} g(\nabla_X Y, Z) = g(h(X, PY), FZ) &- \alpha\,\eta\,(Z)\,g(X, PY) \\ &- \beta\,\eta\,(Z)\;(g(X, Y) - \eta\,(X)\,\eta\,(Y)) \\ &- \eta\,(\nabla_X Z)\,\eta\,(Y) \end{aligned}$$

for any $X, Y \in D^{\perp}$.

Thus we have

$$\begin{aligned} g\,([X, Y], Z) = g(h(X, PY) - h(Y, PX), FZ) \\ - 2\eta\,(Z)\,g(X, PY) + \eta\,(\nabla_Y Z)\,\eta\,(X) - \eta\,(\nabla_X Z)\,\eta\,(Y) \end{aligned}$$

from which our assertion follows.

Corollary 2.8 — Let M be a ξ-horizontal CR-submanifold of a trans-Sasakian manifold $\bar{M}$. Then the horizontal distribution D is integrable if and only if

$$g(h(X, PY) - h(Y, PX), FZ) = 0 \qquad \text{... (2.13)}$$

for any $X, Y \in D$ and $Z \in D^{\perp}$.

3. Geometry of Leaves on CR-Submanifolds

In this section we obtain results on the immersions of leaves of distributions in M. For the leaves of involutive distributions we shall obtain necessary and sufficient conditions in order that their immersions in M be totally geodesic.

Proposition 3.1 — Let M be a ξ-horizontal CR-submanifold of a trans-Sasakian manifold $\bar{M}$. Then the distribution D is integrable and the leaf of D is totally geodesic in M if and only if

$$g(h(D, D), \phi\, D^{\perp}) = 0. \qquad \text{... (3.1)}$$

PROOF : Let $X, Y \in D$ and $Z \in D^{\perp}$. If the distribution D is integrable and its leaf is totally geodesic in M. Then $\nabla_X \phi Y \in D$.

For $X, Y \in D$, $Z \in D^{\perp}$ using (2.11) we have

$$0 = g(\nabla_X \phi Y, Z) = -g(\nabla_X Z, \phi Y) = g(P\nabla_X Z, Y)$$

$$= -g(A_{FZ} X, Y) - g(th(X, Z), Y)$$

$$= -g(h(X, Y), \phi Z)$$

from which we get (3.1).

Conversely, if (3.1) holds, then the distribution D is integrable by virtue of (2.13). Moreover, we have, using (1.3)

$$0 = g(h(X, \phi Y), \phi Z) = g(\bar{\nabla}_X \phi Y, \phi Z)$$

$$= g(\phi \bar{\nabla}_X Y, \phi Z)$$

$$= g(\nabla_X Y, Z).$$

Thus $\nabla_X Y \in D$ for any $X, Y \in D$ and the leaf of D is totally geodesic in M which completes the proof.

For the leaf $M^{\perp}$ of $D^{\perp}$, we prove the following :

Proposition 3.2 — Let M be a CR-submanifold of a trans-sasakian manifold $\bar{M}$. Then the leaf $M^{\perp}$ of $D^{\perp}$ is totally geodesic in M if and only if

$$g(h(Y, W) FZ) + \alpha\, \eta(Y)\, g(W, Z) = 0 \qquad \text{... (3.2)}$$

for any $Y \in D$ and $W, Z \in D^{\perp}$.

PROOF : Putting $X = W \in D^{\perp}$ in (2.11) we have

$$P \nabla_W Z = -A_{FZ} W - th(W, Z) + \alpha\, \{\eta(Z) W - g(W, Z)\xi\}.$$

Now making an innerproduct with $Y \in D$, we get

$$g(\nabla_W Z, PY) = g(h(Y, W), FZ) + \alpha\, \eta(Y)\, g(W, Z)$$

from which our assertion follows immediately.

Corollary 3.3 — Let M be a ξ-vertical CR-submanifold of a trans-Sasakian manifold $\bar{M}$. Then the leaf $M^{\perp}$ of $D^{\perp}$ is totally geodesic in M if and only if

$$g(h(D, D^{\perp}),\ \phi D^{\perp}) = 0.$$

Proposition 3.4 — Let M be a CR-submanifold of a trans-Sasakian manifold $\bar{M}$ and each leaf $M^{\perp}$ and $D^{\perp}$ is totally geodesic in M. If the endomorsymbol fsm P satisfies

$$(\nabla_X P)(Y) = \alpha\, \{g((X, Y)\xi - \eta(Y) X\} + \beta\, \{g(PX, Y)\xi - \eta(Y) PX\} \qquad \text{... (3.3)}$$

for any X, Y tangent to M. Then $\dim D_x = 0$.

PROOF : The equation (3.2) gives

$$g(A_{FZ}Y + \alpha\,\eta(Y)Z, W) = 0 \qquad \ldots (3.4)$$

for any $Y \in D$, $W, Z \in D^{\perp}$.

Next from (2.5) and (3.3), we have

$$A_{FY}X + th(X, Y) = 0$$

for any X, Y tangent to M. From this we obtain $th(X, Y) = 0$ for any X tangent to M and $Y \in D$. Thus we have

$$g(h(X, Y), \phi W) = g(A_{FW}Y, X)$$

$$= -\, g(\phi h(X, Y) . W) = 0$$

that is $A_{\phi W}Y = 0$ for $Y \in D$, $W \in D^{\perp}$.

Putting this equation into (3.4), we get dim. $D_x^{\perp} = 0$.

4. CR-SUBMANIFOLDS WITH PARALLEL STRUCTURES

In this section, first we prove :$^{\perp}$

Proposition 4.1 — Let M be a CR-submanifold of a trans-Sasakian manifold $\overline{M}$. Then $\overline{\nabla} F = 0$ if and only if $\overline{\nabla} t = 0$.

PROOF : Suppose t is parallel i.e. $\nabla t = 0$, then from (2.7) we have

$$g(A_{fN}X, Y) = g(PA_N X, Y) - \beta\, g(\phi X, N)\, g(Y, \xi)$$

for any X, Y tangent to M and any vector field N normal to M.

This equation means

$$g(fh(X, Y), N) = g((h(X, PY), N) + \beta\, g(\phi X, N)\, g(Y, \xi)$$

which is equivalent to

$$fh(X, Y) = h(X, PY) + \beta\,\eta(Y)\, FX$$

i.e. $\overline{\nabla} F = 0$.

We next have :

Lemma 4.2 — Let M be a CR-submanifold of a trans-Sasakian manifold $\overline{M}$. Then P is parallel i.e. $\overline{\nabla} P = 0$ if and only if

$$A_{fX}Y - A_{FY}X = \alpha\,\{\eta(X)Y - \eta(Y)X\} + \beta\,\{\eta(Y)PX - \eta(X)PY\}$$

for any X, Y tangent to M.

PROOF : From (2.5) we have

$$g((\overline{\nabla}_X P)(Y), Z) = g(A_{FY}X, Z) + g(th(X, Y), Z)$$

$$+ \alpha\,\{g(X, Y)\eta(Z) - \eta(Y)g(X, Z)\}$$

$$+ \beta\,\{g(PX, Y)\eta(Z) - \eta(Y)g(PX, Z)\}$$

$$\begin{aligned}
&= g(A_{FY}X, Z) - g(A_{FZ}X, Y) \\
&\quad + \alpha \ \{g(X, Y)\,\eta\,(Z) - \eta\,(Y)\,g(X, Z)\} \\
&\quad + \beta \ \{g(PX, Y)\,\eta\,(Z) - \eta\,(Y)\,g(PX, Z)\} \\
&= g(A_{FY}Z, X) - g(A_{FZ}Y, X) \\
&\quad + \alpha \ \{g(X, Y)\,\eta\,(Z) - \eta\,(Y)\,g(X, Z)\} \\
&\quad + \beta \ \{g(PX, Y)\,\eta\,(Z) - \eta\,(Y)\,g(PX, Z)\}
\end{aligned}$$

from which the assertion follows.

Now we have :

Proposition 4.3 — Let M be a ξ-horizontal CR-submanifold of a trans-Sasakian manifold $\bar{M}$. If $\bar{\nabla} P = 0$ then

(i) The holomorsymbol fc distribution D is integrable,

(ii) $A_{FU}X = \alpha \ \{\eta(U)X - \eta(X)U\} + \beta \ \{\eta(X)PU - \eta(U)PX\}$

for $X \in D$ and $U \in TM$.

PROOF : If P is parallel, then Lemma 4.2 implies

$$A_{FU}X = \alpha \ \{\eta(U)X - \eta(X)U\} + \beta \ \{\eta(X)PU - \eta(U)PX\}$$

for $X \in D$, U tangent to M.

Now for $U = Z \in D^{\perp}$, X, $Y \in D$ we have

$$g(A_{FZ}X, Y) = 0 \text{ i.e. } g(h(X, Y), FZ) = 0$$

which shows that D is integrable.

Finally we have :

Proposition 4.4 — Let M be a ξ-horizontal CR-submanifold of a trans-Sasakian manifold $\bar{M}$ such that F is parallel. Then the horizontal distribution D is integrable and the leaf of D is totally geodesic in M.

PROOF : From (2.2) we have

$$(\bar{\nabla}_X F)(Y) = 0$$

which gives

$$- F\,\nabla_X Y = 0$$

for any X, $Y \in D$.

Hence $\nabla_X Y \in D$ for any $X, Y \in D$ and our proposition follows immediately.

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