Indian J. pure appl. Math., 25(3); 319-330, March 1994

THE ORDER STRUCTURE OF WALLMAN NEARNESSES

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(Received 28 *September* 1992; *after revision* 26 *May* 1993;
accepted 30 *August* 1993*)*

Reed[8] introduced Wallman nearnesses. In this article we investigate the order structure of the set $(\mathcal{M}_W(X), \subset)$ of all Wallman nearnesses compatible with a given T_1-space X. It is well-known that the set $(\mathcal{M}(X), \subset)$ of all Lodato nearnesses compactible with the space X is a complete lattice. We describe explicitly the least and largest elements of $(\mathcal{M}_W(X), \subset)$ and prove that it is also a complete lattice. Though $\mathcal{M}_W(X), \subset \mathcal{M}(X)$, an example is given to show that $(\mathcal{M}_W(X), \subset)$ is not a sublattice of the lattice $(\mathcal{M}(X), \subset)$.

1. INTRODUCTION

In order to introduce Wallman type (W-T) extensions, Reed[8] defined Wallman nearnesses. She proved that there is a 1-1 match between the set $\mathcal{M}_W(X)$ of all compatible Wallman nearnesses on a T_1-space X and the set $\mathcal{E}_W(X)$ of all W-T extensions of X. Reed proved that all T_2 compactifications, in particular all Wallman-Frink compactifications, classical Wallman compactifications of T_1 spaces and one-point compactifications of T_1 spaces are W-T extensions. Recently present authors[4] introduced a concept of generalised Wallman type compactifications and proved that they are also W-T extensions. Thus there is a large class of well-known compactifications belonging to the class of W-T extensions. Since for a T_1-space X there is a 1-1 correspondence between $\mathcal{M}_W(X)$ and $\mathcal{E}_W(X)$, the order structure of $(\mathcal{M}_W(X), \subset)$ has some bearing on that of $(\mathcal{E}_W(X), \leq)$.

In this article, the investigation of the order structure of $(\mathcal{M}_W(X), \subset)$ for a T_1-space X has just been begun which may be thought to be far from being completed. It is well-known that the set $(\mathcal{M}(X), \subset)$ of all Lodato nearnesses

*The research of the second author is supported by CSIR, New Delhi.

compatible with the space X is a complete lattice. We describe explicitly the least and the largest elements of $(\mathcal{M}_W(X), \subset)$ and prove that it is also a complete lattice. Though $\mathcal{M}_W(X) \subset \mathcal{M}(X)$, an example is given to show that $(\mathcal{M}_W(X), \subset)$ is not a sublattice of the lattice $(\mathcal{M}(X), \subset)$.

Preliminaries 1.1 — In what follows, in this section, we collect some definitions and results without proof from works of different authors which will be used in this article.

Theorem 1.2 — If in a partially ordered set every nonempty subset that has an upper bound has a least upper bound, then every nonempty subset which has a lower bound has a greatest lower bound. (For proof see Thron[10]).

Definition 1.3 — A 'grill' $\mathcal{G}$ on X is a family of subsets of X satisfying : $A \in \mathcal{G}$ and $A \subset B \Rightarrow B \in \mathcal{G}$, $A \bigcup B \in \mathcal{G} \Rightarrow A \in \mathcal{G}$ or $B \in \mathcal{G}$ and $\phi \notin \mathcal{G}$.

Another way to look at grills is that they are the union of ultrafilters. The collection of all grills on X will be denoted by $\Gamma(X)$ (For more details on these concepts we refer to Thron[9]).

A grill $\mathcal{G}$ on a topological space X is called a c-grill if

$$\forall A \subset X, \bar{A} \in \mathcal{G} \Rightarrow A \in \mathcal{G}.$$

For each point x of a topological space X, set,

$$\mathcal{G}_X = \{ A \subset X : x \in \bar{A}\}.$$

One can easily check that $\mathcal{G}_x$ is a c-grill on X for each $x \in X$ and $\mathcal{G}_x$ is called the 'adherence grill' of the point x.

Definition 1.4 — A 'closed filter' on a topological space X shall be a filter on X having a base consisting of closed sets only. An 'ultraclosed filter' on X shall be maximal closed filter on X. Using Zorn's lemma it can be proved that every closed filter is contained in an ultraclosed filter.

Proposition 1.5 — If F is a closed set in X and F intersects each member of a base of an ultraclosed filter $\mathcal{U}$, consisting of only closed sets, then $F \in \mathcal{U}$.

For a topological space X we shall use the notation

$$\Psi(X) = \{\mathcal{U} : \mathcal{U} \text{ is an ultraclosed filter on } X\}.$$

We note that if X is a T_1-space then for every x in X, $\mathcal{U}_x = \{A : x \in A\}$ is an ultraclosed filter on X and each such ultraclosed filter is called a principal ultraclosed filter. An ultraclosed filter is known as a nonprincipal (n.p.) ultraclosed filter if it is not a principal ultraclosed filter.

Results concerning closed filters may be found in Chattopadhyay[1] and Thorn[10].

Definition 1.6 — A (basic) proximity π on X is a symmetric binary relation on the power set $\mathcal{P}(X)$ of X satisfying the following conditions :

$P_1 : A \bigcap B \neq \phi \Rightarrow (A,B) \in \pi$

$P_2 : (A, B \bigcup C) \in \pi \Leftrightarrow (A,B) \in \pi$ or $(A,C) \in \pi$

$P_3 : (X, \phi) \notin \pi.$

A (basic) proximity π on X induces a closure operator

$$c_\pi(A) = \{ x \in X : (\{x\}, A) \in \pi \}.$$

A (basic) proximity π is called a LO-(Lodato) proximity if $(c_\pi(A), c_\pi(B)) \in \pi \Rightarrow (A,B) \in \pi$. A grill $\mathcal{G}$ on X is called a π-clan if

$$(A, B) \in \pi \ \forall \ \{A, B\} \subset \mathcal{G}.$$

Definition 1.7 — A (basic) nearness ν on X is a collection of families $\mathcal{A} \subset \mathcal{P}(X)$ satisfying :

$N_1 : \bigcap \mathcal{A} \neq \phi \Rightarrow \mathcal{A} \in \nu$

$N_2 : \mathcal{A} \in \nu \Rightarrow \phi \notin \nu$

$N_3 : \mathcal{A} \in \nu \Rightarrow \mathcal{B} \in \nu$ if $\mathcal{B}$ satisfies $B \in \mathcal{B} \Rightarrow \exists \ A \in \mathcal{A}, B \supset A$

$N_4 : \mathcal{A} \notin \nu, \ \mathcal{B} \notin \nu \Rightarrow \{A \bigcup B : A \in \mathcal{A}, B \in \mathcal{B}\} \notin \nu.$

A (basic) nearness ν induces a (basic) proximity

$$\pi_\nu = \{(A, B) : \{A, B\} \in \nu\}.$$

A (basic) nearnesses induces a closure operator

$$c_\nu(A) = \{x : \{ \{x\}, A\} \in \nu\}.$$

The (basic) nearness ν is called a LO-(Lodato) nearness (or simply a nearness due to Herrlich[7] if

$$\{c_\nu(A) : A \in \mathcal{A}\} \in \nu \Rightarrow \mathcal{A} \in \nu.$$

A grill $\mathcal{G}$ on X is a ν-clan if $\mathcal{G} \in \nu$. Thus $\nu \bigcap \Gamma(X)$ is the set of all ν-clans. A maximal family in ν is called a ν-cluster. Every ν-cluster is a grill and is maximal ν-clan. A nearness is called cluster generated (or concerte) if $\mathcal{A} \in \nu \Rightarrow \mathcal{A} \subset \mathcal{G}$ for some ν-cluster $\mathcal{G}$. The collection of all ν-cluster is denoted by X^ν.

Details on proximities and nearnesses may be found in Gagrat and Thron[5], Herrlich[6] and Thron[9]. Now we state Theorem 2.7 of Gagrat and Thron[5] which will frequently be used in this article.

Theorem 1.8 — Let $\gamma = \{\mathcal{G}_i : i \in I$ be a family of grills on X with the property that for every $x \in X$ there exists an i such that $\{x\} \in \mathcal{G}_i$. Then ν_γ, defined by $\mathcal{A} \in \nu_\gamma$ if and only if $\mathcal{A} \subset \mathcal{G}_i$ for some $i \in I$, is a basic nearness on X. If in addition no two members of γ are comparable then ν_γ is cluster generated and $X^{\nu_\gamma} = \gamma$.

Let X be a topological space. A grill $\mathcal{G}$ on X is called a 'Wallman grill' if there exists an ultraclosed filter $\mathcal{U}$ on X such that $\mathcal{U} \subset \mathcal{G}$. The set of all Wallman grills on X shall be denoted by $\Gamma^W(X)$. Thus

$$\Gamma^W(X) = \{\mathcal{G} \in \Gamma(X) : \exists\ \mathcal{U} \in \Psi(X),\ \mathcal{U} \subset \mathcal{G}\}.$$

Henceforth all spaces are assumed to be T_1 unless stated otherwise. Let π be a compatible LO-proximity on a space X; that is, $c_\pi(A) = \bar{A}$ for every $A \subset X$ where $\bar{A}$ denotes the closure of A in X. A π-clan $\mathcal{G}$ is called a Wallman π-clan if $\mathcal{G} \in \Gamma^W(X)$. One can easily check that each adherence grill $\mathcal{G}_x$ is a Wallman π-clan. Thus in view of Theorem 1.8, the set of all Wallman π-clans generates a nearness which is denoted by $\nu_W(\pi)$. Thus

$$\nu_W(\pi) = \{\mathcal{A} \subset \mathcal{P}(X) : \mathcal{A} \subset \mathcal{G} \text{ for some wallman } \pi\text{-clan } \mathcal{G}\}.$$

One can easily check that $\nu_W(\pi)$ is a compatible LO-nearness, that is, $c_{\nu_W(\pi)}(A) = \bar{A}$ for all $A \subset X$. A nearness of the form $\nu_W(\pi)$ is called a 'Wallman nearness' on X. Thus a Wallman nearness on X is a nearness generated by the set of all Wallman π-clans for some compatible LO-proximity π on X.

Proposition 1.9 — A compatible LO-nearness ν on a space X is a Wallman nearness if and only if $\nu = \nu_W(\pi_\nu)$. (For the proof see Reed[8]).

We conclude this section by collecting a few more known definitions and results with exact references for they will frequently be used in the rest of the paper.

Definition 1.10 (Chattopadhyay and Guin[3], Definition 2.1) — A family of grills

$$\Gamma_1 = \{\mathcal{G}_i : i \in I\} \subset \Gamma(X)$$

is called a binary collection with respect to Γ_2 (a second family in $\Gamma(X)$) if the following holds :

If $\mathcal{G} \in \Gamma_2$ has the property that each two-element subfamily $\mathcal{A}$ of $\mathcal{G}$ implies the existence of $i(\mathcal{A}) \in I$ such that $\mathcal{A} \subset \mathcal{G}_{i(\mathcal{A})} \in \Gamma_1$ then there exists a $j \in I$ such that $\mathcal{G} \subset \mathcal{G}_j \in \Gamma_1$.

In what follows we give a simple but useful characterization of Wallman nearnesses.

Theorem 1.11[3] — Let ν be a compatible cluster generated LO-nearness on a space X. Then ν is a Wallman nearness if and only if $X^\nu \subset \Gamma^W(X)$ and X^ν is a binary collection with respect to $\Gamma^W(X)$.

Definition 1.12[3] — For a topological space X and $\mathcal{A} \subset \mathcal{P}(X)$

$$b(\mathcal{A}) = \{A \subset X : \bar{A} \in \mathcal{A}\}$$

then following properties of the operator 'b' can be easily verified.

Theorem 1.13[3] — For a topological space X, the following results hold.

(i) $\mathcal{F} \subset b(\mathcal{F})$ for every filter $\mathcal{F}$ on X.

(ii) If $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n$ are filters on X then

$$b(\mathcal{F}_1 \cup \mathcal{F}_2 \cup \ldots \cup \mathcal{F}_n) = b(\mathcal{F}_1) \cup b(\mathcal{F}_2) \cup \ldots \cup b(\mathcal{F}_n).$$

(iii) $b(\mathcal{U})$ is a c-grill whenever $\mathcal{U}$ is an ultrafilter or ultraclosed filter on X.

(iv) $b(\mathcal{U}_x) = \mathcal{G}_x$ for every principal ultrafilter $\mathcal{U}_x$ on X.

(v) If $\mathcal{U}$ is an ultrafilter and $\mathcal{V}$ is an ultraclosed filter on X such that $\mathcal{V} \subset \mathcal{U}$ then $\mathcal{U} \subset b(\mathcal{V})$.

(vi) Let $\mathcal{V}_1$, $\mathcal{V}_2$ be two ultraclosed filters on X. Then

$$\mathcal{V}_1 \subset b(\mathcal{V}_2) \Leftrightarrow \mathcal{V}_1 = \mathcal{V}_2 \Leftrightarrow b(\mathcal{V}_1) \subset b(\mathcal{V}_2).$$

(vii) Let $\mathcal{U}$, $\mathcal{U}_1$, $\mathcal{U}_2$, ..., $\mathcal{U}_n$ be ultraclosed filters on X such that $\mathcal{U} \subset b(\mathcal{U}_1 \cup \mathcal{U}_2 \cup \ldots \cup \mathcal{U}_n)$ then $\mathcal{U} = \mathcal{U}_i$ for some $i = 1, 2, \ldots, n$.

Remark 1.14 : Though more general results can be proved than those stated in the above theorem, in view of our purpose we do not wish to do so.

Proposition 1.15[3] — Let $\mathcal{A}$ be a family of subsets of a topological space X such that $\overline{\mathcal{A}}$ has the finite intersection property (F.I.P.). Then there exists an ultraclosed filter $\mathcal{U}$ on X such that $\mathcal{A} \subset b(\mathcal{U})$ where $\overline{\mathcal{A}} = \{\overline{A} : A \in \mathcal{A}\}$.

Proposition 1.16[3] — Let ν be a compatible LO-nearness on a space X and $\mathcal{U}$ be an ultraclosed filter on X. Then $b(\mathcal{U})$ is a Wallman π_ν-clan.

Above results immediately yield the following

Theorem 1.17[3] — Let ν be a compatible Wallman nearness on a space X. Then

(i) $b(\mathcal{U}) \in \nu$ for every $\mathcal{U} \in \Psi(X)$.

(ii) $\mathcal{A} \in \nu$ if $\mathcal{A} \subset \mathcal{P}(X)$ and $\overline{\mathcal{A}}$ has the F.I.P.

(iii) $b(\mathcal{F}) \in \nu$ for all filters $\mathcal{F}$ on X.

(iv) $\mathcal{F} \in \nu$ for all filters $\mathcal{F}$ on X.

Theorem 1.18[3] — Let X be a space and ν_0 be the elementary nearness on the space X, that is

$$\nu_0 = \{\mathcal{A} \subset \mathcal{P}(X) : \bigcap \overline{\mathcal{A}} \neq \phi\}.$$

Then following are equivalent :

(i) X is compact.

(ii) ν_0 is a compatible Wallman nearness on X.

2. The Order Structure of Wallman Nearnesses

This section contains our results concerning the order structure of the partially ordered set $(\mathcal{M}_W(X), \subset)$. Let $\mathcal{M}(X)$ be the set of all LO-nearnesses compatible with the space X. Note that $\mathcal{M}_W(X) \subset \mathcal{M}(X)$. Though it is well-known from Herrlich[6] that $(\mathcal{M}(X), \subset)$ is a complete lattice, for the sake of completeness below we sketch a proof of it.

Note that $\nu_0 = \{\mathcal{A} \subset \mathcal{P}(X) : \bigcap \overline{\mathcal{A}} \neq \phi\}$ is the smallest element of $(\mathcal{M}(X), \subset)$.

Also if $\{\nu_i : i \in I$ is a nonempty subfamily of $\mathcal{M}(X)$ then it is immediate that

$$\nu = \bigcup \{\nu_i : i \in I\}$$

is an element of $\mathcal{M}(X)$ and hence ν is the least upper bound of $\{\nu_i : i \in I\}$ in the partially ordered set $(\mathcal{M}(X), \subset)$ and hence in view of Theorem 1.2, we conclude that $(\mathcal{M}(X), \subset)$ is a complete lattice.

In order to prove that $(\mathcal{M}_W(X), \subset)$ is a complete lattice we need the following two lemmas.

Lemma 2.1 — A cluster generated (basic) nearness ν on a set X is a LO-nearness if and only if each ν-cluster is a c-grill on (X, c_ν).

Proof : If ν is a cluster generated LO-nearness and $\mathcal{G}$ is a ν-cluster then $c_\nu(A) \in \mathcal{G}$ implies that $\{c_\nu(A)\} \bigcup \{c_\nu(B) : B \in \mathcal{G}\} \in \nu$ and since ν is a LO-nearness, $\{A\} \bigcup \mathcal{G} \in \nu$ and hence by the maximality of $\mathcal{G}$, $A \in \mathcal{G}$.

Conversely suppose that ν is a cluster generated (basic) nearness such that each ν-cluster is a c-grill on (X, c_ν). Let $\{c_\nu(A) : A \in \mathcal{A}\} \in \nu$ for some $\mathcal{A} \subset \mathcal{P}(X)$. Since ν is cluster generated, there exists a ν-cluster $\mathcal{G}$ such that $\{c_\nu(A) : A \in \mathcal{A}\} \subset \mathcal{G}$. Since $\mathcal{G}$ is a c-grill, $\mathcal{A} \subset \mathcal{G}$ and hence $\mathcal{A} \in \nu$. Thus ν is a LO-nearness on X.

Lemma 2.2 — Let X be a topological space.

Set

$$\gamma_1 = \{b(\mathcal{U}) : \mathcal{U} \in \Psi(X)\}.$$

Then

$$\nu_{\gamma_1} = \{\mathcal{A} \subset \mathcal{P}(X) : \mathcal{A} \subset \mathcal{G} \text{ for some } \mathcal{G} \in \gamma_1\}$$

is a compatible Wallman nearness on X such that $X^{\nu_{\gamma_1}} = \gamma_1$.

Proof : Obviously for $x \in X$, $\mathcal{U}_x$ is an ultraclosed filter and $\{x\} \in b(\mathcal{U}_x)$. Also in view of Theorem 1.13 (iii) and (vi) it follows that if $\mathcal{U}$, $\mathcal{V}$ are ultraclosed filters then $b(\mathcal{U})$ is a c-grill and $b(\mathcal{U}) \subset b(\mathcal{V})$ implies that $\mathcal{U} = \mathcal{V}$.

Taking into account these facts we conclude, by Theorem 1.8 and Lemma 2.1, that ν_{γ_1} is a cluster generated LO-nearness compatible with the space X and $X^{\nu_{\gamma_1}} = \gamma_1$.

To show that ν_{γ_1} is a Wallman nearness we need to check that $\gamma_1 \subset \Gamma^W(X)$ and γ_1 is binary with respect to $\Gamma^W(X)$ (Theorem 1.11). Since each member of γ_1 is of the form $b(\mathcal{U})$ for some ultraclosed filter $\mathcal{U}$ on X it follows by Theorem 1.13 (i) that $\gamma_1 \subset \Gamma^W(X)$.

Let $\mathcal{G} \in \Gamma^W(X)$ such that each two-element subfamily of $\mathcal{G}$ is contained in some element of γ_1. Choose $\mathcal{U}_0 \in \Psi(X)$ such that $\mathcal{U}_0 \subset \mathcal{G}$. Let $A \in \mathcal{G}$ and F be any arbitrary closed set in $\mathcal{U}_0$. Then since $\{A, F\}$ is contained in osme member $b(\mathcal{U})$ of γ_1 it follows that $\bar{A} \bigcap F \neq \phi$. Since the the closed set F has been choosen arbitrarily in $\mathcal{U}_0$ and $\mathcal{U}_0$ is ultraclosed, in view of Proposition 1.5, $\bar{A} \in \mathcal{U}_0$ and hence $A \in b(\mathcal{U}_0)$, consequently $\mathcal{G} \subset b(\mathcal{U}_0)$. Thus γ_1 is a binary collection with respect to $\Gamma^W(X)$.

Theorem 2.3 — Let X be a space. Then $(\mathcal{M}_W(X), \subset)$ is a complete lattice with

$$\{\mathcal{A} \subset \mathcal{P}(X) : \mathcal{A} \subset b(\mathcal{U}),\ \mathcal{U} \in \Psi(X)\}$$

as the least element of $(\mathcal{M}_W(X), \subset)$.

PROOF : Denote by

$$\gamma_1 = \{b(\mathcal{U}) : \mathcal{U} \in \Psi(X)\}.$$

Then by the above lemma ν_{γ_1} is an element of $\mathcal{M}_W(X)$. If $\nu \in \mathcal{M}_W(X)$ then $b(\mathcal{U}) \in \nu$ for all $\mathcal{U} \in \Psi(X)$ (see Theorem 1.17(i)). Hence ν_{γ_1} is the least element of $(\mathcal{M}_W(X), \subset)$.

Let $\{\nu_i : i \in I\}$ be a nonempty subfamily of $\mathcal{M}_W(X)$. Let $\nu = \bigcup \{\nu_i : i \in I\}$. Since $\{\nu_i : i \in I\} \subset \mathcal{M}(X)$ according to the discussion at the beginning of the section

$$\nu = \bigcup \{\nu_i : i \in I\} \in \mathcal{M}(X).$$

In general, ν need not be a Wallman nearness (see Example 2.6). Let us set

$$\nu_L = \nu_W(\pi_\nu).$$

In view of proposition 1.9, ν_L belongs to $\mathcal{M}_W(X)$. We shall show that ν_L is the l.u.b. of $\{\nu_i : i \in I\}$ in $(\mathcal{M}_W(X), \subset)$.

Let $i \in I$ and $\mathcal{A} \in \nu_i$. Then there exists a Wallman π_{ν_i}-clan $\mathcal{G}$ such that $\mathcal{A} \subset \mathcal{G}$. Since $\nu_i \subset \nu$, $\pi_{\nu_i} \subset \pi_\nu$ and hence $\mathcal{G}$ is a Wallman π_ν-clan, consequently $\mathcal{A} \in \nu_L$. Thus ν_L is an upper bound of $\{\nu_i : i \in I\}$ in $(\mathcal{M}_W(X), \subset)$.

Let ν^* be an element of $\mathcal{M}_W(X)$ such that $\nu_i \subset \nu^*$ for all $i \in I$. Then $\nu \subset \nu^*$ and hence $\pi_\nu \subset \pi_{\nu^*}$. Thus any Wallman π_ν-clan is a Wallman π_{ν^*}-clan. Hence

$$\nu_L = \nu_W(\pi_\nu) \subset \nu_W(\pi_{\nu^*}) = \nu^*.$$

Thus in view of Theorem 1.2, we conclude that $(\mathcal{M}_W(X), \subset)$ is a complete lattice.

Remark 2.4 : We have seen that $(\mathcal{M}(X), \subset)$ and $(\mathcal{M}_W(X), \subset)$ are both complete lattices whenever X is a space. Also $\mathcal{M}_W(X) \subset \mathcal{M}(X)$. We shall give an example to show that $(\mathcal{M}_W(X), \subset)$ is not a sublattice of $(\mathcal{M}(X), \subset)$. To accomplish it we begin with the following example.

Example 2.5 — Existence of uncountably many n.p. ultraclosed filters on a noncompact space.

Let X be an uncountable set with the co-countable topology. From a well-known result of set theory we assert that there exists a partition $\{X_x : x \in X\}$ of X, i.e., $X = \bigcup \{X_x : x \in X\}$ and $X_x \bigcup X_y = \phi$ if $x \neq y$ satisfying the additional property

$$|X_x| = |X| \ \forall\ x \in X$$

where $|X|$ denotes the cardinal number of the set X.

Now let us choose $F_x \subset X_x\ \forall\ x \in X$ such that $|F_x| = \mathcal{H}_0$. Since each F_x is countable, it can be expressed as

$$F_x = \{Y_{x,1}, Y_{X,2}, Y_{X,3}, \ldots\}\ \forall\ x \in X..$$

Now set

$$B_{x,n} = \{Y_{x,n}, Y_{x,n+1}, Y_{x,n+2}, \ldots\}\ \forall\ n \in \mathrm{N},\ x \in X..$$

and

$$\mathcal{B}_x = \{B_{x,1}, B_{x,2}, B_{x,3}, \ldots.\}\ \forall\ x \in X..$$

Then for each $x \in X.$, $\mathcal{B}_x$ is a family of closed sets with the F.I.P. Hence there exists an ultraclosed filter $\mathcal{U}(\mathcal{B}_x)$ such that $\mathcal{B}_x$ such that $\mathcal{B}_x \subset \mathcal{U}(\mathcal{B}_x)$. Since $\bigcap \mathcal{B}_x = \phi$, $\mathcal{U}(\mathcal{B}_x)$ is n.p.

Finally note that if $x, y \in X$ and $x \neq y$ then $X_x \bigcap X_y = \phi$. Then $\mathcal{U}(\mathcal{B}_x) \neq \mathcal{U}(\mathcal{B}_y)$ for $X_x \in \mathcal{U}(\mathcal{B}_x)$ and $X_y \in \mathcal{U}(\mathcal{B}_y)$. Thus $|\mathcal{U}(\mathcal{B}_x) : x \in X| = |X|$.

Example 2.6 — Union of Wallman nearnesses on a space X may not be a Wallman nearness.

Let X be a concompact space and Φ be a set of n.p. ultraclosed filters on X such that Φ contains at least three elements (such spaces exist), see the example given above. Let ξ be a collection of finite proper subsets of Φ such that ξ contains each two-element subset of Φ.

Now set

$$\gamma_{\mathcal{A}} = \{b(\mathcal{U}) : \mathcal{U} \in \Psi(X) - \mathcal{A}\} \cup \{b(\cup \mathcal{A})\} \ \forall \ \mathcal{A} \in \xi.$$

In view of Theorems 1.8, 1.13 (iii) and Lemma 2.1, it immediately follows that $\nu_{\gamma_{\mathcal{A}}}$ is a cluster generated *LO*-nearness with $\gamma_{\mathcal{A}}$ as the set of $\nu_{\gamma_{\mathcal{A}}}$-clusters for each $\mathcal{A} \in \xi$.

Let $\mathcal{A}_0 \in \xi$. Since $\mathcal{A}_0$ is a finite subset of Φ, $\mathcal{A}_0$ can be expressed as $\mathcal{A}_0 = \{\mathcal{U}_1, \mathcal{U}_2, ..., \mathcal{U}_n\}$. We now show that $\nu_{\gamma_{\mathcal{A}_0}}$ is a Wallman nearness. Clearly $\gamma_{\mathcal{A}_0} \subset \Gamma^W(X)$. Thus to check that $\nu_{\gamma_{\mathcal{A}_0}}$ is a Wallman nearness it is sufficient to check that $\gamma_{\mathcal{A}_0}$ is a binary collection with respect to $\Gamma^W(X)$.

Let $\mathcal{G} \in \Gamma^W(X)$ such that each two-element subfamily of $\mathcal{G}$ is contained in some element of $\gamma_{\mathcal{A}_0}$. Since $\mathcal{G} \in \Gamma^W(X)$, $\mathcal{G}$ contains an ultraclosed filter $\mathcal{U}_0$ on X.

Case 1 — $\mathcal{U}_0 \in \Psi(X) - \mathcal{A}_0$. Then we claim that $\mathcal{G} \subset b(\mathcal{U}_0)$. If not, by Proposition 1.5, there is a closed set $F \in \mathcal{U}_0$ and $A \in \mathcal{G}$ such that $\bar{A} \cap F = \phi$. Also since $\mathcal{U}_0 \notin \mathcal{A}_0$, there are closed sets $F_1, F_2, ..., F_n$ in $\mathcal{U}_0$ such that $F_k \notin \mathcal{U}_k \ \forall \ k = 1, 2, ..., n$. Let $B = F \cap F_1 \cap F_2 \cap \dots \cap F_n$. Then $B \in \mathcal{U}_0 \subset \mathcal{G}$. Thus $\{A, B\} \subset \mathcal{G}$. Since $\bar{A} \cap B = \phi$, $\{A, B\}$ is not contained in any element of $\gamma_{\mathcal{A}_0}$-a contradiction to the fact that each two-element subfamily of $\mathcal{G}$ is contained in an element of $\gamma_{\mathcal{A}_0}$.

Case 2 — $\mathcal{U}_0 \in \mathcal{A}_0$. Then we claim that $\mathcal{G} \subset b(\cup \mathcal{A}_0)$. If not, there is $A \in \mathcal{G}$ such that $\bar{A} \notin \cup \mathcal{A}_0$ and hence in particular $\bar{A} \notin \mathcal{U}_0$. Then there is a closed set $B \in \mathcal{U}_0$ such that $\bar{A} \cap B = \phi$. Clearly $\{A, B\} \not\subset b(\cup \mathcal{A}_0)$ because $A \notin b(\cup \mathcal{A}_0)$. Also since $\bar{A} \cap B = \phi$, $\{A, B\} \not\subset b(\mathcal{U})$ for all $\mathcal{U} \in \Psi(X) - \mathcal{A}_0$. Note that $\{A, B\}$ is a two-element subfamily of $\mathcal{G}$ which is not contained in any element of $\gamma_{\mathcal{A}_0}$-a contradiction.

Since $\mathcal{A}_0$ has been chosen arbitrarily in ξ, $\nu_{\gamma_{\mathcal{A}}}$ is a Wallman nearness for each $\mathcal{A} \in \xi$.

Set

$$\gamma = \{ b(\mathcal{U}) : \mathcal{U} \in \Psi(X) - \cup \xi \} \cup \{ b(\cup \mathcal{A}) : \mathcal{A} \in \xi \}.$$

Note that $\gamma \subset \cup \{\gamma_{\mathcal{A}} : \mathcal{A} \in \xi\}$ and each element of $\cup \{\gamma_{\mathcal{A}} : \mathcal{A} \in \xi\}$ is contained in some element of γ. Thus

$$\nu_\gamma = \cup \{\nu_{\gamma_{\mathcal{A}}} : \mathcal{A} \in \xi\}$$

It can be easily proved that ν_γ is a cluster generated *LO*-nearness on X with γ as the set of ν_γ-clusters. In fact each $\nu_{\gamma_{\mathcal{A}}}$ and ν_γ are compatible with the space X.

Now we show that ν_γ fails to be a Wallman nearness. Since each n.p. ultraclosed filter is contained in a n.p. ultrafilter, we can choose a n.p. ultrafilter $\mathcal{U}^*$ for each $\mathcal{U} \in \Phi$ such that $\mathcal{U} \subset \mathcal{U}^*$. Set

$$\mathcal{G}^* = \bigcup \{\mathcal{U} : \mathcal{U} \in \Phi\}.$$

By Theorem 1.13 (v), $\mathcal{U}^* \subset b(\mathcal{U})$ for all $\mathcal{U} \in \Phi$. Note that $\mathcal{G}^* \subset \Gamma^W(X)$ and each two-element subfamily of $\mathcal{G}^*$ is contained in some element of $\{b(\bigcup \mathcal{A}) : \mathcal{A} \in \xi\}$ but in view of 1.13(vi), (vii), $\mathcal{G}^*$ is not contained in any element of γ because $\mathcal{A} \neq \Phi$ for each $\mathcal{A} \in \xi$. Thus γ fails to be a binary collection with respect to $\Gamma^W(X)$. Since $\gamma = X^{\nu_\gamma}$, by Theorem 1.11, ν_γ is not a Wallman nearness.

Now as a corollary of the above example it follows that the l.u.b. of $\{\nu_{\gamma\mathcal{A}} : \mathcal{A} \in \xi\}$ in $(\mathcal{M}_W(X), \subset)$ is different from $\nu_\gamma = \bigcup \{\nu_{\gamma\mathcal{A}} : \mathcal{A} \in \xi\}$ which is the l.u.b. of $\{\nu_{\gamma\mathcal{A}} : \mathcal{A} \in \xi\}$ in $(\mathcal{M}(X), \subset)$ (see the discussion at the beginning of this section). Thus $(\mathcal{M}_W(X), \subset)$ is not a sublattice of the lattice $(\mathcal{M}(X), \subset)$.

Following two theorems give an explicit description of the largest element in $(\mathcal{M}_W(X), \subset)$. In order to prove the theorems we need the following lemma.

Lemma 2.7 — A space X is compact if and only if each ultraclosed filter is principal.

PROOF : Suppose that X is compact. Let $\mathcal{U}$ be an ultraclosed filter on X. Since X is cpompact, there exists an $x \in X$ such that $x \in \overline{A}$ for all $A \in \mathcal{U}$. Since $\mathcal{U}$ is ultraclosed and X is T_1, by Proposition 1.5 $\{x\} \in \mathcal{U}$ and hence $\mathcal{U} \subset \mathcal{U}_x$. Since $\mathcal{U}_x$ is closed filter, by the maximality $\mathcal{U} = \mathcal{U}_x$.

Conversely suppose that X is not compact. Then there exists a nonempty family $\mathcal{F}$ of closed sets in X with the finite intersection property such that $\bigcap \mathcal{F} = \phi$. Clearly $\mathcal{F}$ is a subbase of a closed filter on X. Using Zorn's lemma one can prove that there exists an ultraclosed filter $\mathcal{U}$ such that $\mathcal{F} \subset \mathcal{U}$. Since $\bigcap \mathcal{F} \supset \mu$ and $\bigcap \mathcal{F} = \phi$ it follows that $\bigcap \mathcal{U} = \phi$ and hence $\mathcal{U}$ is nonprincipal.

Theorem 2.8 — If X is compact then the elementary nearness ν_0 is the only compatible Wallman nearness on X. In particular ν_0 is the largest element in $(\mathcal{M}_W(X), \subset)$.

PROOF : Let X be compact and ν be a compatible Wallman nearness on X. Since ν is a LO-nearness, $\mathcal{G}_x \in \nu$ for each $x \in X$. If $A \subset X$ and $\mathcal{G}_x \bigcup \{A\} \in \nu$ then $\{\{x\}, A\} \in \nu$ which implies that $x \in \overline{A}$, consequently $A \in \mathcal{G}_x$ and hence $\mathcal{G}_x$ is a ν-cluster for each $x \in X$. Let $\mathcal{G}$ be a ν-cluster. Then $\mathcal{G}$ contains an ultraclosed filter on X. Since X is compact, in view of Lemma 2.7, there exists $x \in X$ such that

$\mathcal{U}_x \subset \mathcal{G}$ which implies, as above, $x \in \overline{A}$ for all $A \in \mathcal{G}$, consequently $\mathcal{G} \subset \mathcal{G}_x$ and hence $\mathcal{G} = \mathcal{G}_x$. Thus $\{\mathcal{G}_x : x \in X\}$ is the set of all ν-clusters, $\nu \subset \nu_0$. Since ν_0 is the smallest element in $\mathcal{M}(X), \subset), \nu = \nu_0$. It is also known that ν_0 is a compatible Wallman nearness on X (Theorem 1.18). Thus ν_0 is the only element of $(\mathcal{M}_W(X), \subset)$.

Theorem 2.9 — If X is a noncompact T_1-space then

$$\nu_G = \{\mathcal{A} \subset P(X) : \bigcap \overline{\mathcal{A}} \neq \phi\} \bigcup \{\mathcal{A} \subset P(X) : A \in \mathcal{A} \Rightarrow |A| \geq \mathcal{H}_0\}$$

is the largest element of $(\mathcal{M}_W(X), \subset)$.

PROOF : Set

$$\gamma = \{\mathcal{G}_x : x \in X\} \bigcup \{\mathcal{G}^*\}$$

where $\mathcal{G}^* = \{A \subset X : |A| A \geq \mathcal{H}_0\}$.

Then it is easily verified that $\nu_G = \nu_\gamma$. Clearly ν_G is a cluster generated LO-nearness compatible with the space X and $X^{\nu_G} = \gamma$.

Since X is T_1, $\mathcal{U}_x$ is ultraclosed filter for each $x \in X$. Also $\mathcal{U}_x \subset \mathcal{G}_x$ for each $x \in X$. Also since X is noncompact, in view of lemma 2.7, there is a non-principal ultraclosed filter $\mathcal{U}$ on X. Clearly $\mathcal{U} \subset \mathcal{G}^*$. Thus $x^{\nu_g} (= \gamma) \subset \Gamma^W(X)$.

We now prove that X^{ν_G} is binary with respect to $\Gamma^W(X)$. Let $\mathcal{G} \in \Gamma^W(X)$ such that each two-element subfamily of $\mathcal{G}$ is contained in some element of $X^{\nu G}$. Choose $\mathcal{U} \in \Psi(X)$ such that $\mathcal{U} \subset \mathcal{G}$.

Case 1 — If $\mathcal{U} = \mathcal{U}_x$ for some $x \in X$ then $\mathcal{G} \subset \mathcal{G}_x$.

Case 2 — If $\mathcal{U}$ is n.p. then we claim that $\mathcal{G} \subset \mathcal{G}^*$. If not, then there is $A \in \mathcal{G}$ such that $|A| < \mathcal{H}_0$. Since $\mathcal{G}$ is a grill there is $x \in X$ such that $\{x\} \in \mathcal{G}$ nu $\{x\} \notin \mathcal{U}$ and hence there is a closed set $F \in \mathcal{U}$ such that $\{x\} \bigcap F = \phi$ (Proposition 1.5). Note that $\{\{x\}, F\} \subset \mathcal{G}$ but $\{\{x\}, F\}$ is not contained in any element of $X^{\gamma G} (= \gamma)$ – a contradiction to the fact that each two-element subfamily of $\mathcal{G}$ is contained in some element of $X^{\nu G}$.

Thus ν_G is Wallman.

Next let ν be any element of $\mathcal{M}_W(X)$.

Let $\mathcal{G} \in X^\nu$. If $|A| \geq \mathcal{H}_0$ for each $A \in \mathcal{G}$

then $\mathcal{G} \subset \mathcal{G}^*$.

If $\mathcal{G}$ contains a finite set then there exists $x \in X$ such that $\{x\} \in \mathcal{G}$. Then for each $A \in \mathcal{G}$, $\{\{x\}, A\} \in \nu$. Since ν is compatible with the space X, $x \in \overline{A}$ for each $A \in \mathcal{G}$ and hence $\mathcal{G} \subset \mathcal{G}_x$. Since ν is cluster generated, $\nu \subset \nu_G$.

Thus ν_G is the largest element of $(\mathcal{M}_W(X), \subset)$.

ACKNOWLEDGEMENT

Authors would like to thank the referee for useful suggestions.

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