

CHAINS OF ARCHIMEDEAN SEMIGROUPS (SEMIPRIMARY SEMIGROUPS)

STOJAN BOGDANOVIC AND MIROSLAV CIRIC

Institute of Mathematics, Knez Mihailova 35, Beograd, Yugoslavia

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We describe chains of Archimedean (right Archimedean, t -Archimedean) semigroups by radicals of ideals (right ideals, bi-ideals) and we prove that these semigroups are exactly semiprimary semigroups.

1. INTRODUCTION AND PRELIMINARIES

By \mathbb{Z}^+ we denote the set of all positive integers. If $a, b \in S$, then $a \mid b$ iff $b = xay$ for some $x, y \in S^{-1}$, $a \mid_r b$ iff $xa = b$ for some $x \in S^{-1}$, $a \mid_l b$ iff $xa = b$ for some $x \in S^1$, $a \mid_i b$ iff $a \mid_r b$ and $a \mid_l b$, $a \rightarrow b$ iff $a \mid b^i$ for some $i \in \mathbb{Z}^+$ and $a \xrightarrow{h} b$ iff $a \mid b^i$ for some $i \in \mathbb{Z}^+$, where h is r, l or t . A semigroup S is Archimedean

(right Archimedean), t -Archimedean iff for all $a, b \in S$, $a \rightarrow b$ ($a \xrightarrow{r} b$, $a \xrightarrow{l} b$). By the radical of the subset A of a semigroup S we mean the set \sqrt{A} defined by

$$\sqrt{A} = \{ x \in S \mid (\exists n \in \mathbb{Z}^+) x^n \in A \}.$$

A subset A of a semigroup S is completely prime iff

$$(\forall a, b \in S) ab \in A \Rightarrow a \in A \vee b \in A.$$

A subsemigroup B of a semigroup S is a bi-ideal of S if $BSB \subseteq B$.

For undefined notions and notations we refer to Gilmer and Mott⁷ and Satyanarayana¹².

Semilattices of Archimedean semigroups well known as Putcha semigroups, were first completely described by Putcha¹¹ and afterwards by the authors^{4, 6}. In this paper we consider chains of Archimedean semigroups and their relationship with semiprimary semigroups.

In our investigations the following result will be very useful

Theorem A⁶ — The following conditions on a semigroup S are equivalent :

- (i) S is a semilattice of Archimedean semigroups;
- (ii) $(\forall a, b \in S) a^2 \rightarrow ab$;
- (iii) \sqrt{A} is an ideal, for every ideal A of S .

Satyanarayana^{12, 13} studied primary semigroups and rings. For the related results see also Aubert¹, Gilmer and Mott⁷, Lal⁸ and Zariski and Samuel¹⁴. Bogdanovic² introduced the following notion : A subset A of a semigroup S is semiprimary iff

$$(\forall x, y \in S) (\exists n \in \mathbb{Z}^+) xy \in A \Rightarrow x \in A \vee y^n \in A.$$

S is a *semiprimary semigroup* if all of its ideals are semiprimary. A simple characterization of semiprimary semigroups is given by the following

*Lemma A*² — A semigroup S is semiprimary if and only if

$$(\forall a, b \in S) ab \rightarrow a \vee ab \rightarrow b. \square$$

2. THE MAIN RESULTS

In this paper we prove that semiprimary semigroups are exactly chains of Archimedean semigroups.

Theorem 1 — The following conditions on a semigroup S are equivalent :

- (i) S is a chain of Archimedean semigroups;
- (ii) S is semiprimary;
- (iii) \sqrt{A} is a completely prime ideal, for every ideal A of S ;
- (iv) \sqrt{A} is a completely prime subset of S , for every ideal A of S ;
- (v) S is a semilattice of Archimedean semigroups and completely prime ideals of S are totally ordered.

PROOF : (i) \Rightarrow (ii). Let S be a chain Y of Archimedean semigroups $S_\alpha, \alpha \in Y$. Let $x \in S_\alpha, y \in S_\beta, \alpha, \beta \in Y$. Then $x, xy \in S_\alpha$ or $y, xy \in S_\beta$, whence

$$x^n \in S_\alpha xy S_\alpha \subseteq SxyS \text{ or } y^n \in S_\beta xy S_\beta \subseteq SxyS$$

for some $n \in \mathbb{Z}^+$. By Lemma A we have that the assertion holds.

(ii) \Rightarrow (iii). Let S be a semiprimary semigroup. Let $a, b \in S$. Then by Lemma A there exists $n \in \mathbb{Z}^+$ such that

$$(ba)^n \in S(ba)(ab)S \text{ or } (ab)^n \in S(ba)(ab)S$$

whence

$$(ab)^{n+1} \in Sa^2S \tag{1}$$

so by Theorem A it follows that \sqrt{A} is an ideal, for every ideal A of S . Let $ab \in \sqrt{A}$. Then from

$$a^n \in SabS \subseteq \sqrt{A} \text{ or } b^n \in SabS \subseteq \sqrt{A}$$

for some $n \in \mathbb{Z}^+$, it follows that $a \in \sqrt{A}$ or $b \in \sqrt{A}$. Therefore, \sqrt{A} is a completely prime ideal.

(iii) \Rightarrow (iv). This implication follows immediately.

(iv) \Rightarrow (ii). Let \sqrt{A} be a completely prime subset of S , for every ideal A of S . Then \sqrt{SabS} is a completely prime subset of S , for every $a, b \in S$. Since $a^2 b^2 \in SabS \in \sqrt{SabS}$, we then have that

$$a^2 \in \sqrt{SabS} \text{ or } b^2 \in \sqrt{SabS}$$

i.e.

$$a^{2k} \in SabS \text{ or } b^{2k} \in SabS$$

for some $k \in \mathbb{Z}^+$, whence by Lemma A it follows that S is a semiprimary semigroup.

(ii) \Rightarrow (i). Let S be a semiprimary semigroup. Then we have that the condition (1) holds and by Theorem A, S is a semilattice Y of Archimedean semigroups $S_\alpha, \alpha \in Y$. Let $\alpha, \beta \in Y$. Assume that $a \in S_\alpha, b \in S_\beta$. Then by Lemma A there exists $n \in \mathbb{Z}^+$ such that

$$a^n \in SabS \text{ or } b^n \in SabS$$

whence $\alpha \leq \beta$ or $\beta \leq \alpha$. Thus Y is a chain.

(iii) \Rightarrow (v). Let P_1 and P_2 be completely prime ideals of S . Suppose that $P_1 \not\subseteq P_2$ and $P_2 \not\subseteq P_1$. Then there exists $a \in P_1 - P_2$ and $b \in P_2 - P_1$, whence $ab \in P_1 \cap P_2 = \sqrt{P_1 \cap P_2}$, and by (iii), $a \in P_1 \cap P_2$ or $b \in P_1 \cap P_2$, which is not possible. Therefore, completely prime ideals of S are totally ordered. That S is a semilattice of Archimedean semigroups follows by Theorem A.

(v) \Rightarrow (iii). By Theorem A we have that \sqrt{A} is an ideal, for every ideal A of S . By Theorem II 3.7 of Petrich⁹ we have that $\sqrt{A} = \bigcap P_i, A \subseteq P_i$, where P_i are completely prime ideals. Assume that $a \notin \bigcap P_i, b \notin \bigcap P_i$. Then there exist P_j, P_k such that $a \notin P_j$ and $b \notin P_k$. By hypothesis, $P_j \subseteq P_k$ or $P_k \subseteq P_j$. Assume that $P_j \subseteq P_k$ (the case $P_k \subseteq P_j$ can be similarly treated). Then $a \notin P_j$ and $b \notin P_j$ and $ab \notin P_j$, since P_j is completely prime. Thus $ab \notin \bigcup P_i$ and by contradiction we have the assertion. \square

Proposition 1 — Every right ideal of a semigroup S is semiprimary if and only if

$$(\forall a, b \in S) ab \xrightarrow{r} a \vee ab \xrightarrow{r} b. \quad \dots (2)$$

PROOF : Let every right ideal of a simgroup S be semiprimary. Let $a, b \in S$. Then abS is semiprimary and $ab^2 \in abS$, whence $ab \xrightarrow{r} a$ or $ab \xrightarrow{r} b$. Thus the condition (2) holds.

Conversely, let R be a right ideal of S . Assume that $ab \in R$. Then by hypothesis there exists $n \in \mathbb{Z}^+$ such that $a^n \in abS \subseteq RS \subseteq R$ or $b^n \in abS \subseteq RS \subseteq R$. Therefore, R is semiprimary. \square

Lemma 1 — Let R be a right ideal of a semigroup S . If \sqrt{R} is completely prime, then R is semiprimary.

PROOF : Let $a, b \in S$. then

$$ab \in R \subseteq \sqrt{R} \Rightarrow a \in \sqrt{R} \vee b \in \sqrt{R} \Rightarrow (\exists n \in \mathbb{Z}^+) a^n \in R \vee b^n \in R \square$$

By the next theorem chains of right Archimedean semigroups will be described by radicals of right idelas.

Theorem 2 — The following conditions are equivalent on a semigroup S :

- (i) S is a chain of right Archimedean semigroups;
- (ii) \sqrt{R} is a completely prime ideal, for every right ideal R of S ;
- (iii) S is a semilattice of right Archimedean semigroups and every right ideal of S is semiprimary;
- (iv) S is a semilattice of right Archimedean semigroups and the condition (2) holds.

PROOF : (i) \Rightarrow (ii) — By Theorem 5 we have that \sqrt{R} is an ideal of S , for every right ideal R of S . Let $ab \in \sqrt{R}$. Since S is a chain of right Archimedean semigroups, we then have that $ab \xrightarrow{r} a$ or $ab \xrightarrow{r} b$, whence

$$a^n \in abS \subseteq \sqrt{R} \text{ or } b^n \in abS \subseteq \sqrt{R}$$

for some $n \in \mathbb{Z}^+$, and therefore, $a \in \sqrt{R}$ or $b \in \sqrt{R}$, i.e. \sqrt{R} is completely prime.

(ii) \Rightarrow (iii). That every right ideal of S is semiprimary follows by Lemma 1. That S is a semilattice of right Archimedean semigroups follows by Theorem 5 of Bogdanovic and Ciric⁴.

(iii) \Rightarrow (iv) — It is clear that $a^n \in abS$ or $b^n \in abS$ for some $n \in \mathbb{Z}^+$. Assume that $a^n = abx$, for some $x \in S$. Then by Theorem 5 of Bogdanovic and Ciric⁴ we have that $(a^n)^k \in bS$ for some $k \in \mathbb{Z}^+$. Therefore, $b \xrightarrow{r} a$. Similarly we treat the case $b^n \in abS$. Thus the condition (2) holds.

(iv) \Rightarrow (i) — Let S be a semilattice Y of right Archimedean semigroups S_α , $\alpha \in Y$. Let $\alpha, \beta \in Y$. Assume that $a \in S_\alpha$, $b \in S_\beta$. Then from $a^n \in bS$ or $b^n \in aS$ for some $n \in \mathbb{Z}^+$, it follows that $a^{n+1} \in abS$ or $b^{n+1} \in baS$, whence $\alpha \leq \beta$ or $\beta \leq \alpha$, i.e. Y is a chain. \square

Chains of t -Archimedean semigroups will be described via bi-ideals by the next theorem.

Theorem 3 — The following conditions are equivalent on a semigroup S :

- (i) S is a chain of t -Archimedean semigroups;
- (ii) \sqrt{B} is a completely prime ideal, for every bi-ideal B of S ;
- (iii) S is a semilattice of t -Archimedean semigroups and the condition (2) holds.

PROOF : (i) \Rightarrow (ii) — By Theorem 9 of we have that \sqrt{B} is an ideal of S , for every bi-ideal B of S . Let $ab \in \sqrt{B}$. Since S is a chain of t -Archimedean semigroups, we have that $ab \xrightarrow{r} a$ or $ab \xrightarrow{r} b$, whence

$$a^n \in abS \subseteq \sqrt{B} \text{ or } b^n \in abS \subseteq \sqrt{B}$$

for some $n \in \mathbb{Z}^+$. Thus $a \in \sqrt{B}$ or $b \in \sqrt{B}$, i.e. \sqrt{B} is completely prime.

(ii) \Rightarrow (iii) — That S is a semilattice of t -Archimedean semigroups follows by Theorem 9 of Bogdanović and Ćirić⁴. The rest follows by Theorem 2.

(iii) \Rightarrow (i). Let S be semilattice Y of t -Archimedean semigroups $S_\alpha, \alpha \in Y$. Let $\alpha, \beta \in Y$. Assume $a \in S_\alpha, b \in S_\beta$. Then by (2) there exists $n \in \mathbb{Z}^+$ such that

$$b^n = ax \text{ or } a^n = by$$

for some $x \in S_\gamma, y \in S_\delta, \gamma, \delta \in Y$, so

$$b^{n+1} = bax \text{ or } a^{n+1} = aby.$$

From this it follows that $\beta = \beta\alpha\gamma$ or $\alpha = \alpha\beta\delta$, whence $\beta = \alpha\beta$ or $\alpha = \alpha\beta$, i.e. $\alpha \leq \beta$ or $\beta \leq \alpha$. Therefore, Y is a chain. \square

Finally we characterize semigroups in which radicals of all semigroups are completely prime.

Theorem 4 — The radical of every subsemigroup of semigroup S is completely prime if and only if

$$(\forall a, b \in S) (\exists n \in \mathbb{Z}^+) a^n \in \langle ab \rangle \vee b^n \in \langle ab \rangle. \quad \dots (3)$$

PROOF : Let \sqrt{A} be completely prime for every subsemigroup A of S . Then for all $a, b \in S$ we have that

$$ab \in \langle ab \rangle \subseteq \sqrt{\langle ab \rangle} \Rightarrow a \in \sqrt{\langle ab \rangle} \vee b \in \sqrt{\langle ab \rangle}.$$

Therefore, condition (3) holds.

Conversely, let A be a subsemigroups of S . Let $ab \in \sqrt{A}$. Then $(ab)^k \in A$, for some $k \in \mathbb{Z}^+$. Since

$$a^n = (ab)^r \vee b^n = (ab)^t,$$

for some $n, r, t \in \mathbb{Z}^+$, we then have that

$$a^{nk} = (ab)^{rk} \in A \vee b^{nk} = (ab)^{tk} \in A$$

whence $a \in \sqrt{A}$ or $b \in \sqrt{A}$. Therefore, \sqrt{A} is completely prime. \square

It is clear that condition (3) is equivalent to

$$(\forall a, b \in S) (\exists n \in \mathbb{Z}^+) (ab)^n \in \langle a \rangle \cup \langle b \rangle. \quad \dots (4)$$

Semigroups which satisfy (4), called generalized \mathcal{U} -semigroups or, shortly, \mathcal{GU} -semigroups, are considered by the authors^{3, 5}. The structure of periodic \mathcal{GU} -semigroups is completely described in Ćirić and Bogdanović⁵.

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